Unperturbed measurements of the *n*-quantum state of an harmonic oscillator

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The fundamental conditions of an essentially unperturbed measurement of the *n*-quantum state of an harmonic oscillator with the aid of an electron beam are analyzed. The limits of sensitivity of the measurements and the probability of perturbing the oscillator are determined.

Earlier[1] a method of unperturbed measurement of the energy of the vibrational modes of an electromagnetic resonant cavity was proposed, permitting one in principle to determine the corresponding quantum number n, including n = 0, with high accuracy. The probability of perturbation during the measurement of the initial n-quantum state may be extremely small. The urgency of the development of such a method is related, for example, to the requirements imposed on electromagnetic amplifier-transducers in gravitational wave experiments.

The feasibility of such a measurement is determined, on the one hand, by the rapid growth in the quality of electromagnetic superconducting UHF cavities and, on the other hand, by successes in the development of electron microscopes. The quality factor of superconducting cavities at $\omega = 2 \times 10^{10}$ sec^{-1} reaches $Q = 5 \times 10^{11}$.[2] This implies a relaxation time $\tau^* = 2Q/\omega = 50$ sec. If such a cavity is in equilibrium with a thermostat with a temperature T = 2°K, the lifetime τ_1 of the vibrational mode in the level n = 0 will be equal to $\tau_1 \cong \hbar\omega\tau^*/kT \cong 3$ sec, but the lifetime in the most probable level $n_T = kT/\hbar\omega \cong 15$ will be $\tau_1 \approx \hbar\omega\tau^*/kTn_T \approx 0.2$ sec. Therefore, even for $kT > \hbar\omega$ but $Q \gg 1$, the width of the energy levels at not too large values of n will be much smaller than $\hbar\omega$ (see[3] for more details). Under such conditions the electromagnetic oscillations in the cavity should be analyzed from the viewpoint of quantum theory. We note that in the example under consideration $\hbar\omega \approx 2 \times 10^{-17}$ erg, and if the measurement takes place during a time interval $\hat{\tau} \approx 0.1$ sec $< \tau_1$, then $\Delta\epsilon \approx \hbar/\hat{\tau} \approx 10^{-26}$ erg.

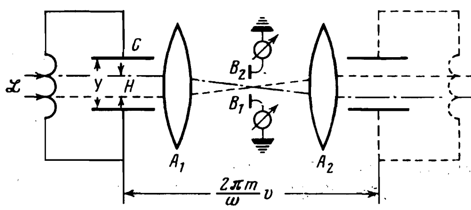
The fundamental conditions of an unperturbed measurement are known (see, for example, [4]): the matrix of the interaction must be diagonal in the same representation in which the quantity to be measured is diagonal. However, the question of how exactly this fundamental condition can be satisfied, or what limitations may appear as a consequence of the uncertainty relations in connection with a measurement of the energy of the natural modes of a resonant cavity, was not analyzed. Let us consider the example of a measurement of the energy of the vibrational modes of an electromagnetic resonant cavity with respect to the scattering by the field of an electron beam (see the figure).

A beam of electrons with horizontal velocity v_x passes through the capacitance of an microwave cavity of klystron type. Then the electrons enter the system of lenses A_1 and A_2, which brings about mirror reflection of the electron trajectories with respect to the symmetry plane of the capacitor. After this the electrons again enter the capacitance of the resonant cavity under investigation. In this case the resonant cavity must have two spatially separated capacitances. If the design of the cavity is such that it only has a single capacitance, then the scheme depicted in the figure should be supplemented by a mirror system.

The second flight of the electrons that do not hit screens B_1 and B_2 through the resonant cavity is utilized for the compensation of that effect on the oscillations in the cavity which they produce by passing through the cavity the first time. In order for such a compensation to occur, the time between the first and second flights must correspond to a change in the phase of the oscillations by $2\pi m$, where m is an integer. This condition assumes rather exact knowledge of the frequency ω and a sufficiently monoenergetic spectrum of the electrons in the beam.

