Langmuir condensate turbulence spectrum

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We show that the strong Langmuir turbulence produced by a Langmuir condensate leads to a spectrum $W_k \propto 1/k^2$ in agreement with existing results from numerical simulation of strong turbulence.

1. Weak Langmuir turbulence spectra have been studied recently in detail.[1,2] These spectra are formed as a result of a nonlinear spectral transfer to the region of small wave numbers k. It has also been made clear that as the result of such a transfer oscillations may be stored in the small k region where, as a rule, $k < k_* = \frac{1}{3}k_d(m_e/m_i)^{1/2}$, where $k_d = \omega_{pe}/v_{Te}$ is the inverse of the Debye radius, and m_e and m_i are the electron and ion masses. If the level of oscillations in that region exceeds a critical value, given by the inequality

$$W/nT > 12k^2/k_d^2. \tag{1}$$

(W is the total turbulent energy density per unit volume and nT the thermal energy density in the plasma) the so-called modulation instability occurs which has been studied in[3-5].

A simple analysis shows that it is in practice difficult to satisfy the criterion (1) for the modulation instability in the region $k > k_*$ because the nonlinear scattering by ions rapidly smears out the energy over the whole region up to small $k \sim k_*$ so that the modulation instability occurs in the region $k < k_*$. If we substitute $k = k_*$ into (1), we get

$$W/nT > W_c/nT = \frac{4}{3}m_e/m_i, \tag{2}$$

where W_c is the critical value of W. The criterion (2) is easily satisfied in many existing experiments.

When the modulational instability becomes well-developed and reaches the nonlinear stage, the modulational oscillations begin to affect the Langmuir oscillations distribution. Such a state is sometimes called the state of strong Langmuir turbulence. Two approaches have been proposed to describe it.

On the one hand, Rudakov[6] suggested that the nonlinear stage of the modulational instability corresponds to the formation of Langmuir "solitons", i.e., Langmuir wave packets with an amplitude which has the form of solitary pulses. In the framework of this hypothesis developed strong turbulence is an ensemble of interacting "solitons."

In the other approach, developed by the present authors,[7,8] it is proposed to describe the developed modulational oscillations statistically. In that case the interacting Langmuir oscillations form, when random modulational perturbations are present, a Langmuir condensate which in turn determines the interaction between the waves which do not belong to the condensate. Such an approach does not enable us to give the detailed actual forms of the nonlinear modulational perturbations which may be of a different character and, moreover, can interact strongly with one another. It is necessary for a description of these processes to know the correlation functions of the modulational perturbations. From the point of view of applications of most interest is not so much the nature of the interaction of the Langmuir oscillations in the condensate as well the influence of the condensate on the interaction between the other Langmuir waves.

Recently particular attention was drawn to these problems in connection with one-dimensional numerical calculations[9] which showed that under conditions of developed Langmuir turbulence the energy flux changes its direction and the oscillations are transformed to the large wave number (k) region and a spectrum $W_k \propto k^{-2}$ is established, where $W = \int W_k dk$. This result is important for various mechanisms for heating a plasma by particle beams, microwave decay instabilities, and laser beams, and other ways since the Langmuir oscillations when converted into the large k region are absorbed in the Maxwell distribution tail and this leads to the production of fast particles. One needs to know the actual form of the turbulence spectrum for different variants of the acceleration and heating of fast particles. Attempts[10,11] have been made to explain the results of the numerical calculations in the framework of the "soliton" hypothesis of turbulence and the assumption that developed turbulence corresponds to an equipartition of energy between solitons of different amplitudes. In that case a spectrum of the form k^{-2} was obtained from qualitative considerations. The dynamics of the formation of solitons was studied numerically in more detail in[12] and a somewhat different spectrum was found. It was shown in[8] that the presence of a Langmuir condensate changes the direction in which energy is transferred in an essential manner.

The aim of the present paper is to discover the stationary spectrum of the Langmuir oscillations outside the condensate. We shall show here that the spectrum $W_k \propto k^{-2}$ occurs necessarily in the approach which we develop.

We note also that the condition (2) is in fact satisfied in most experiments on laser-plasma or beam-plasma interactions. However, the formation of the condensate does not occur instantaneously but through a quasi-stationary weak Langmuir turbulence which exists for a time which varies slightly depending on the mechanisms for exciting the turbulence.

