Spectrum of acoustic turbulence

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Interaction processes in a system of random, weakly nonlinear, acoustic waves are discussed. It is shown that a system of this kind exhibits both self-interaction and appreciable wave mixing, which leads to the smearing out of the leading edge of a sawtooth wave and, in this sense, is similar to turbulent viscosity. It is shown that the system spectrum in the inertial frequency band can, in general, be divided into two parts. The first of these is determined by the spectrum of the sawtooth wave and the second corresponds to the acoustic turbulence spectrum.

Nonlinear effects in an ensemble consisting of a large number of finite-amplitude acoustic waves manifest themselves in two ways. Firstly, in the distortion of each of the waves as a result of self-interaction and, secondly, in the redistribution of energy over the spectrum as a result of interaction between the waves.

If we ignore the self-interaction process, we may suppose that all the harmonic waves are uncorrelated. A system of this kind, called acoustic turbulence, was discussed by Zakharov and Sagdeev.[1] Assuming that, in a certain region lying at sufficiently low frequencies, wave excitation takes place and absorption is appreciable only at very high frequencies, we can establish the character of the energy spectrum in the intermediate frequency band, i.e., in the so-called inertial interval. It turns out that the spectral energy density $\mathscr{E}_k$ in the inertial interval satisfies the three-halves law:[1]

$$\mathscr{E}_k \sim k^{-3/2}. \qquad (1)$$

In reality, however, the self-interaction processes, which are typical for acoustics in the absence of dispersion, and lead to the transformation of a harmonic wave into a sawtooth wave, are very effective and this has, in fact, been noted by Kadomtsev and Petviashvili.[2] They have pointed out that the generation of sawtooth waves leads to an expression for the energy density which is different from that given by (1), i.e.,

$$\mathscr{E}_k \sim k^{-2}. \qquad (2)$$

The present paper is concerned with the analysis of both nonlinear effects, i.e., self-interaction and mixing, in an ensemble of weakly nonlinear waves, and the elucidation of the character of the energy spectrum of a system of this kind. We shall use an approximate approach, based on the assumption that the wave interaction is weak. This will enable us, in the first approximation, to ignore the interaction between the waves and determine the parameters of sawtooth waves from the solution of the self-interaction problem for a single wave. We shall then consider wave packets corresponding to sawtooth waves which, in the next approximation, will enable us to take into account their self-interaction.

Let us begin by considering the evolution in time of the following system of weakly nonlinear waves: at the initial time, the wave vectors are close in magnitude within a small range in the neighborhood of k_0, but can have any direction so long as the solid angle $\Omega_0 = k_\theta^2/k_0^2$ contains one wave. The wave phases are assumed to be random. Therefore, k_θ^2 is the area on the surface of a sphere of radius k_θ in k space, which is defined by the characteristic scale of the angular correlation $\Delta\theta = k_\theta/k_0$.

The equations describing the propagation of acoustic waves with allowance for nonlinear effects in the second approximation take the form

$$\frac{\partial v_i}{\partial t}+\frac{1}{\rho_0}\frac{\partial p'}{\partial x_i}=-v_k\frac{\partial v_i}{\partial x_k}+\frac{1}{\rho_0}\frac{\partial p'}{\partial x_i}\frac{p'}{\rho_0 c_0^2},$$

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x_k}(\rho v_k)=0,$$

$$\rho'=\frac{p'}{c_0^2}-\frac{\gamma-1}{2}\frac{p'^2}{\rho_0 c_0^4},\qquad \gamma=\left(\frac{\partial c^2}{\partial \rho}\right)_s\frac{\rho_0}{c_0^2}+1. \qquad (3)$$

In this expression v, ρ', and p' are the velocity, density, and pressure perturbations due to the acoustic wave, and c is the velocity of sound. The zero subscript indicates equilibrium values of the hydrodynamic parameters. In terms of the dimensionless variables

$$\bar v = v/c_0,\qquad \bar p' = p'/\rho_0 c_0^2$$

this system can be written in the form

$$\frac{\partial \psi_i}{\partial t}+H_{ik}\psi_k=\{\psi^2\}_i; \qquad (4)$$

$$\psi=\begin{vmatrix}\bar p\\ \bar v\end{vmatrix},\qquad H_{ik}=\begin{vmatrix}0 & c_0\dfrac{\partial}{\partial x_k}\\ c_0\dfrac{\partial}{\partial x_k} & 0\end{vmatrix},$$

$$\{\psi^2\}=\begin{vmatrix}\dfrac{(\gamma-1)}{2}\dfrac{\partial \bar p'^2}{\partial t}-c_0\dfrac{\partial}{\partial x_k}\bar v_k\bar p'\\ c_0\bar v_k\dfrac{\partial \bar v_i}{\partial x_k}+c_0\dfrac{\partial \bar p'}{\partial x_i}\bar p'\end{vmatrix}. \qquad (5)$$

