Thermodynamic phenomena on vortex filaments in helium

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(Submitted July 10, 1974)
Zh. Eksp. Teor. Fiz. 67, 2293–2306 (December 1974)

It is shown that localization of thermal and impurity excitations on a vortex filament in helium leads to an experimentally observable dependence of the effective radius $b^*(c, T)$ of the vortex filament core on the temperature and the solution concentration. General formulas are derived which relate $b^*(c, T)$ with the value of the thermodynamic potential $\alpha(c, T)$ per unit filament length for excitations localized on the filament. Calculations of $\alpha(c, T)$ for a vortex filament in pure He^4 and for a vortex in a weak He^3–He^4 solution are presented for extremely low temperatures. In the first case the explicit form of the dispersion law for flexural vibrations of the vortex filament is employed in the calculations. In the second case a phenomenological calculation of $\alpha(c, T)$ is found to be possible (with an accuracy to two constants) and is independent of the details of the spectrum of the impurity excitations localized on the filament. The results of the theory for weak solutions are compared with the available experimental data. Qualitative agreement between theory and experiment is noted. The problem of motion of vortex rings along the vapor-liquid interface in liquid helium is formulated in the Appendix and solved for some limiting cases. It is shown that the proximity of the boundary leads to a number of observable effects. It is suggested, in particular, that under ceratin conditions the formation of vortex semi-rings whose ends emerge to the free surface of helium should be possible.

INTRODUCTION

It is well known (Atkins[1], Andreev[2]) that the presence in liquid helium of surface thermal excitations leads to the appearance of an experimentally observable temperature dependence of the surface-tension coefficient on the flat liquid-vapor surface, and to other thermodynamic effects. Similar phenomena also should accompany the adsorption of thermal excitations on vortex filaments. Until recently, however, this question was not discussed seriously, in view of the lack of really observable effects of thermodynamic origin due to vortex motion.

The use of helium ions has greatly extended the experimental capabilities for the study of different localized thermal excitations in liquid helium. In particular, a definite scheme was developed for the observation of thermodynamic phenomena on a vortex filament, which makes it possible to extract the necessary quantitative information on the value of the adsorption on the filament from the experimental data on the motion of the charged vortex rings in superfluid helium and its weak solutions. The first concrete results in this direction were obtained experimentally by Kuchnir, Ketterson, and Roach[3], who have shown that the effective size of the "core" of the vortex filament in weak solutions of helium is a function of the solution concentration and of the temperature. They presented a variant of the explanation of their experimental data. This explanation is based on the assumption that the excess concentration of the impurity excitations in the vicinity of the vortex filament leads to a local stratification of the solution. The position of the stratification boundary, which depends on the temperature and concentration of the solution, determines the value of the new effective radius b* of the vortex-filament core, which reaches a value $b^* \approx 4$ Å at low temperatures (we recall that in pure helium at low temperatures the radius of the vortex-filament core is $b_0 = 1.2$ Å[4]).

In spite of the physical nature of this interpretation, it must be recognized that the introduction of the concept of the stratification boundary under sharply inhomogeneous conditions, when the linear dimensions of the phase enriched with the He^3 atoms have a scale comparable with the thickness of the boundary itself, is somewhat artificial. Another possibility for the interpretation of the experimental data[3] and related phenomena on vortices was indicated by the author earlier[5]. It stems from the deep analogy between the adsorption phenomena on the flat surface of helium and vortex filaments, makes it possible to take into account in similar fashion the contribution made to the adsorption phenomena on the filament by the phonons, rotons, impurity excitations, and, quite importantly, is free of the model assumptions used in[3].

In this paper we present a systematic exposition of the theory of thermodynamic phenomena on vortex filaments in pure He^4 and its weak solutions in the low-temperature limit.

BASIC DEFINITIONS

1. We consider a vortex ring of radius R in liquid helium. The production of such rings with the aid of helium ions entails no difficulty at present[4]. The total energy of the ring and its momentum are expressed in the form[6, 7]

$$W_0=\frac{1}{2}\rho_s\Gamma^2 R\left(\ln\frac{8R}{b_0}-\frac{3}{2}\right),\quad P_0=\pi\rho_s\Gamma R^2, \tag{1}$$

where Γ is the circulation of the vortex filament making up the ring, b_0 is the radius of its core ($\Gamma = 2\pi\hbar/m_4 = 10^{-3}$ cm^2/sec, $b_0 = 1.2$ Å), and ρ_s is the density of the superfluid component of the helium.

The form of expressions (1) is known from classical hydrodynamics[6], apart from the constant C which enters in the definition of W_0. The numerical value of this constant, C = 3/2, was obtained by Roberts and Donelly[7].

It is shown in their paper that the velocity V of the vortex ring can be expressed in terms of W_0 and P_0 in the usual manner: $V = \partial W_0/\partial P_0$. As a result we have

$$V=\frac{\Gamma}{4\pi R}\left(\ln\frac{8R}{b_0}-\frac{1}{2}\right). \tag{2}$$

We note now that under real conditions the total ring energy W should include also the thermal component of the energy, which is connected with the presence on the vortex filament of adsorbed excitations:

$$W=W_0+W_T,\quad W_T=-2\pi R\alpha(c,\,T), \tag{3}$$

where α(c, T) is the energy of adsorption of a given sort of thermal or impurity excitations, calculated per unit length of the vortex filament. It is assumed in (3) that $R \gg a$, where a is the radius of the ion localized on the vortex filament.