We get an estimate $\tau = 1/\gamma$ from the non-linear scattering by ions, where

$$\gamma \approx \omega_{pe}(W/nT)(k_*/k)^2. \tag{3}$$

The quantity γ is rather small when $k > k_* \sim k_d$ and relatively small when $k \sim k_*$ and when W/nT is rather large (for estimates of the effect of uhf oscillations or laser radiation on the plasma the parameter W/nT can be replaced by $|E_0|^2/4\pi nT$ where E_0 is the amplitude of the excited electric field).

2. In the present discussion we restrict ourselves to one-dimensional Langmuir turbulence corresponding to an infinitely strong magnetic field (see [5,7,8]). We introduce the correlation function for Langmuir oscillations

$$\langle E_{k_1,\omega_1} E_{k_2,\omega_2}\rangle_{\rm hf} = I_{k_1,\omega_1}\delta(k_1+k_2)\delta(\omega_1+\omega_2), \tag{4}$$

where ω_i is the wave frequency and k_i the wave number in the preferred direction (along the strong magnetic field). We shall denote the correlation function for the Langmuir oscillations inside the condensate by $I^{(0)}_{k,\omega}$, while $I_{k,\omega}$ without an upper index will refer to the oscillations outside the condensate. We denote the correlation function of the low-frequency oscillations by

$$\langle E_{k',\omega'} E_{k'',\omega''}\rangle_{\rm lf} = 4\pi w_{k',\omega'}\delta(k'+k'')\delta(\omega'+\omega''). \tag{5}$$

The averaging in (4) is over a statistical ensemble of high-frequency oscillations and in (5) over a statistical ensemble of low-frequency oscillations (hence the indices hf and lf at the averaging sign).

We found the spectrum $w_{k',\omega'}$ of the modulational oscillations in [7]. The spectrum of the modulational oscillations inside the condensate turns out to be in practice of little importance as the basic effects are determined by the total magnitude of the energy density of the oscillations in the condensate:

$$W_0 = \frac{1}{4\pi}\int I^{(0)}_{k,\omega}dk\,d\omega. \tag{6}$$

To find the spectrum of the Langmuir oscillations outside the condensate we can use the equation for the correlation function $I_{k,\omega}$:

$$(\tilde\varepsilon_{k,\omega} + \varepsilon^{R}_{k,\omega} + \varepsilon^{RN}_{k,\omega})I_{k,\omega} = 0. \tag{7}$$

Here k and ω are, respectively, the wave numbers and frequencies outside the condensate,

$$\tilde\varepsilon_{k,\omega} = \varepsilon^{l}_{k,\omega} - \frac{\omega_{pe}^2}{4\pi n_0 m_e}\int \tilde\Sigma_{k,\omega,k_1\omega_1} I_{k_1,\omega_1}dk_1\,d\omega_1, \tag{8}$$

while $\epsilon^{l}_{k,\omega}$ is the linear part of the dielectric permittivity of the plasma, and $\tilde\Sigma_{k,\omega,k_1,\omega_1}$ is the kernel of the nonlinear part of the dielectric permittivity which can easily be determined from the expression for the nonlinear plasma currents (see [1]).

The quantity $\epsilon^{R}_{k,\omega}I_{k,\omega}$ describes the effect of the modulational oscillations of the condensate which is linear in $I_{k,\omega}$ ($\epsilon^{R}_{k,\omega} = \int G_{k,\omega,k',\omega'}w_{k',\omega'}dk'\omega'$) and the contribution of $\epsilon^{R}_{k,\omega}$ to (7) is small, as was shown in [8]. We shall therefore drop the term $\epsilon^{R}_{k,\omega}$ in (7) in what follows. Finally, $\epsilon^{RN}_{k,\omega}I_{k,\omega}$ describes the effects of the modulational oscillations which are nonlinear in $I_{k,\omega}$, and inside the condensate

$$\varepsilon^{RN}_{k,\omega} = \frac{\omega_{pe}^8}{4\pi n_0^2 T_e^2}\int \tilde\Sigma_{k,\omega,k_1\omega_1}\frac{I^{(0)}_{k_1,\omega_1}dk_1\,d\omega_1 w_{k',\omega'}dk'\,d\omega'\psi^{(0)}_{k',\omega'}}{|\tilde\varepsilon(k_1-k',\omega_1-\omega')|^2 k'^2(\omega_1-\omega')^4} \tag{9}$$