If we seek the solution of (4) in the form of an expansion in terms of the solutions of its linear part, which is orthogonal in a unit volume, i.e.,

$$\psi_i=\sum_{\mathbf k} C_{\mathbf k}\psi_{0i}^{k}\exp\{i(\omega_{\mathbf k}t-\mathbf{kx})\},$$

then, following Vedenov,[3] we obtain the following equation for the slowly-varying amplitudes $C_k(t)$:

$$\frac{\partial C_{\mathbf k}}{\partial t}=\sum_{\mathbf k} V_{\mathbf{kk'k''}}C_{\mathbf k'}C_{\mathbf k''}\exp\{i(\omega_{\mathbf k'}+\omega_{\mathbf k''}-\omega_{\mathbf k})t\},\qquad \mathbf k''=\mathbf k-\mathbf k'. \qquad (6)$$

The interaction potential is given by

$$V_{\mathbf{kk'k''}}=i\left[\frac{\gamma-1}{2}(\omega_{\mathbf k'}+\omega_{\mathbf k''})+\frac{c_0^2}{\omega_{\mathbf k'}}(\mathbf k'^2+\mathbf k'\mathbf k'')-c_0^4\frac{(\mathbf k'\mathbf k'')(\mathbf{kk}'')}{\omega_{\mathbf k'}\omega_{\mathbf k''}\omega_{\mathbf k}}-c_0^2\frac{\mathbf k'\mathbf k}{\omega_{\mathbf k}}\right]$$

$$\times\frac{\omega_{\mathbf k'}^{1/2}\omega_{\mathbf k''}^{1/2}\omega_{\mathbf k}^{1/2}}{2\omega_{\mathbf k}(1+\mathbf k''^2c_0^2/\omega_{\mathbf k''}^2)^{1/2}(1+\mathbf k'^2c_0^2/\omega_{\mathbf k'}^2)^{1/2}(1+\mathbf k^2c_0^2/\omega_{\mathbf k}^2)^{1/2}} \qquad (7)$$

with the following normalization:

$$\sum_{\mathbf k} C_{\mathbf k}^2\omega_{\mathbf k}=\sum_{\mathbf k} N_{\mathbf k}\omega_{\mathbf k}=\sum_{\mathbf k}\mathscr{E}_{\mathbf k}=\frac{E}{\rho_0 c_0^2}=M^2, \qquad (8)$$

where N_k is the "number" of waves with wave vector $\mathbf{k}$ and E is their energy per unit volume. This equation describes the amplitude variation for a wave with vector $\mathbf{k}$, due to its interaction with waves having wave vectors $\mathbf{k}'$ and $\mathbf{k}''$. In particular, if there is only one initial wave with wave vector $\mathbf{k}' = \mathbf{k}''$, this equation describes the growth of the second harmonic of the wave due to the self-interaction:

$$C_{2\mathbf{k}'} = V_{\mathbf{k}\mathbf{k}'\mathbf{k}''} C_{\mathbf{k}'}^2 t.$$

The condition that the second harmonic becomes comparable with the first leads directly to an estimate for the characteristic wave distortion time:

$$\frac{1}{\tau_n} \sim V_{\mathbf{k}\mathbf{k}'\mathbf{k}''} C_{\mathbf{k}'}.$$

Since, in accordance with (7), we have $V_{\mathbf{k}\mathbf{k}'\mathbf{k}''} \approx \frac{1}{2}(\gamma + 1)\omega_{\mathbf{k}}^{3/2}$, we obtain the following well-known expression for the time taken by the wave to be transformed from a sinusoidal waveform of frequency ω to the sawtooth waveform: [4]

$$1/\tau_n \approx \alpha\omega M, \quad \alpha = \frac{1}{2}(\gamma+1). \tag{9}$$

The width δ of the low-intensity break in the sawtooth wave is determined by dissipation, and the order-of-magnitude estimate for it is

$$\delta/\lambda = 1/\mathrm{Re}, \tag{10}$$

where λ is the wavelength of the original wave, $\mathrm{Re} = \alpha v_0 \lambda/\nu$ is the acoustic Reynolds number, ν is the effective viscosity which determines the attenuation of the acoustic wave,

$$\rho\nu = \frac{4}{3}\eta + \eta' + \varkappa\left(\frac{1}{C_v} - \frac{1}{C_p}\right),$$

η and η' are the shear and volume viscosities, κ is the thermal conductivity, and C_v and C_p are the specific heats. From the spectral standpoint, the transformation of the wave from a sinusoidal to a sawtooth waveform corresponds to the appearance of correlated harmonics such that the break width determines an important wave characteristic (see [5]), namely, the number of harmonics $N = k_N/k_0$, where k_0 and k_N are, respectively, the wave numbers of the fundamental wave and its high-frequency harmonic corresponding to the spectrum limit, i.e.,

$$N = \mathrm{Re}. \tag{11}$$

The appearance of breaks leads to strong attenuation of the wave. Its amplitude falls by an order of magnitude in the time given by (9) to within a numerical factor. We may therefore suppose that $\tau = \tau_n$ determines the lifetime of wave packets corresponding to sawtooth waves.