The total momentum of the ring also changes in principle as a result of the contribution of the excitations adsorbed on the ring. This contribution, however, just as W_T, is proportional to the radius of the ring, whereas the hydrodynamic momentum P_0 in (1) is proportional to the square of the radius. As aresult, for large vortex rings with $R \gg a$, which will be discussed later on, the change of the momentum as a result of the adsorption of the excitations can be neglected.

Taking (3) into account, the vortex velocity determined from the equation $V = \partial W/\partial P_0$ turns out to be

$$V=\frac{\Gamma}{4\pi R}\left(\ln\frac{8R}{b^*}-\frac{1}{2}\right),\quad b^*=b_0\exp\left(\frac{4\pi}{\rho_s\Gamma^2}\alpha\right). \tag{2a}$$

This expression is equivalent in its structure to (2). However, the radius of the core of the filament, b*, turns out to be effectively dependent on the temperature and on the concentration of the impurity excitations to the same extent that $\alpha(c, T) \neq 0$. It should be noted that the definition of b* in (2a) is general in form and does not depend on any model assumptions. This definition was obtained earlier in somewhat different form[5].

The task of the theory reduces now to the calculation of the adsorption energy α(c, T) in accordance with the known general rules[8]. Before we proceed to solve this problem, however, it is meaningful to discuss the correctness of the determination of b* from the experimental data on the motion of vortex rings in helium.

2. The most lucid situation, from all points of view, is the one in which a charged ring moves through the superfluid helium in a stationary regime, i.e., under conditions when the external force eE applied to the charge is fully balanced by the friction force exerted on the ring by the normal component of the helium

$$eE=F_{\mathrm{fr}}=2\pi R\eta V,\quad R\gg a, \tag{4}$$

η is a dissipative coefficient calculated per unit filament length. At a given temperature and external field intensity E, relation (4) determines the equilibrium radius of the ring as a function of E and V. Substituting R from (4) in the definition of the velocity (2a), we easily obtain the explicit dependence of V on E:

$$V=\frac{4eE}{\pi b^*\eta}\exp\left(-\frac{2eE}{\Gamma\eta}-\frac{1}{2}\right), \tag{5}$$

which was observed experimentally. If we now choose some pair of experimental points E_m and V_m (corresponding, for example, to the maximum electric field used in a given experiment), denoting the remaining pairs of points by E and V, and assuming that the experimental data on the mobility of the charged vortex rings are described by expression (5), then we obtain for the connection between E, V, and E_m, V_m the following result:

$$\ln\frac{V}{E}\frac{E_m}{V_m}=\frac{2e}{\Gamma\eta}(E_m-E). \tag{5a}$$

It follows from (5a) that $\ln(VE_m/EV_m)$ is a linear function of the electric field, and the slope of this line determines directly the coefficient η independently of b*. By determining the numerical value of η from the experimental data, we can then easily calculate b* from any of the E, V pairs. This was precisely the method used to determine the radius of the core of a filament in pure He^4[4] and in its weak solution[3]. In both cases the experimental data on the mobility of the charged vortex rings can be fitted with good accuracy to straight lines plotted in the coordinates

$$\ln\frac{V}{E}\frac{E_m}{V_m},\qquad E_m-E.$$

It should be noted, however, that the definitions (4) and (5) are perfectly rigorous only so long as we are dealing with a ring uniformly charged along its perimeter. Actually, however, the charge is localized on the ring practically in δ-function manner. As a result, the points where the force eE is applied no longer coincide with the total friction force, which is applied to the center of the ring, and a force couple $|K| = eER$ begins to act on the ion-ring complex. The question is: how is the action of this couple to be neutralized? And what is the justification for writing down Eq. (4)?

If we disregard the local deformation of the ring in the vicinity of the localized ion, then the answer is the following. Under the influence of the force couple K, the total velocity of the ring V ceases to be perpendicular to the ring plane (Fig. 1). At a given radius R, the velocity $V_\perp$ retains, of course, its value (2). But together with $V_\perp$ there appears a velocity component $V_\parallel$ corresponding to the motion of the ring in its own plane. To the extent that $V_\parallel \neq 0$, additional Magnus forces $\mathbf{G} = -\rho_s\mathbf{\Gamma}\times\mathbf{V}$ appear, directed in this case in such a way as to produce a couple that is negative relative to K. The condition that these two couples be equal, together with the other obvious relations, enables us to write down a closed system of equations for the quantities $V_\parallel$, $V_\perp$, $E_\parallel$, and $E_\perp$. The explicit form of this system is given inthe Appendix. It turns out that the ratios $E_\parallel/E_\perp$ and $V_\parallel/V_\perp$, which characterize the tilting of the ring plane by the couple K and the need for its can-