The expression for $\epsilon^{RN}_{k,\omega}$ which was actually found in [8] refers to the case where I_{k_1,ω_1} corresponds to waves inside or close to the condensate so that in Eq. (9) the index (0) occurs, while in (9)

$$\psi^{(0)}_{k',\omega'} = \frac{(1+\xi_0)^2-1}{(1+\xi_0+\xi_0')^2} \approx 2\xi_0 \text{ when } \xi_0, \xi_0' \ll 1, \tag{10}$$

where

$$\xi_0(k) = \frac{\omega_{pe}^4}{4\pi n_0(T_e+T_i)}\int\frac{I^{(0)}_{k_1,\omega_1}dk_1\,d\omega_1}{\omega_1^2(\omega_1-\omega)^2\tilde\varepsilon(k_1-k,\omega_1-\omega)},$$

$$\xi_0'(k) = \frac{\omega_{pe}^4}{4\pi n_0 T_e}\int\frac{I^{(0)}_{k_1,\omega_1}dk_1\,d\omega_1}{\omega_1(\omega_1-\omega)^3\tilde\varepsilon(k_1-k,\omega_1-\omega)}. \tag{11}$$

This result makes the factor obtained in [8] exact (in [8] there occurs a factor $2\xi_0$ instead the correct $2\xi_0 + \xi_0^2$ in the numerator of Eq. (10); however, the main contribution to the integral comes from $\xi_0 \ll 1$ so that this correction is a small one).

In the case when k and ω correspond to the region outside the condensate it is necessary to re-evaluate $\epsilon^{RN}_{k,\omega}$, recognizing that the sum of the renormalized diagrams occurring in [5] is carried out only for waves for which the frequency difference satisfies the inequality

$$\omega-\omega_1 \ll |k\pm k_1|v_{Ti}. \tag{12}$$

This calculation differs from the one performed in [5,7,8] only in that it is necessary to clearly separate waves referring to different regions for which (12) is satisfied, while (12) is violated if the values of ω and k without index correspond to the wave we are studying and ω_1 and k_1 to the condensate.

Bearing this in mind we find $\epsilon^{RN}_{k,\omega}$ for a wave outside the condensate in the form

$$\varepsilon^{RN}_{k,\omega} = \frac{\omega_{pe}^8}{4\pi n_0^2 T_e^2}\int\tilde\Sigma_{k,\omega;k_1,\omega_1}\frac{w_{k',\omega'}dk'\,d\omega'}{|\tilde\varepsilon(k_1-k',\omega_1-\omega')|^2 k'^2(\omega_1-\omega')^4}\cdot \tag{13}$$

where

$$\cdot\{\psi_{k',\omega'}I^{(0)}_{k_1,\omega_1} + \psi'_{k',\omega'}I_{k_1,\omega_1}\}dk_1 d\omega_1,$$

$$\psi_{k',\omega'} = \xi_0 - \frac{T_e}{T_e+T_i}\xi_0' + \xi + \frac{T_e}{T_e+T_i}\xi', \tag{14}$$

$$\psi'_{k',\omega'} = \xi + \xi_1. \tag{15}$$

The last two formulae have been written down in the approximation ξ, ξ_0, $\xi_0' \ll 1$. It is clear that when $\xi_0 = \xi$, $\xi_0' = \xi'$ the result changes to the earlier known result (10). The quantities ξ, ξ' determine the intensity of the oscillations outside the condensate:

$$\xi = \frac{\omega_{pe}^4}{4\pi n_0(T_e+T_i)}\int_\Delta\frac{I_{k_2,\omega_2}dk_2\,d\omega_2}{\omega_2^2(\omega_2-\omega')^2\tilde\varepsilon(k_2-k',\omega_2-\omega')}, \tag{16}$$

$$\xi' = \frac{\omega_{pe}^4}{4\pi n_0 T_e}\int_\Delta\frac{I_{k_2,\omega_2}dk_2\,d\omega_2}{\omega_2(\omega_2-\omega')^3\tilde\varepsilon(k_2-k',\omega_2-\omega')}. \tag{17}$$

The integration over k_2, ω_2 is here performed over the regions Δ for ξ and ξ' and Δ_1 for ξ_1 which are determined by the inequalities

$$\omega_2-\omega \ll |k_2\pm k|v_{Ti} \text{ for } \Delta,$$
$$\omega_2-\omega_1 \ll |k_2\pm k_1|v_{Ti} \text{ for } \Delta_1.$$