Let us now consider the interaction between wave packets. We recall that the harmonics making up the packet are correlated with one another. Because of the large number of interacting waves, this process has a random character and this was reflected in the above assumption that the phases of waves traveling at different angles were uncorrelated. This process can be described by the kinetic equation [6]

$$\frac{\partial N_{\mathbf{k}}}{\partial t} = 2\pi \sum_{\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'\mathbf{k}''}^2 (N_{\mathbf{k}'}N_{\mathbf{k}''} - N_{\mathbf{k}}N_{\mathbf{k}'} - N_{\mathbf{k}}N_{\mathbf{k}''}) R(\omega), \tag{12}$$

where $\mathbf{k}'' = \mathbf{k} - \mathbf{k}'$ and

$$R(\omega) = \int_0^\tau \exp\{i(\omega_{\mathbf{k}'} + \omega_{\mathbf{k}''} - \omega_{\mathbf{k}})t\} dt.$$

Using the assumption that the interaction time was sufficiently long, the authors of [6] assume in the derivation of the kinetic equation that

$$R(\omega) = \delta(\omega_{\mathbf{k}'} + \omega_{\mathbf{k}''} - \omega_{\mathbf{k}}).$$

In our case, the wave interaction time is determined by the wave lifetime, so that we have the order-of-magnitude result $R(\omega) \sim \tau$.

We now estimate the time τ_x during which there is an appreciable change in the energy of the k-th harmonic ($k = nk_0$) of one of the sawtooth waves as a result of its interaction with other wave packets. To do this, we can, in principle, sum in (12) over all terms describing the interaction of the given harmonic with all the harmonics of another sawtooth wave, and then sum over all interacting sawtooth waves. Since, however, the main contribution is made by the interaction with the lowest-frequency harmonics of the sawtooth wave, in which its energy is largely concentrated, it is sufficient in estimating the interaction with one of the sawtooth waves to take into account the contribution of only one term $N_{\mathbf{k}}N_{\mathbf{k}'}$ for $\mathbf{k}' \sim \mathbf{k}_0$, and then sum over the angles in order to take into account the number of interacting waves. The result is

$$1/\tau_x \approx V_{\mathbf{k}\mathbf{k}'\mathbf{k}''}^2 \tau N_{\mathbf{k}'} \Omega/\Omega_0. \qquad \mathbf{k}' \sim \mathbf{k}_0, \quad \mathbf{k}'' = \mathbf{k} - \mathbf{k}'. \tag{13}$$

In this expression, the factor Ω/Ω_0 corresponds to summation over the angles and takes into account the number of interacting waves. Moreover,

$$\Omega = (k_\perp/k_\parallel)^2, \tag{14}$$

where $k_\perp$ is the transverse component of the wave vector of the harmonic and $k_\parallel$ is the longitudinal component. The angle Ω defines the size of the "interaction cone," i.e., the region in k space which corresponds to interacting waves. The quantity $k_\perp$ can be determined as follows. The departure of the wave from the k_0 direction by an angle characterized by $k_\perp$ leads to a change in the wave number modulus given by

$$\Delta k = \sqrt{k_\parallel^2 + k_\perp^2} - k_\parallel \approx k_\perp^2/2k_\parallel,$$

and this, in turn, leads to a relative dephasing given by

$$\Delta k/k_\parallel \approx k_\perp^2/k_\parallel^2, \tag{15}$$

which is equivalent to frequency detuning by $\Delta\omega = c\Delta k$. On the other hand, the lifetime τ allows an efficient interaction between waves for which $\Delta\omega$ does not exceed the value

$$\Delta\omega \approx 1/\tau \approx \omega M = \omega_n M_n, \quad M_n = M/n,$$

and hence $\Delta\omega/\omega \approx M_n$. Equating this expression to that given by (15), we obtain an estimate for $k_\perp$:

$$k_\perp^2/k_\parallel^2 = M_n. \tag{16}$$

Since, in accordance with (7)

$$V_{\mathbf{k}\mathbf{k}'\mathbf{k}''} \approx \alpha^2 \omega_{\mathbf{k}'}^{1/2} \omega_{\mathbf{k}''}^{1/2} \omega_{\mathbf{k}}^{1/2}$$

we have for $\mathbf{k}' \approx \mathbf{k}_0$ and $\mathbf{k}'' = \mathbf{k} - \mathbf{k}'$

$$V_{\mathbf{k}'\mathbf{k}\mathbf{k}}^2 \approx \alpha^2 \omega_{\mathbf{k}}^2 \omega. \tag{17}$$

Substituting in (13) the expressions for $\tau = \tau_n$ and E given by (8) and (9), we obtain

$$1/\tau_x \approx \alpha^2 \omega M_n^2 (k_0/k_\theta)^2 = \omega M_n^2 N_\theta, \tag{18}$$

where $N_\theta = (k_0/k_\theta)^2 = 1/\Omega_0$ is the number of waves per unit solid angle.