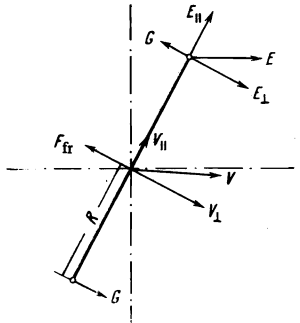

FIG. 1

cellation, are determined by the equalities

$$E_{\parallel}/E_{\perp}=2\eta_{\parallel}/\rho_s\Gamma, \quad V_{\parallel}/V_{\perp}=2\eta_{\perp}/\rho_s\Gamma. \tag{6}$$

Here $\eta_{\parallel}$ and $\eta_{\perp}$ are the friction coefficients that arise when the ring moves in its plane and perpendicular to it, calculated per unit length of the vortex filament. These coefficients seem to have the same order of magnitude, but can differ numerically from each other. Expressing $\eta_{\perp}$ in terms of E and R, we rewrite the expression for $V_{\parallel}/V_{\perp}$ in a different form:

$$\frac{V_{\parallel}}{V_{\perp}} \approx 4eE/\rho_s\Gamma^2 \ln\frac{8R}{b^*}, \quad \rho_s\approx\rho. \tag{6a}$$

For all reasonable values of E encountered in experiments ($E \lesssim 10-10^2$ cgs esu) and the quantities R that are uniquely related to them. the ratio $V_{\parallel}/V_{\perp}$ given by (6a) turns out to be very small. Thus, at $E \sim 10$ cgs esu, $\Gamma \approx 10^{-3}$ cm^2/sec, and $\rho = 0.146$ g/cm^2 we have $V_{\parallel}/V_{\perp} \sim 10^{-2}-10^{-4} \ll 1$. The smallness of this ratio justifies the use of the definitions (4) and (5) to describe the motion of real ion-ring complexes under the influence of an external field in a medium with finite viscosity.

CALCULATION OF α(T) IN PURE He^4

According to[8], and also[1, 2], the starting point for the calculation of the adsorption energy on the surface of helium or of a vortex filament is the determination of the spectra of the corresponding localized thermal or impurity excitations. Once this problem is solved, we can use the general formulas of statistical physics to calculate α(c, T).

1. The most thoroughly investigated of the Bose excitations localized on a vortex filament in pure He^4 are the flexural oscillations of the filament core. The dispersion of oscillations of this type was known already to Kelvin

$$\omega = \frac{\Gamma}{4\pi} q^2 \left(\ln\frac{2}{qb_0} - C\right), \tag{7}$$

where q is the wave number of the excitation along the vortex-filament axis, ω is its frequency, and $C \approx 1$.

Having the definition (7), we calculate (cf.[1]) the contribution of the flexural oscillations to the thermodynamic potential Ω, and by the same token also to α(T)

$$\alpha(T) = \Omega/L, \quad \Omega = \frac{LT}{2\pi}\int_0^\infty \ln(1 - e^{-\hbar\omega/T})\,dq, \tag{8}$$

L is the total length of the vortex filament. In the expression for Ω, the chemical potential of the flexural oscillations is, as usual, equal to zero. Substituting (7) in (8), we get, with logarithmic accuracy,

$$\alpha(T) = T^{3/2}\zeta\left(\frac{3}{2}\right) \Big/ 2\left(\pi\hbar\Gamma\ln\frac{2}{q_Tb_0}\right)^{1/2}, \quad \frac{\hbar\Gamma q_T^2}{4\pi}\ln\frac{2}{q_Tb_0} \approx T, \tag{9}$$

where ζ(x) is the Riemann function.

The combination

$$\frac{4\pi}{\rho\Gamma^2}\alpha(T) = 2\pi^{1/2}T^{3/2}\zeta\left(\frac{3}{2}\right) \Big/ \rho\Gamma^2\left(\hbar\Gamma\ln\frac{2}{q_Tb_0}\right)^{1/2}, \tag{9a}$$

which determines the renormalization of the radius of the core of the vortex filament b* from (2a) at temperatures $T \lesssim 1$°K turns out to be smaller than or of the order of unity, i.e., the effect of the renormalization of b* as a result of the flexural oscillations of the vortex-filament core is quite readily observable.

2. The flexural oscillations of the vortex-filament core make the main contribution to the temperature dependence of α(T) only at low temperatures. With increasing T, it becomes necessary to take into account other Bose excitations localized on the vortex filaments, namely phonons and rotons. A consistent calculation of the spectra of these excitations calls for the use of numerical methods and have not yet been performed, although certain approximate results are available in this direction[9]. We confine ourselves therefore to the remark that the presence of such excitations, together with the perturbation of the distribution function of the volume phonons and rotons by the field of the vortex filament, leads to the appearance of additional terms proportional to T^λ ($\lambda > 3/2$) and $\exp(-\Delta/T)$ in the temperature dependence α(T). In other words, with increasing temperature the function α(T) in pure He^4 becomes extremely more complicated in comparison with the low-temperature limit, when $\alpha(T) \propto T^{3/2}$.