The receiving electrodes, realized in the form of two screens, are located at right angles to the axis of the beam near the focus of the lenses and symmetrically with respect to their optical axis. Under the influence of the field in the left condenser, the focal spot will be smeared in the plane of the screens. As a result the average number of electrons incident on the screens varies. Smearing of the focal spot will be observed when the increase $\delta\langle N_B\rangle$ in the average number of electrons incident on the screen exceeds the fluctuation ΔN_B in the number of electrons which hit the screens in the absence of a field in the condenser. Since $\delta\langle N_B\rangle \sim N$ and $\Delta N_B \sim N^{1/2}$ (N is the total number of electrons passing through the condenser), the observed smearing of the spot will be smaller the larger the value of N. However, the probability of a change in the state of the resonant cavity increases upon increasing the value of N. The point is that the matrix characterizing the interaction of the electrons with the resonant cavity is not diagonal, and the discussed method of measurement is not completely unperturbing. However, as will be shown below, the extent of the perturbation may be extremely small.

Let us note the fundamental role of the focusing of the electron beam. It enables one to eliminate the influence of the uncertainty in the initial coordinate y of the electrons in the beam on the sensitivity of the measurements,[1] which in principle will be limited only by the uncertainty in the momentum of the electrons in their time of flight through the resonant cavity. In a rough but quite reasonable approximation, the resonant cavity with a clearly expressed capacitance spacing is an oscillatory system with one degree of freedom with lumped parameters $\mathscr{L}$ and C. If the voltage U on the



capacitor is chosen as the generalized coordinate, the generalized momentum $p = \mathscr{L}C^2\dot{U}$. In this connection the Hamiltonian operator takes the form

$$\hat{H}=\hat{p}^2/2\mathscr{L}C^2+CU^2/2. \tag{1}$$

It follows from the Ramo-Shockley theorem about induced currents that the force of excitation of the resonant cavity associated with an electron's transit has the form of a pulse whose amplitude is equal to ey/Y (e is the electron charge, Y is the distance between the plates of the parallel plate capacitor, and y is the electron coordinate measured from the plane of symmetry of the capacitor, i.e., from the zero equipotential). If Y is much smaller than the length of the plates, the duration of the pulse fronts will be much smaller than the total duration τ_{tr} of a transit pulse, and their influence can be neglected.

In the general case $y = y_0 + v_y\tau + \delta(\tau)$, where y_0 is the electron coordinate upon entering the capacitor, v_y is the component of the electron velocity normal to the plates at the moment of entrance, and $\delta(\tau)$ is the electron displacement due to the influence of the field inside the capacitor. In many practical cases (for example, for $Y \approx 10^{-1}$ cm, $\tau_{tr} \approx 10^{-10}$ sec, $U \approx 10^{-6}$ V) the rms value is $(\langle y_0^2\rangle)^{1/2} \gg \delta(\tau_{tr})$; therefore, the quantity $\delta(\tau)$ can be neglected. Therefore, the force perturbing the resonant cavity during a single transit of an electron through the capacitor will be given by

$$f(\tau)=\begin{cases} e(y_0+v_y\tau)/Y, & 0\leq\tau\leq\tau_{tr} \\ 0, & \tau\leq 0,\quad \tau\geq\tau_{tr} \end{cases}. \tag{2}$$

As is well known, if a force $f(\tau)$ ($f = 0$ as $\tau \to \pm\infty$) acts on a quantum oscillator, the probability for a transition of the oscillator from the ground state into an excited state is given by[5]

$$P=1-e^{-w}, \tag{3}$$

where

$$w=\frac{\omega}{2\hbar C}\left|\int_{-\infty}^{+\infty} f(\tau)e^{i\omega\tau}d\tau\right|^2. \tag{4}$$

If the electron does not hit the receiver screens and, after a time which is a multiple of the oscillation period, it again passes through the capacitor region of the cavity, then from Eq. (4) we obtain $w = 0$. Only those electrons which pass through the capacitor region only once will cause excitation of the resonant cavity. Let us determine this perturbation.

Since the quantities y_0 and v_y have random values within the limits determined by the cross section of the beam and its divergence, it is necessary to average expression (4) over the ensemble of electrons. Then, putting $\omega\tau_{tr} = \pi$, we obtain the following result from Eqs. (2) and (4):

$$w=\frac{e^2}{2\hbar\omega C}\left[\frac{H^2}{3Y^2}+\frac{(4+\pi^2)\overline{v_y^2}}{Y^2\omega^2}\right]. \tag{5}$$

In formula (5) it is considered that, for equiprobable values of y_0 within the limits $\pm H/2$, one will have $\overline{y_0^2} = H^2/12$. For simplicity of calculation we, shall neglect the second term in formula (5), since in the majority of cases of practical interest it is relatively small. For $C = 0.3$ pF, $\omega = 2 \times 10^{10}$ sec^{-1}, and $Y \approx H$, the quantity w is equal to 10^{-2}. That is, a single electron slightly perturbs the ground state of the resonant cavity. However, if αN electrons hit the screens out of N electrons passing through the resonant cavity, the value of w is increased by αN times.