The first term in the braces in (13) describes the interaction of a wave outside the condensate ($I_{k,\omega}$) with a wave of the condensate ($I^{(0)}_{k_1,\omega_1}$) and the second term the interaction of a wave outside the condensate with another wave also outside the condensate. The same interaction of waves outside the condensate is described by the second term in (8), but the signs of these interactions are each other's opposites, which shows up in that the interaction described by (13) corresponds to a spectral transfer in the opposite direction, namely, from small wave numbers k to large ones. While the first term in (13) describes the transformation of energy straight from the condensate to the region of large k-values, the second term describes the effect of the condensate on the process of the transformation to large k-values from the region outside the condensate.

We write Eq. (14) in the form

$$\psi_{k',\omega'}=\psi_1+\psi_2, \tag{18}$$

where

$$\psi_1=\xi_0-\frac{T_e}{T_e+T_i}\xi_0', \quad \psi_2=\xi+\frac{T_e}{T_e+T_i}\xi'.$$

An analysis which we shall expound elsewhere has shown that ψ_1 need be taken into account only when the energy in the condensate is appreciably larger than the threshold value W_c—this is connected with the different signs of the two terms in ψ_1.

However, the outflow of energy from the condensate due to the change in the direction of the transfer diminishes the energy of the oscillations in the condensate so that we may think that the case when the energy of the condensate does not exceed the threshold value appreciably is of most interest. We restrict ourselves here to that case and assume that

$$\psi_{k',\omega'}=\psi_2=\xi+\frac{T_e}{T_e+T_i}\xi'\approx 2\xi, \tag{19}$$

i.e., the case when $\psi_1 \ll \psi_2$.

The quantity $\xi = \xi' T_e/(T_e + T_i)$ can be found in the form of the following approximate expression:

$$\xi'=\frac{\omega_{pe}^4}{4\pi n_0 T_e}\int\frac{I_{k_2,\omega_2}\,dk_2\,d\omega_2}{\omega_2(\omega_2-\omega')^3\tilde{\varepsilon}(k_2-k',\omega_2-\omega')}\approx\frac{I_k k_*}{4\pi n_0 T_e}\frac{k'^2 v_{Te}^2}{\omega'^2},$$

$$I_k=\int_{\omega>0} I_{k,\omega}\,d\omega. \tag{20}$$

We have written this approximate expression for ξ' for the case when the inequalities

$$\omega'<\omega_2;\quad k'<k_2,\quad \omega'>k'\frac{d\omega_{k_2}}{dk_2}. \tag{21}$$

are satisfied.

The quantity $\epsilon_{k,\omega}^{RN}$ can be expressed in terms of the integral

$$J(k)=\int\frac{w_{k',\omega'}\,dk'\,d\omega'}{k'^2}\psi_{k',\omega'}, \tag{22}$$

which for the given case when we use the spectrum of the modulational oscillations $w_{k',\omega'}$ found in [7] is equal to

$$J(k)\approx\frac{4\pi n_0 T_e v_{Te}^2}{\sqrt{3}\,\omega_{pe}^2}\Lambda\left(\frac{W_0}{nT}\right)^{-1/2}\frac{I_k k_*}{4\pi n_0 T_e}, \tag{23}$$

where Λ is a numerical factor which depends weakly on the spectrum of the modulational oscillations ($\Lambda \approx \ln(k'_{max}/k'_{min}) \gg 1$).

3. We consider first of all the problem of the nonlinear wave dispersion. To fix the ideas one can sometimes assume that the spectrum formed by the condensate decreases as a power of k from the condensate in the direction of large k, namely

$$I_k\approx I_0/k^\alpha. \tag{24}$$

However, the further results are independent of this assumption. In that case we get, neglecting the second term in Eq. (13),

$$\varepsilon_{k,\omega}^{RN}=\frac{\omega_{pe}^4}{4\pi n_0^2 T_e^2}\int\tilde{\Sigma}_{k,\omega;k_1,\omega_1}\frac{I_{k_1,\omega_1}^{(0)}\,dk_1\,d\omega_1}{|\varepsilon_{k_1,\omega_1}|^2}J(k). \tag{25}$$