VALUE OF α(c, T) IN WEAK He^3-He^4 SOLUTIONS

Let us proceed to calculate α(c, T) in weak He^3–He^4 solutions. A solution is regarded as weak if it satisfies the Boltzmann condition $T \gg T_F$, or

$$\Lambda(c, T) = \frac{2m_4}{c\rho}\left(\frac{m_3^*T}{2\pi\hbar^2}\right)^{3/2} \gg 1, \tag{10}$$

where ρ is the density of the solvent, c is the relative volume concentration of the solutions, and m_3^* and m_4 are the effective masses of the He^3 and He^4 atoms in the solution. The chemical potential of the dissolved particles under similar conditions is

$$\mu = -T\ln\Lambda(c, T) \ll 0. \tag{11}$$

The localization of the impurities in the vicinity of the vortex filaments (such impurities will be called later on *l*- impurities) is due to the action exerted on the impurity excitation by the attracting potential V(r), which behaves like $v(r) \propto r^{-2}$ at large distances r. In the problem of the spectrum of particles with a potential of this type, the particle should fall on the axis of the vortex filament. For this reason, to obtain finite results it is necessary to have a model of the potential V(r) at short distances. Choosing V(r) in the form[10]

$$V(r) = \begin{cases} -\gamma_j r^{-2} & r \geqslant b_3 \\ U_0 \to +\infty, & b_3 \geqslant r \geqslant 0 \end{cases},$$

$$\gamma_j = \frac{2\pi\hbar^2}{m}\left[(\delta m^2 - \delta m)\left(\frac{m}{m_4}\right)^2 \pm \delta m\frac{m}{m_4}j\right], \quad \delta m = \frac{m - m_3}{m_3}; \quad j = 0, 1, 2\ldots \tag{12}$$

(m is the mass of the *l*-impurity, m_3 is the mass of the bare He^3 atom, b_3 is the characteristic cutoff radius of the attraction potential at short distances, j is the orbital quantum number, the plus and minus signs in (12) correspond to different signs in the scalar product $p \cdot v_s$, where $v_s(r)$ is the vortex-filament velocity field and p is the momentum of the *l*-impurity), and solving the problem of the spectrum of the *l*-impurities in the quasiclassical approximation, we obtain

$$\begin{gathered} \varepsilon_{nj}(p) = -\Delta_{nj} + \frac{p^2}{2m}, \quad \Delta_{nj} = \frac{8\pi^2\hbar^2}{mb_3^2}\lambda_j\exp\left(-\frac{2n}{\lambda_j^{1/2}}\right), \\ \lambda_j = m\gamma_j/2\pi^2\hbar^2 - j^2/(2\pi)^2, \quad p = p_z, \quad 0 \leqslant p^2 \leqslant 2m\Delta_{nj}, \\ n = n_0,\ n_0+1, \ldots, \quad n_0 \gg 1, \quad \lambda_j > 0, \quad \varepsilon_{nj}(p) \leqslant 0. \end{gathered} \tag{13}$$

The z axis is directed along the vortex filament, and n is the radial quantum number. According to (13), the numerical values of n can be arbitrarily large. In other words, the l-impurity spectrum has a condensation point at the boundary of the continuous spectrum. The limits of integration with respect to p in (13) follow from the condition $\epsilon_{nj}(p) \leq 0$, and the limits of the variation of the quantum number j are determined by the requirement $\lambda_j > 0$. As to the deep levels that do not lend themselves to the quasiclassical description, their explicit position will not be needed in what follows.

The presence of a condensation point in the l-impurity spectrum greatly complicates the analysis of the temperature dependence of $\Omega_l(c, T)$ for the vortex filament in comparison with the analogous problem for the plane surface. The point is that the degeneracy temperature T_F^S of the impurity excitations on the surface levels is a finite quantity. Therefore, in the region $T \ll T_F^S$, the following expansion of the thermodynamic potential is valid[11]:

$$\Omega_s(\mu, T) \approx \Omega_s(\mu, 0) = \Omega_s(\mu_0, 0) + \delta\mu\,\partial\Omega_s/\partial\mu \tag{14}$$

($\Omega_S(\mu_0, 0)$ and $\partial\Omega_S/\partial\mu = -N_S^0$ are the maximum adsorption energy and the maximum number of impurity excitations localized on a flat surface), which solves the problem of the concentration and temperature dependences of the surface tension in the low-temperature region, accurate to terms $(T/T_F^S)^2 \ll 1$.

A similar expansion of $\Omega_l(\mu, T)$ for l impurities, owing to the presence of a large number of shallow levels and the ensuing absense of a clear-cut definition of the Fermi temperature, should contain, in comparison with (14), an additional last term in

$$\Omega_l(\mu, T) \approx \Omega_l(\mu_0, 0) + \frac{\partial\Omega_l}{\partial\mu}\delta\mu + \frac{\partial\Omega_l}{\partial T}\delta T, \tag{14a}$$

the smallness of which relative to the first two terms cannot be established from general considerations. Nonetheless, the main result of a more detailed analysis, which takes into account the concrete structure of the quasiclassical part of the l-impurity spectrum, reduces to the fact that at low temperatures the third term of (14a) is small in comparison with the second. Consequantly, as in the surface problem, the temperature dependence of $\Omega_l(c, T)$ is determined in the low-temperature limit mainly by the variation of the chemical potential μ with temperature:

$$\alpha(c, T) = \Omega_l/L = \alpha_0 - z_0 T \ln \Lambda. \tag{15}$$

The constants α_0 and z_0, which have the meaning of maximum values of the adsorption energy and the number of impurity excitations localized on the unit length of the vortex, remain parameters of the theory in the phenomenological analysis and should be determined experimentally.