We note that the mechanical degrees of freedom of the resonant cavity play an important role in the described scheme. The change of the electron momentum is accompanied by a change in the mechanical momentum of the resonant cavity. But since the mass of the latter is suffuciently large, its position remains well determined.

The number N of electrons sufficient for the measurements is determined by the voltage potential in the capacitor gap and by the distribution of the probability density for the electron coordinates in the plane of the receiving electrodes. If the distribution of the probability densities corresponds to diffraction, the minimal root-mean-square value of the potential of the observable field in the case of narrow screens, located in the region of the first diffraction minimum, will be given by (we omit the uncomplicated but tedious calculation)

$$(\langle U_B^2\rangle)^{1/2}\geq\frac{1.5\hbar\omega}{\sqrt{2}e}\frac{Y}{H}\left(\frac{B}{N}\right)^{1/4}. \tag{6}$$

Here

$$B=HpA/2\hbar L, \tag{7}$$

p is the momentum of the electrons, A is the width of the screens, and L is the focal length of the lens. Since the ground state of the cavity resonator corresponds to $\langle U^2\rangle = \hbar\omega/2C$, the number of electrons required for its observation must be given by

$$N\geq\frac{5(\hbar\omega C)^2}{e^4}\left(\frac{Y}{H}\right)^4 B. \tag{8}$$

(Relations (6) and (8) are valid for $B \leq 1$). Out of the total number of electrons, the following number is incident on the average on both screens

$$\langle N_B\rangle=N\pi^{-3}({}^1/{}_6B^3+2B\langle\xi^2\rangle), \tag{9}$$

where

$$\langle\xi^2\rangle=e^2\langle U^2\rangle H^2/(\hbar\omega)^2Y^2. \tag{10}$$

Equation (10) corresponds to the optimal time of flight, $\tau_{tr} = \pi/\omega$, through the capacitor space.

The ground state of the resonant cavity corresponds to

$$\langle\xi^2\rangle=e^2H^2/2\hbar\omega CY^2. \tag{11}$$

Multiplying (5) by (9) and taking (8) and (11) into consideration, we obtain

$$w_B=\frac{e^2}{6\hbar\omega C}\frac{H^2}{Y^2}\frac{N}{\pi^3}\left(\frac{B^3}{6}+B\frac{e^2H^2}{\hbar\omega CY^2}\right)\geq\frac{5}{6\pi^3}\left[\frac{B^4}{6}\frac{\hbar\omega C}{e^2}\left(\frac{Y}{H}\right)^2+B^2\right]. \tag{12}$$

For $\omega = 2 \times 10^{10}$ sec^{-1}, $C = 0.3$ pF, $B = 0.5$, and $Y = H$, the value of the right hand side in expression (12) will be equal to 1.3×10^{-2}. Thus, after the measurement the oscillator remains in the ground state with a probability of 0.98. For such a measurement it is necessary to pass $N = 1600$ electrons through, of which only two hit the screens. (We note that the parameters must be chosen in such a way that no less than one electron hits the screens).

In the case of the utilization of semi-infinite screens, the minimal potential of the observable UHF field is determined by the relation

$$(\langle U_\infty^2\rangle)^{1/2} \geqslant \frac{\hbar\omega}{e}\frac{Y}{H}\frac{1,7}{N^{1/4}}. \tag{13}$$

If, in this connection, the screens are nontransparent for the electrons, the perturbation of the ground state will be appreciable since close to 20% of the total number of electrons will hit the screens.