If we assume that $\mathrm{Re}\,\tilde{\epsilon}_{k,\omega} = -\mathrm{Re}\,\epsilon_{k,\omega}^{RN(0)} \approx -\Lambda^{1/2}(W_0/nT)^{1/2}$, we find

$$\varepsilon_{k,\omega}^{RN}\approx\frac{1}{(3\pi^2)^{1/4}\Lambda^{1/2}}\frac{\omega_{pe}^4}{n_0 T_e}\tilde{\Sigma}_{k,\omega;0,\omega_{pe}}J(k). \tag{26}$$

Using the relation

$$\mathrm{Re}\,\tilde{\Sigma}_{k,\omega;0,\omega_{pe}}\approx-\frac{m_e}{m_i}\frac{k^2}{\omega_{pe}^2(\Delta\omega)^2}, \tag{27}$$

we get for the real part of this expression for $\Delta\omega = \omega - \omega_1 \approx \omega - \omega_{pe}$

$$\mathrm{Re}\,\varepsilon_{k,\omega}^{RN}\approx-4\pi^{1/2}\Lambda^{1/2}\left(\frac{nT}{W_0}\right)^{1/2}\left(\frac{m_e}{m_i}\right)^{3/2}\frac{I_k I_d}{4\pi n_0 T_e}\frac{k^2 v_{Te}^2}{(\Delta\omega)^2}. \tag{28}$$

Substituting the value (24) of the turbulence spectrum and using $\mathrm{Re}\,\epsilon_{k,\omega}^{l} \approx 2\Delta\omega/\omega_{pe}$, we get the following value for the nonlinear change in the frequency:

$$\Delta\omega_k^N=\omega_{pe}\eta(I_k k^2)^{1/3}=\mu k^{(2-\alpha)/3}, \tag{29}$$

where

$$\mu=(2\pi^{1/2})^{1/3}\omega_{pe}\Lambda^{1/6}\left(\frac{W_0}{nT}\right)^{-1/6}\left(\frac{m_e}{m_i}\right)^{1/2}\left(\frac{I_0}{4\pi nT}\right)^{1/3}k_d^{-1/3}$$

or

$$\eta=\frac{\pi^{-5/3}}{16}\frac{\Lambda^{1/6}}{k_d^{1/3}n_0^{1/3}T_e^{1/3}}\left(\frac{W_0}{nT}\right)^{-1/6}\left(\frac{m_e}{m_i}\right)^{1/2}, \tag{30}$$

$$\Delta\omega_k^N=\nu_k(k/k_d)^{1/3}, \tag{31}$$

where

$$\nu_k\approx\Lambda^{1/6}\omega_{pe}\left(\frac{W_0}{nT}\right)^{-1/6}\left(\frac{m_e}{m_i}\right)^{1/2}\left(\frac{W(k)}{nT}\right)^{1/3}, \tag{32}$$

while $W(k) = I_k k/4\pi$.

Comparing $\Delta\omega_k^N$ with the magnitude of the frequency shift for linear dispersion, $\Delta\omega_k^l = \frac{3}{2}(k/k_d)^2\omega_{pe}$, we find the value of k for which the nonlinear dispersion will dominate over the linear one, namely, for $k < k_c$ where

$$k_c=k_d\left(\frac{nT}{W_0}\right)^{1/10}\left(\frac{m_e}{m_i}\right)^{3/10}\left(\frac{W(k_c)}{nT}\right)^{1/5}. \tag{33}$$

4. We consider now the effect of the condensate on the spectral transfer processes for waves outside the condensate which will be determined by $\mathrm{Im}\,\epsilon_{k,\omega}^{RN}$. As $\mathrm{Im}\,\epsilon_{k,\omega}^{RN}$ is determined by the quantity $\mathrm{Im}\,\tilde{\Sigma}_{k,\omega;k_1,\omega_1}$ which contains a factor $\delta(\omega - \omega_1 - (k - k_1)v)$ we can at once conclude that effects determined by ions are exponentially small (when there are no accelerated ions) for the interaction of waves in the condensate with the waves considered since we have assumed that $\omega - \omega_1 \gg |k - k_1|v_{Ti}$. To find the nonlinear transformation through the electrons from the condensate into the given value of (k, ω) we can use the expression for $\mathrm{Im}\,\epsilon_{k,\omega}^{RN}$ which we get from (25) by susbsituting $\mathrm{Im}\,\tilde{\Sigma}_{k,\omega;k_1,\omega_1}$ for $\mathrm{Re}\,\tilde{\Sigma}_{k,\omega;k_1,\omega_1}$:

$$\mathrm{Im}\,\varepsilon_{k,\omega}^{RN}\approx\frac{\pi^{-2/3}}{(3)^{3/4}}\Lambda^{1/2}\left(\frac{W_0}{nT}\right)^{-1/2}\frac{I_k k_*}{4\pi n_0 T_e}\frac{\omega_{pe}^2 v_{Te}^2}{n_0 T_e}\int\mathrm{Im}\,\tilde{\Sigma}_{k,\omega;k_1\omega_1}I_{k_1\omega_1}^{(0)}\,dk_1\,d\omega_1. \tag{34}$$