2. We present a comparative estimate of the last term of (14a). To this end, we express Ω_l in terms of $\epsilon_{nj}(p)$

$$\Omega_l = -\frac{LT}{\pi\hbar}\sum_{nj}\int dp\,(2j+1)\ln[1+\exp(\varkappa_{nj}(p))],$$
$$\varkappa_{nj}(p) = \frac{\mu_l - \varepsilon_{nj}(p)}{T} \equiv \frac{2m(\Delta_{nj} - |\mu|) - p^2}{2mT}, \qquad \mu_l \equiv \mu, \tag{16}$$

where μ_l is the chemical potential of the l- impurities. The value of the latter, according to the general premises[8, 2], should coincide with the value of the chemical potential μ of the impurity excitations in the volume of the liquid, i.e., in the particular case of Boltzmann statistics, with the value of μ from (11).

Furthermore, we have

$$\frac{\partial\Omega_l}{\partial T} = -\frac{L}{\pi\hbar}\sum_{nj}\int dp\,(2j+1)\left[\ln(1+e^{\varkappa}) - \frac{\varkappa}{1+e^{-\varkappa}}\right] \tag{17}$$

(where $\kappa \equiv \kappa_{nj}(p)$). We see therefore that the square bracket, and with it also $\partial\Omega_l/\partial T$, differ noticeably from zero, on account of states located in the vicinity of

$$-1 \leq \varkappa_{nj}(p) \leq 1. \tag{17a}$$

Neither the deep states with $\kappa \gg 1$, nor the shallow ones with $\kappa \ll -1$ (in particular, also the condensation point near the boundary of the continuous spectrum) contribute to $\partial\Omega_l/\partial T$.

At the same time, the quantity

$$N_l = -\frac{\partial\Omega_l}{\partial\mu} = \frac{L}{\pi\hbar}\sum_{nj}\int dp\,\frac{2j+1}{e^{-\varkappa}+1} \tag{18}$$

contains contributions from all states: deep, intermediate, and shallow. For this reason, we can conclude on the basis of (17), (17a), and (18) that the relation between N_l and $\partial\Omega_l/\partial T$ should take the form $N_l > \partial\Omega_l/\partial T$. If we now recognize that $|\mu| \gg T$ in accordance with (11), then the following inequality becomes obvious

$$N_l|\mu| \equiv N_l T \ln \Lambda \gg T\,\partial\Omega_l/\partial T, \tag{19}$$

q.e.d.

Of course, the foregoing arguments do not make it possible to determine distinctly the actual order of smallness of the ratio $(\partial\Omega_l/\partial T)/N_l \ln \Lambda$, as was done for the case of adsorption of impurity excitations on a flat surface. However, the arguments presented are apparently sufficient to establish the very fact that the inequality (19) exists.

3. Let us discuss the content of the limiting asymptotic form (15) for $\alpha(c, T)$. By definition, the limiting value of the adsorption energy α_0 and the number $z_0 = N_l/L$ do not depend on the concentration of the solution. This result should not be surprising, since we are dealing with filling of all possible states in the vicinity of the vortex vilament with l-impurities in the limit as $T \to 0$. The number of such states is determined only by the character of the interaction of the impurity excitations with the velocity field of the vortex filament, and does not depend on the concentration of the solution.

Another characteristic feature of formula (15) is that the concentration dependence of $\alpha(c, T)$ has a logarithmic character. This circumstance allows us to disregard the contribution made to $\alpha(c, T)$ by volume impurity concentrations, which also interact with the velocity field of the vortex filament. The inhomogeneous Boltzmann distribution of volume impurity excitations was investigated at one time by Reut and Fisher[12]. The magnitude of this inhomogeneity, which makes a contribution to the total number of impurities adsorbed on the filament, turns out to be a linear function of the volume concentration, so that it plays no noticeable role in the low-temperature limit in comparison with the logarithmic contribution made by the l-impurities to α.

4. It is of interest to note the analogy between problems concerning the thermodynamic properties of a system of l-impurities, on the one hand, and a gas of free electrons in a strong magnetic field, on the other. In both cases we are dealing with Fermi particles that contain in the spectrum discrete quantum numbers and

one continuous quantum number. In either problem, when the external parameters are changed, the chemical potential is shifted relative to the levels of the spectrum. In the magnetic problem this shift is due to the fact that the position of the chemical potential is fixed while the Landau levels are shifted with changing magnetic field. In the l-impurity problem, the position of the l-levels is fixed, but the chemical potential shifts with changing temperature. At a given concentration of the solution, the potential μ drops monotonically "downward" with increasing temperature (see (11)), and intersects deeper and deeper l- levels. Consequently, just as in the magnetic problem, the thermodynamic potential of the l-impurities should contain both a monotonic part and an increment that oscillates with the temperature. The separation of these oscillations lies beyond the limits of the accuracy of the approximations used in the present paper to analyze the derivative $\partial\Omega_l/\partial T$. We have therefore obtained no concrete information on the oscillations, with the exception of the fact that their scale (unlike in the magnetic problem) is small in comparison with the monotonic part of $\alpha(c, T)$.