It is clear that the interaction time decreases with increasing wave frequency, so that the harmonics with higher n undergo faster mixing than the initial waves. Let us now determine which harmonics succeed in mixing during the lifetime τ of the wave packet. For this, it is clearly necessary to set $\tau_n = \tau_x$ which yields $\alpha^2 M^2 \omega_n N_\theta \approx M\omega$, and hence

$$n=1/\alpha MN_\theta. \tag{19}$$

Comparing this result with (11), we conclude that when $1/\alpha MN_\theta < \mathrm{Re}$ we have $n < N$, and this means that harmonics whose numbers lie between n and N do succeed in mixing. In the opposite case, i.e., $(\alpha MN_\theta)^{-1} > \mathrm{Re}$, we have $n > N$ and harmonics of the sawtooth wave having the maximum number N do not succeed in interacting. Therefore, (19) defines the maximum number of sawtooth-wave harmonic in the first case. In other words, in this case, the width of the shock wave is no longer determined by viscosity, but by the interaction of high-frequency harmonics, and is given by

$$\delta_T/\lambda \sim \alpha MN_0. \tag{20}$$

If we introduce the turbulent viscosity ν_T by the condition

formula $\delta_T/\lambda = \mathrm{Re}_T^{-1}$,

where $\mathrm{Re}_T = v\lambda/\nu_T$, we obtain

$$\nu_T \approx \alpha v\lambda MN_0. \tag{21}$$

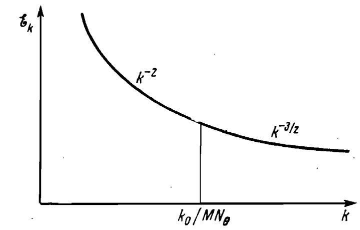
We note that the expression for τ_x and the turbulent viscosity coefficient can also be obtained in another way, without bringing in the kinetic equation (12). If we consider the propagation of a wave with oscillatory velocity is v in a given noise field with fluctuation velocity ϵ, using the method of Lifshitz and Rozentsveĭg, [7] and substitute for v in (3) the sum of the mean velocity u and the fluctuation velocity w, we can solve the equation for w. If the result $w = w(\epsilon u)$ is then substituted into the equation for the mean velocity, this results in the appearance of the viscous term $\partial^3 u/\partial t \partial x^2$ in this equation. The coefficient of this term is identical with the turbulent viscosity given by (21).

We thus find that for sufficiently large initial wave amplitudes (when M is not too small) and sufficiently large N_θ, the nonlinear wave mixing processes turn out to be appreciable. It is also clear that the effectiveness of the process depends on the number N_θ of independent waves propagating in the interaction cone and, as N_θ increases, there is an increase in the number of harmonics which succeed in mixing during the lifetime of the wave packet.

When $N_\theta M \ll 1$, we can have two cases:

1. When $(\alpha MN_\theta)^{-1} > \mathrm{Re}$, the width of the sawtooth wave front is determined by the spectrum of the sawtooth wave and is given by (2).

2. When $(\alpha MN_\theta)^{-1} < \mathrm{Re}$, the formula given by (19) determines the maximum number of coherent sawtooth-wave harmonics characterizing the limit of the regular part of the spectrum in the inertial frequency interval.



At higher frequencies, we have intensive mixing of waves, and the spectral energy density in this band is determined by (1) (see figure).

When $N_\theta \gtrsim M^{-1}$ or $\Omega_0 = M$, i.e., when the independent (in the sense of absence of correlation) wave corresponds to a solid angle $\sim M$, the time τ is sufficient to allow all the harmonics to mix, and there are no sawtooth waves.

We note in conclusion that the above results can be used to analyze the time-independent problem of the spectrum of acoustic turbulence considered in [1,2]. In particular, we shall suppose that, in the low-frequency part of the spectrum corresponding to $\mathbf{k} \approx \mathbf{k}_0$, there is continuous wave excitation whereas, at higher frequencies, we have wave damping. In the intermediate frequency band, the character of the spectrum is determined by nonlinear effects.

The foregoing discussion can also be used for this problem although one must then introduce the concept of the wave range $l \sim c_0\tau$ rather than lifetime.

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