This expression has the opposite sign determining the transfer from the small k-value to the large k-value region.

Using the expression for $\mathrm{Im}\,\tilde{\Sigma}_{k,\omega;k_1,\omega_1}$ for the case of induced scattering by electrons when the nonlinear scattering dominates over the Compton scattering we can find that

$$\gamma_k^{RNe}\approx\frac{\pi^{-1/6}}{5.6}\omega_{pe}\Lambda^{1/2}\left(\frac{W_0}{nT}\right)^{-1/2}\left(\frac{m_e}{m_i}\right)^{3/2}\frac{I_k k_d}{4\pi n_0^2 T_e^2}\int\frac{k^3 v_{Te}^3}{(\Delta\omega)^3}I_{k_1,\omega_1}^{(0)}\,dk_1\,d\omega_1, \tag{35}$$

and we find, substituting for $\Delta\omega_k^N$ from (29), that

$$\gamma_k^{RNe}\approx\frac{\pi^{2/3}}{2{,}8}\omega_{pe}\frac{m_e}{m_i}\frac{k}{k_d}. \tag{36}$$

In the region where the Compton scattering dominates

$$\gamma_k^{RNe} \approx \Lambda^{4/9} \omega_{pe} \frac{m_e}{m_i} \left(\frac{W_0}{nT}\right)^{-2/3} \left(\frac{W(k)}{nT}\right)^{4/3} \left(\frac{k}{k_d}\right)^{1/3} \sim k^{(5-4\alpha)/3}. \tag{37}$$

In the region $k > k_c$ where the dispersion is the usual (linear) one we have the following expression for the nonlinear scattering growth rate:

$$\gamma_k^{RNe} \approx \frac{\pi^{1/6}}{46.7} \omega_{pe} \Lambda^{1/3} \left(\frac{m_e}{m_i}\right)^2 \left(\frac{W_0}{nT}\right)^{-1/2} \left(\frac{k_d}{k}\right)^3 \frac{W(k)}{nT}. \tag{38}$$

We must note that this quantity although not strongly, all the same is appreciably (by a factor $\Lambda^{1/3}$) larger than the quantity γ_k^{Ne} for the case of the normal nonlinear scattering which determines the transfer in the direction of small k.

5. We now turn to a consideration of the interaction of two waves outside the condensate. Such an interaction is for the case $T_e \gg T_i$ described by γ_k^{Ni} and γ_k^{RNi}, i.e., it is due to an interaction process with ions which requires the inequality $\omega - \omega_1 \ll |k \pm k_1| v_{Ti}$. We discuss first of all how the normal scattering by ions γ_k^{Ni} is changed with the change in the character of the dispersion of the Langmuir waves. In the case studied

$$\gamma_k^{Ni} = \frac{\pi}{2} \frac{\omega_{pe}}{(1+T_e/T_i)^2 n_0 m_e} \int (k^2+k_1^2) I_{k_1} \delta'(\mu(k^{(2-\alpha)/3} - k_1^{(2-\alpha)/3}))\, dk_1 \tag{39}$$

$$= \beta k^{(8+2\alpha)/3} \left\{ \frac{\partial I_k}{\partial k} + \frac{4+\alpha}{3} \frac{I_k}{k} \right\},$$

where

$$\beta \approx \frac{9\pi \Lambda^{-2/3}}{(2-\alpha)^2 n_0 m_e \omega_{pe}} \frac{m_i}{m_e} \left(1 + \frac{T_e}{T_i}\right)^{-2} \left(\frac{W_0}{nT}\right)^{1/3} \left(\frac{I_0}{4\pi n_0 T_e}\right)^{-2/3} k_d^{2/3}.$$

Estimating expression (39) we have

$$\gamma_k^{Ni} \approx \omega_{pe} \frac{m_i}{m_e} \left(\frac{W_0}{nT}\right)^{1/3} \left(\frac{I_0}{4\pi n_0 T}\right)^{1/3} \left(\frac{k}{k_d}\right)^{(5-\alpha)/3}. \tag{40}$$

One can easily see from Eq. (39) which we have just obtained that the wave interaction vanishes when

$$I_k \propto k^{-(4+\alpha)/3}, \tag{41}$$

while, on the other hand, $I_k \propto 1/k^{\alpha}$, whence $4 + \alpha = 3\alpha$ or $\alpha = 2$.