COMPARISON OF THEORY WITH EXPERIMENT

As noted in the Introduction, the discussed effects were experimentally observed[3] for weak solutions of helium. In accordance with the results of the theory, the experiment offers evidence of the existence of a limiting value, as $T \to 0$, of the effective vortex-filament core radius b_0^*, the value of which does not depend on the concentration of the solution and noticeably exceeds the limiting value of b_0 in pure He^4. The numerical value is $b_0^* \approx 3.8$–4 Å. The observed temperature dependence of $b^*(c, T)$ also agrees qualitatively with the theory. This is a power-law dependence and responds weakly to the concentration of the solution. However, the number of experimental points is too small for a reliable verification of the theory.

An exception is the estimate of the constant α_0. Taking into account the definition (2a) of the radius b^* and the numerical value of b_0^* which follows from the experimental data, we readily obtain

$$\alpha_0 = \frac{\rho_s \Gamma^2}{4\pi} \ln \frac{b_0^*}{b_0}, \qquad \frac{b_0^*}{b_0} \approx 3.25. \tag{20}$$

An estimate of the parameters z_0, based on the data of[3], and using the relation

$$z_0 = \frac{\rho_s \Gamma^2}{4\pi} \ln \frac{b_0^*}{b^*(c, T)} \Big/ T \ln \Lambda(c, T), \tag{21}$$

derived on the basis of (2a) and (15), yields the results gathered in the table. The calculation was performed for all the suitable experimental points. We see from the table that if we confine ourselves to points with $\Lambda > 20$ (we recall that the parameter Λ must satisfy the requirement $\ln \Lambda \gg 1$), then we get $z_0 \approx (0.7$–$1.0) \times 10^8$ cm^{-1}. The approximate constancy of z_0, which takes place when the impurity concentration and the temperature are varied by a factor of almost 5, is evidence that the definition (21) of z_0 is reasonable.

$10^3 c$	b^*, Å	T, °mK	$\Lambda(c, T)$	$10^{-8} z_0$, cm^{-1}
0.5	3.7	18	2.34	2.74
	3.6	28	4.55	1.69
	3.05	85	24.0	0.75
0.17	3.5	28	13.6	1.30
	3.3	43	25.4	1.0
	3.0	78	62.0	0.71
0.106	2.8	85	113	0.70

As to the small tendency of z_0 to decrease at a given concentration with rising temperature, it is due to the next higher terms of the expansion in (15), the explicit of which cannot be established in a phenomenological treatment.

CONCLUSION

Let us present some summaries. The feasibility of observing of the localization of the thermal and impurity excitations on the vortex filaments allows us to speak of an entire class of phenomena in which excitations are localized in liquid helium. These include the rather well investigated process of localization of excitations on a free flat surface of helium and its solutions[1, 2], the adsorption of excitations on small-radius spherical surfaces[3, 13, 14], and finally, localization of excitations on a vortex filament. All these problems have much in common, but differ in a large number of details.

We have derived in this paper general formulas that enable us to connect the observed dependence of the effective radius of the vortex-filament core on the temperature and concentration of the solution with the dynamic characteristics of the thermal and impurity concentrations localized on the vortex. We investigated the asymptotic properties of the quantitative characteristics of adsorption on a vortex filament in the low-temperature limit, both in pure He^4 and in its weak solutions. Rather simple formulas were obtained for the energy adsorption on a filament (see formulas (9), (15)), which admit of comparison with the experimental data. We have discussed the degree of correspondence between the theory and the available experimental data on the adsorption of impurity excitations on filaments in weak solutions of helium. The theory describes qualitatively correctly the presence in weak solutions of a limiting effective radius b_0^* of the core of the vortex filament, with a value that does not depend on the solution concentration. The theory explains the power-law dependence of $b^*(c, T)$ on the temperature in the region of low temperatures and the weak (logarithmic) dependence on the solution concentration. For a more reliable determination of the constants contained in the low-temperature limit of the theory, however, further experimental study of the problem is needed.

In the Appendix we investigate the singularities of the motion of vortex rings along the vapor-liquid surface of liquid helium. This situation can be easily realized in experiment if it is recalled that the helium ions can be fixed at a definite depth x_0 near the free surface of the helium[15]. The value of the parameter $\delta = R/x_0$ can vary in a wide range $0 < \delta \lesssim 1$. In particular, it is suggested that if $\delta > 1$ then stable vortex half-rings can be produced, the ends of which emerge to the surface of the helium. We discuss the possibility of observing similar half-rings in practice.

The author is grateful to A. F. Andreev for a discussion of the results.

APPENDIX

1. The stationary motion of an ion-ring complex in a field E, in the presence of friction forces and neglecting the small deformation of the ring in the vicinity of the ion localized on the filament[1)], can be determined by assuming that the vectors of the total field intensity E and of the velocity V of the ring cease to be collinear.

Using the symbols ∥ and ⊥ to designate the components of the vectors **V** and **E** in the plane of the ring and in a direction normal to it (Fig. 1), we write down in analogy with (2) and (4) the following definitions:

$$V_\perp=\frac{\Gamma}{4\pi R}\left(\ln\frac{8R}{b_0}-\frac{1}{2}\right),\quad eE_\perp=2\pi R\eta_\perp V_\perp,$$
$$eE_\parallel=2\pi R\eta_\parallel V_\parallel,\quad E^2=E_\parallel{}^2+E_\perp{}^2. \tag{A.1}$$

Here $\eta_\parallel$ and $\eta_\perp$ are the effective friction coefficients per unit length of the ring. The coefficient η from (4) is identically equal to $\eta_\perp$. It is obvious that the coefficients $\eta_\perp$ and $\eta_\parallel$ need not exactly equal each other, although they seem to be of the same order of magnitude.