Let us discuss the feasibility of such a measurement, in which one can distinguish the n-state of the oscillator from the (n + 1)-state. These states will be distinguishable if the difference between the average numbers of electrons incident on the screens for these states ($\langle N_n\rangle$, $\langle N_{n+1}\rangle$) will satisfy the inequality

$$\langle N_{n+1}\rangle - \langle N_n\rangle \geqslant (\langle N_n\rangle)^{1/2}. \tag{14}$$

Fulfilment of the following condition [5,6] is necessary in order that the probability of perturbing the n-state shall be small:

$$(2nw_N)^{1/2} < 1. \tag{15}$$

Taking into consideration that $\langle U^2\rangle = E_n/C$, from Eqs. (9) and (10) we find

$$\langle N_{n+1}\rangle - \langle N_n\rangle = \frac{2N}{\pi^3}B\frac{e^2}{\hbar\omega C}\frac{H^2}{Y^2}. \tag{16}$$

Having substituted (16) and (9) into (14), we obtain one of the conditions of measurement:

$$N \geqslant \frac{\pi^3}{24}B\left(\frac{\hbar\omega C}{e^2}\frac{Y^2}{H^2}\right)^2 + n\pi^3\hbar\omega CY^2/2Be^2H^2. \tag{17}$$

Since $w_n = w\langle N_n\rangle$ in the inequality (15), by utilizing expressions (5), (9), and (17) we find that the probability of perturbing the n-state will be small provided that

$$\frac{e^2}{\hbar\omega C}\frac{H^2}{Y^2} < \frac{3}{n^3}, \text{ i.e. } w < \frac{1}{2n^3} \tag{18}$$

and

$$B < 4/n. \tag{18a}$$

The conditions (18) and (18a) for the smallness of the perturbation indicate that there is no limitation in principle on the possibility of determining an arbitrary n-state of an oscillator. However, the technical difficulties will grow with increasing values of n. In order to fulfill condition (18a) for a width of the screens no smaller than several atomic layers, it is necessary to have a rather broad diffraction maximum. For this it is necessary to reduce the energy of the electrons or to obtain magnification of the picture with the aid of electron lenses.

We note that, although it is necessary to utilize all the achievements of electron optics in the discussed experiment, the posed problem differs substantially from the problem of the design of an electron microscope. If in the latter case it is required to obtain the smallest possible electron spot, in our case it is sufficient that the size of the spot be determined basically by the diffraction effect, and its absolute magnitude does not play a fundamental role. In practice it is more convenient to have a broad focal spot. It is already possible in the current state of electron optics to realize the conditions under which the size of the spot will be basically determined by the diffraction effect.

The influence of the nonmonoenergetic nature of the electrons, the diffraction perturbation of the electron beam by the screens, and the interaction between the electrons were not taken into consideration in the estimates cited above. From relations (4), (5), and (9) one can easily find the requirement on the relative spread of the electron velocities:

$$\left(\frac{\Delta v_x}{v_x}\right)^2 < \frac{2}{(2\pi m)^2}\frac{\langle N_B\rangle}{N} = \frac{1}{2\pi^3 m^2}\left(\frac{B^3}{6} + 2B\langle\xi^2\rangle\right). \tag{19}$$

For the above used numerical values of the parameters, in the case n = 0 it follows from formula (19) that $\Delta v_x/v_x < 1.5 \times 10^{-2}/m$. One can easily satisfy this condition up to $m \approx 10^3$. In the case $n \gg 1$, from Eqs. (19), (18), and (18a) we obtain

$$\Delta v_x/v_x < 0.25/mn^{3/2}. \tag{20}$$

Diffraction perturbation of the electron beam by the screens depends on the distance between the screens, their widths, and on the quantity $\lambda = h/p$. Since the absolute size of the focal spot can be increased with the aid of electron optics, the distance between the screens can be made much larger than λ. Therefore, in principle the diffraction perturbation can be made quite small.

Let us note one more important property: in the calculation of the values of w for n = 0 and $n \neq 0$ the receiving screens were assumed to be nontransparent. In other words, the electrons incident on them a second time, did not pass through the resonant cavity. However, if sufficiently "transparent" screens are used (for $v_x \approx 1 \times 10^{10}$ cm/sec, the thin layers in contemporary transmission electron microscopes significantly scatter a small fraction of the electrons), the value of w can be made even smaller than in the examples cited above.

In conclusion let us once again emphasize the two important conditions for the experimental realization of a determination of the n-quantum state of a resonant cavity with a small probability of perturbation after the measurement:

1. A second flight of the electrons through the resonant cavity, without hitting the screens in order to compensate for the perturbation. Such compensation is possible only if the frequency of the cavity oscillations is known and the beam is sufficiently monoenergetic.

2. Focusing of the electron beam with the aid of an electron-optic system and the utilization of narrow screens.

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