In actual fact the value $\alpha = 2$ is not exactly a solution, since the growth rate becomes undetermined for that value. However, if the value of α is not exactly equal to two, but approaches that value, the interaction of the waves increases very strongly. A more exact analysis shows that the turbulent spectrum has the form

$$I_k = C/k^2 + \delta I_k,$$

but the correction $\delta I_k \ll C/k^2$, where C is a constant quantity.

We turn to Eq. (13) for the determination of the growth rate γ_k^{RNi} of the nonlinear scattering by ions when the effect of the high-frequency and modulational oscillations is taken into account. We have for the general form for the case of the interaction of two waves outside the condensate

$$\operatorname{Im} \varepsilon_{k,\omega}^{RNi} \approx \frac{1}{\sqrt{3}} \Lambda^{1/3} \left(\frac{W_0}{nT}\right)^{-1/2} \left(\frac{m_e}{m_i}\right)^{1/2} \frac{\omega_{pe}^2 v_{Te}^2 k_d}{4\pi n_0^2 T_e^2} \int \operatorname{Im} \tilde{\Sigma}_{k,\omega;k_1,\omega_1} \frac{I_{k_1}(I_k + I_{k_1})}{|\tilde{\varepsilon}(k_1, \omega_1)|^2}\, dk_1\, d\omega_1. \tag{42}$$

As $\tilde{\epsilon}_{k,\omega} = -\epsilon_{k,\omega}^{RN}$ we get, using Eq. (28) for $\operatorname{Re} \epsilon_{k,\omega}^{RN}$

$$\operatorname{Im} \varepsilon_{k,\omega}^{RNi} \approx \frac{\pi^{1/3}}{4} \Lambda^{1/3} \left(\frac{W_0}{nT}\right)^{1/2} \left(\frac{m_i}{m_e}\right)^{5/2} \frac{\omega_{pe}^2}{k_d v_{Te}^2} \cdot$$

$$\cdot \int \frac{(\Delta\omega_{k_1})^4}{k_1^4} \operatorname{Im} \tilde{\Sigma}^{i}_{k,\omega;k_1\omega_1} \left(1 + \frac{I_k}{I_{k_1}}\right) dk_1\, d\omega_1. \tag{43}$$

Substituting here the expression for $\operatorname{Im} \tilde{\Sigma}^{i}_{k,\omega;k_1,\omega_1}$ and Eq. (29) for $\Delta\omega_k$ we get the following expression for the growth rate of the nonlinear scattering by ions determined by the transformation of energy to the region of large k-values:

$$\gamma_k^{RNi} \approx \frac{\pi^{1/3} \omega_{pe}}{4 n_0 m_e} \frac{\Lambda^{1/3} (W_0/nT)^{1/2} (m_i/m_e)^{5/2} \mu^4}{(1+T_e/T_i)^2 k_d v_{Te}^2} \cdot$$

$$\cdot \int \frac{(k^2+k_1^2) k_1^{(8-4\alpha)/3}}{k_1^4} \left(1 + \frac{I_k}{I_{k_1}}\right) \delta'(\mu[k^{(2-\alpha)/3} - k_1^{(2-\alpha)/3}])\, dk_1 \tag{44}$$

$$= \omega_{pe} \frac{9\pi^{1/3} \Lambda^{5/9}}{(2-\alpha)^2} \left(\frac{W_0}{nT}\right)^{1/6} \left(\frac{m_i}{m_e}\right)^{3/2} \left(\frac{I_0}{4\pi nT}\right)^{2/3} k_d^{1/3} k^{(1-2\alpha)/3} \left(\alpha + \frac{k}{2I_k} \frac{\partial I_k}{\partial k}\right).$$