The definitions (A.1) contain four relations between the five quantities R, $V_\parallel$, $V_\perp$, $E_\parallel$, and $E_\perp$. One more connection between them, which closes the system of equations, can be obtained by stipulating that the moment of the forces K_1, which arises in the ion-ring system as a result of the fact that the points at which the electric force eE and the total friction force $F_{fr}^\perp$ are applied do not coincide,

$$|K_1|=eE_\perp R, \tag{A.2}$$

is offset by the Magnus moment of the forces K_2, which differs from zero to the extent that $V \neq 0$, and which is directed opposite to K_1. The numerical value of K_2 is

$$|K_2|=\int \mathbf{k}_2\,d\mathbf{l}=4R\int_0^{\pi/2} k_2(\varphi)\,d\varphi=\pi R^2\rho\Gamma V_\parallel,$$
$$\mathbf{k}_2=[\mathbf{r}\times\mathbf{G}],\quad \mathbf{G}=\rho[\mathbf{\Gamma}\times\mathbf{V}_\parallel], \tag{A.3}$$
$$r=R\sin\varphi,\quad |k_2|=\rho\Gamma RV_\parallel\sin^2\varphi.$$

The meaning of the dimension r and of the angle φ between the vectors $\mathbf{\Gamma}$ and **V** in (A.3) is clear from Fig. 2.

Putting $K_1 = K_2$ or

$$eE_\perp=\pi\rho\Gamma RV_\parallel \tag{A.4}$$

and solving (A.1) and (A.4) simultaneously, we obtain relations (6) and

$$\frac{E}{(1+\gamma^2)^{1/2}}=\eta_\perp\frac{\Gamma}{2}\left(\ln\frac{8R}{b_0}-\frac{1}{2}\right),\quad \gamma=\frac{2\eta_\parallel}{\rho\Gamma}, \tag{A.5}$$

which were used in the main text.

2. The stability of the ion-ring system against the action of various moments of forces, which tend to upset the orthogonality of the leading electric field to the plane of the ring, gives grounds for hoping to be able to observe the following effect.

Consider a positive ion that is pressed against the surface of helium by an external field $E_\perp^0$. In such a situation, the position of the ion is fixed at a definite depth x_0 from the surface of the helium[15]:

$$x_0=\left[\frac{e}{E_\perp{}^0}\frac{\varepsilon-1}{4(\varepsilon+1)}\right]^{1/2}, \tag{A.6}$$

where ϵ is the dielectric constant of the helium, $\epsilon - 1 = 0.06$. It follows from (A.6) that at $E_\perp^0 \approx 100$–1000 V/cm we have $x_0 \approx 10^{-5}$–10^{-6} cm (the orthogonal ions near the surface cannot be in the stationary state, since the electron has the ability of tunneling into the gaseous phase).

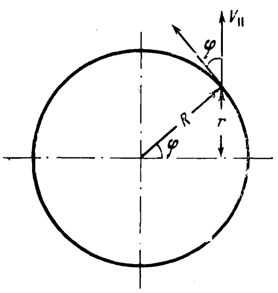

FIG. 2

If we now accelerate the prepared ion along the surface of the helium in such a way as to produce an ion-ring complex, then, owing to the quasimacroscopic dimensions of the ring (typical ring dimensions are $R = 10^{-5}$–10^{-6} cm), it should start to interact with the vapor-liquid boundary. A quantitative analysis of this interaction can be carried out in two limiting cases, $\delta = R/x_0 \ll 1$ and $\delta > 1$.

A. The presence of a boundary is equivalent to introducing into the ring-motion problem an additional vortex ring, which is the mirror image of the initial ring in the vapor-liquid plane (Fig. 3). The auxiliary ring induces around itself, meaning also in the vicinity of the initial ring, a definite velocity field $\mathbf{v}_S(\mathbf{r})$, which falls off with distance in power-law fashion. It is allowance for the influence of this perturbing factor which leads to a dependence of the dynamic characteristics of the initial ring on the distance to the liquid–vapor surface.

In the limiting case $\delta \ll 1$, that part of the field $\mathbf{v}_S(\mathbf{r})$ which is subtended by the area of the initial ring varies little within the limits of this area. For this reason, it is convenient to represent the total velocity in the vicinity of the ring in the form

$$\mathbf{v}_s(\mathbf{r})=\bar{\mathbf{v}}_s+(\mathbf{v}_s-\bar{\mathbf{v}}_s), \tag{A.7}$$

where $\bar{\mathbf{v}}_S$ is the value of $\mathbf{v}_S(\mathbf{r})$ averaged over the area of the initial ring. The components $\bar{\mathbf{v}}_S$ and $\mathbf{v}_S - \bar{\mathbf{v}}_S$ play qualitatively different roles in the problem.

The appearance of $\bar{\mathbf{v}}_S$ is equivalent to a uniform liquid stream in which the initial ring must move. Under such conditions, the usual definition of the ring velocity (see (2)) is valid in a coordinate system moving with the liquid. Consequently, in the laboratory frame, the ring moves with a velocity

$$V(R,x_0)=V_0(R)-\bar{v}_s,\quad V_0(R)=\frac{\Gamma}{4\pi R}\left(\ln\frac{8R}{b_0}-\frac{1}{2}\right). \tag{A.8}$$

This is indeed the observable effect, since the $V(R, x_0)$ relation becomes complicated by the addition of $\bar{\mathbf{v}}_S$, which can be expressed in terms of Γ, R, and x_0. Using the explicit form of $\mathbf{v}_S(\mathbf{r})$[6] and the smallness of the parameter $\delta \ll 1$, we obtain approximately

$$\bar{v}_s=\frac{\Gamma}{2x_0}[1+O(\delta^2)]. \tag{A.9}$$

As a result, expression (A.8) can be recast in a form similar to the definition (2a):

$$V=\frac{\Gamma}{4\pi R}\left(\ln\frac{8R}{b^*}-\frac{1}{2}\right),\quad b^*=b_0\exp\left(\frac{2\pi R}{x_0}\right). \tag{A.8a}$$

In other words, the proximity to the surface of a ring moving along the surface leads in the limit $\delta \ll 1$ to an effective renormalization of the core of the vortex filament forming the ring.

As to the odd increment $\mathbf{v}_S - \bar{\mathbf{v}}_S$, its presence, as can be readily established, causes the appearance of Magnus forces acting on the ring in a direction perpendicular to the surface of the helium. The analysis of the cancellation of these forces is carried out in analogy with Sec. 1 of the present Appendix. As a re-

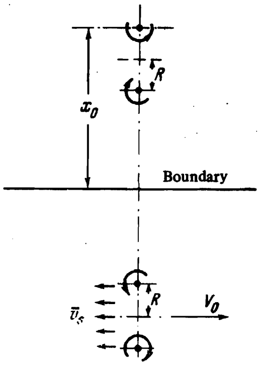

FIG. 3

sult it can be shown that the action of these forces leads to a negligible change in the equilibrium coordinate x_0 and to a certain additional inclination of the plane of the ring compared with the three-dimensional case.

B. In the region $\delta \lesssim 1$, the perturbing velocity field is comparable in intensity with the initial field of the ring, and the problem of the motion of the ring under such conditions is greatly complicated. An exception is the situation $\delta > 1$, for which a rather simple analysis is again possible. As noted above, the presence of an image or more accurately of that part of the perturbing field $v_s(r)$ that enters in the form $v_s - \bar{v}_s$, leads to attraction of the ring to the free surface and to rotation of its plane from a position perpendicular to the surface to a position parallel to it. As a result, of the two possible variants of the ring position near the surface in the limit $\delta > 1$ (Fig. 4), only variant b can be stable, if it is assumed in addition that exactly half the ring is situated in the liquid, and the other half is closed by its hydrodynamic image. Then all the moments of hydrodynamic origin vanish, and the problem becomes essentially a volume problem. The coordinate x_0 drops out from the answer in this case, as is perfectly obvious from Fig. 4b. All that is important is the satisfaction of the requirement $\delta > 1$, which enables the ion to be located somewhere on the perimeter of the ring.

Even though the definition of the velocity V(R) for the half-ring coincides with the volume definition, the situations in the volume and near the surface are nevertheless different. The point is that in the case of a half-ring a vortex filament with dimensions half as small as in the volume of the liquid actually moves through the helium. For this reason, the condition for the stationarity of the motion of the ring in the medium with finite viscosity, which takes in the volume the form $eE = 2\pi R\eta V_0$ [Eq. (4)] (E is the intensity of the driving field and η is the viscosity coefficient per unit length of the filament), should take in the case of a half-ring the somewhat different form

$$eE=\pi R\eta V_1. \qquad \text{(A.10)}$$

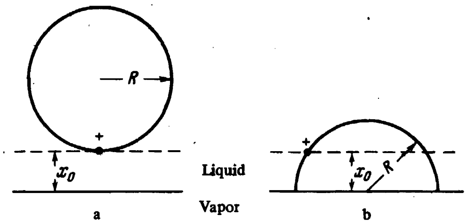

FIG. 4

Consequently, relation (5), which relates the intensity E with the ring V velocity, is somewhat modified for a half-ring:

$$V_1=\frac{8eE}{\pi b_c\eta}\exp\left(-\frac{4eE}{\Gamma\eta}-\frac{1}{2}\right). \qquad \text{(A.11)}$$

In other words, given the temperature and the electric field, the stationary velocities V_0 and V_1 of the ring and of the half-ring are different:

$$V_1/V_0=2\exp(-2eE/\Gamma\eta). \qquad \text{(A.12)}$$

This difference can be observed experimentally.

[1)] The scale of this deformation was estimated by Slyusarev and Strzhemechnyĭ [16]. In principle, a self-consistent approach to the description of the action of external forces on the ion-ring complex, as proposed in this paper, should answer all the questions concerning the character of motion of the ring in the external field. However, the concrete problem of the motion of the ion-ring complex in a stationary field was not investigated to a full extent in [16].

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Translated by J. G. Adashko
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