The interaction between two waves vanishes when

$$I_k \propto k^{-2\alpha}. \tag{45}$$

However, as we are looking for the turbulence spectrum in the form $I_k \propto k^{-\alpha}$, we get, equating the indexes $2\alpha = \alpha$, or $\alpha = 0$, i.e., I_k = const. On the other hand, comparing the expressions γ_k^{RNi} and γ_k^{Ni} we can find the following relation:

$$\gamma_k^{RNi} \approx \gamma_k^{Ni} \Lambda^{7/9} \left(\frac{W_0}{nT}\right)^{-1/6} \left(\frac{m_i}{m_e}\right)^{1/2} \left(\frac{W(k)}{nT}\right)^{1/3} \left(\frac{k_d}{k}\right)^{5/3}; \tag{46}$$

if we put the value of k equal to k_c, we have

$$\gamma_k^{RNi} \approx \gamma_k^{Ni} \Lambda^{7/9} m_i/m_e. \tag{47}$$

Hence we can note that the quantity γ_k^{RNi} which determines the transfer in the direction of large k is appreciably larger than γ_k^{Ni}. In the linear dispersion case comparison of γ_k^{RNi} and γ_k^{Ni} gives

$$\gamma_k^{RNi} \approx \gamma_k^{Ni} \Lambda^{1/3} \left(\frac{W_0}{nT}\right)^{1/2} \left(\frac{m_i}{m_e}\right)^{5/2} \left(\frac{W(k)}{nT}\right)^{-1} \left(\frac{k}{k_d}\right)^{5}, \tag{48}$$

or for k-values equal to k_c

$$\gamma_k^{RNi} \approx \gamma_k^{Ni} \Lambda^{1/3} m_i/m_e. \tag{49}$$

We note that also in this case the quantity γ_k^{RNi} is larger than γ_k^{Ni}.

6. We determine now the total spectrum of the turbulent pulsations taking not only γ_k^{RNi}, but also γ_k^{Ni} and γ_k^{RNe} for the two regions $k < k_c$ and $k > k_c$ into account.

To prove the general character of the spectrum $I_k \propto k^{-2}$ as a solution of nonlinear equations in the linear dispersion region $k > k_c$ we can write down the following balance equation:

$$\alpha \frac{\partial I_k}{\partial k} + \frac{\beta}{k^3} = 0. \tag{50}$$

This equation has a solution in the form

$$I_k \approx \text{const}/k^2, \quad \text{const} = \beta/2\alpha, \tag{51}$$

where

$$\alpha = \frac{\pi \omega_{pe}^3}{9 m_i n_0 v_{Te}^4 (1+T_e/T_i)^2},$$

$$\beta \approx \omega_{pe} \Lambda^{1/3} \left(\frac{m_e}{m_i}\right)^2 \left(\frac{W_0}{nT}\right)^{-1/2} \left(\frac{W(k)}{nT}\right) k_d^3.$$

In the region $k < k_c$ the balance equation takes the following form:

$$\alpha_1 k^{(5-\alpha)/3} + \beta_1 k = 0, \tag{52}$$

where

$$\alpha_1 \approx \omega_{pe} \frac{m_i}{m_e} \left(\frac{W_0}{nT}\right)^{1/3} \left(\frac{I_0}{4\pi nT}\right)^{1/3} k_d^{(\alpha-5)/3},$$

$$\beta_1 \approx \omega_{pe} \frac{m_e}{m_i} k_d^{-1}.$$

Since the indexes in Eq. (52) must be equal, we find that $\alpha = 2$.

We thus get also in that case the spectrum

$$I_k \infty k^{-2}.$$

7. The present considerations show that a spectrum of the form k^{-2}, obtained as the result of numerical calculations, indeed follows from the statistical theory of the Langmuir condensate, proposed earlier by the present authors[7,8] while this form of the spectrum is obtained from the balance of completely different nonlinear interactions and it can therefore be considered to be quite universal.

In contrast to other approaches to this problem the present analysis indicates the actual conditions under which such a spectrum can in reality occur when other kinds of interactions, absorption, and also a buildup of oscillations are taken into account. We must also emphasize that the answer is obtained in a general form without additional assumptions about the structure of strong turbulence. For practical applications one must bear in mind the limitations imposed upon the strong turbulence spectrum by the assumed condition $\psi_1 < \psi_2$ which may be violated when the energy level of the condensate is large, and also the limitations connected with the absorption of the oscillations by fast particles which are accelerated by the Langmuir oscillations. Both these problems need an additional analysis.

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