The slowing down of fast protons in a plasma with a strong magnetic field

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In a plasma with a strong magnetic hield $H > H_0 = 2.2 \times 10^{13}(v/c)^2$ G the fast protons moving along the lines of force have a sharply increased mean free path: the quantization of the energy of the transverse electron motion reduces the Coulomb energy losses. The most important factor then is diffusion in angle space of the protons away from the direction of the magnetic field, since the losses strongly depend on the angle between the field and the direction of motion of the protons. As a result, the Coulomb mean free path increases by a factor of $(M/m)^{1/2}$ compared to the mean free path in a plasma without a magnetic field. Since the Coulomb losses are small, the retardation of the protons is determined by the nuclear collisions. For $16\pi\mu_0 N_e mc^2/kT < H < H_0$ and for high proton velocities, such that $\hbar v/e^2 \gg 1$, the mean free path increases in accordance with the decrease in the Coulomb logarithm, in which the Debye radius should be replaced by the Larmor radius of the zeroth Landau level. Here μ_0 is the Bohr magneton.

The strong magnetic fields used in laboratory facilities slightly change the rate of the relaxation processes in a nonequilibrium plasma and, in particular, the rate of retardation of the fast particles[1, 2]. However, the discovery of pulsars—magnetized, rapidly rotating neutron stars—has shown that the occurrence in astrophysical objects of ultrastrong magnetic fields $H \sim 10^{10}$–10^{14} G is not rare. The energetics of x-ray pulsars is explained by the accretion (the falling) of gas on a neutron star. The strong magnetic field channels the accretion, guiding the matter toward the region of the magnetic poles[3]. The matter moves in the process along the magnetic lines of force. Near the surface of the star (its mass is assumed to be equal to that of the Sun and its radius is assumed to be equal to 10^6 cm), the kinetic energy of the protons is ~100–150 MeV, the particle velocity $v \sim 0.3$–0.5 c, and the surface temperature $T \sim 10^7$–10^8 °K. Of considerable interest is the question of the retardation of the protons, which carry the main portion of the kinetic energy of the accreting flux, in the atmosphere of the magnetized neutron star[1)]. This question is important in the consideration of the mechanisms leading to the directionality of the x rays[6].

In an ultrastrong magnetic field the quantization of the energy of the transverse electron motion leads to a situation in which the transfer of the energy of the fast protons (moving parallel to the direction of the field) to the plasma electrons is hindered and the role of the Coulomb losses in the retardation of the protons is small compared to the role of the nuclear collisions.

1. COULOMB COLLISIONS IN A STRONG MAGNETIC FIELD

Let us consider the retardation of a beam of protons moving in a magnetized plasma along the magnetic lines of force as a result of Coulomb collisions with the electrons of the medium. We recall that the Coulomb losses determine the range of fast charged particles in a plasma without a magnetic field. In this case the particles give up their energy to the plasma electrons mainly after traversing long paths, owing to a change in the transverse component of the momentum (see, for example,[7, 8]). The presence, however, of a strong longitudinal magnetic field can greatly hinder the transfer of transverse momentum, which then leads to a significant increase in the mean free path.

To compute the rate of retardation, we must consider an individual act of scattering of a proton by an electron of the medium in the general case when the proton, prior to the collision, moves at an angle θ to the magnetic field. In such a collision, the energy and momentum component along the direction of the field are conserved. Assuming that the plasma electrons are at rest, which is true when $kT_e \ll mv^2$, we have

$$P\cos\theta = P_{\parallel}\cos\theta + P_{\perp}\sin\theta + p, \tag{1}$$

$$\frac{P^2}{2M} = \frac{P_{\parallel}^2 + P_{\perp}^2}{2M} + \frac{p^2}{2m} + n\hbar\omega_H. \tag{2}$$

Here P is the proton momentum before the collision; $P_{\parallel}$ and $P_{\perp}$ are the post-collision proton-momentum components along two mutually perpendicular directions, $P_{\parallel}$ being the component along the initial direction of motion; p is the post-collision electron momentum along the magnetic field; m and M are the electron and proton masses; $\omega_H = eH/mc$ is the electron gyrofrequency. We neglect the proton–magnetic field interaction, since the proton gyrofrequency is (M/m) times smaller than ω_H. Allowance is made for the fact that the energy of the transverse electron motion is quantized: n = 0, 1, 2, . . . is a whole number. Eliminating $P_{\parallel}$ with the aid of (1) from (2), we find the post-scattering longitudinal electron momentum:

$$p = \frac{mv}{\cos\theta(1 + m/M\cos^2\theta)}\left\{1 - \frac{P_{\perp}}{P}\operatorname{tg}\theta \mp \left[1 - \frac{2n\hbar\omega_H}{mv^2}\left(\cos^2\theta + \frac{m}{M}\right) + \frac{M}{m}\frac{P_{\perp}}{P}\sin 2\theta - \frac{P_{\perp}^2}{P^2}\left(1 + \frac{M}{m}\right)\right]^{1/2}\right\}. \tag{3}$$

The energy lost by the proton in each collision is

$$\varepsilon = n\hbar\omega_H + p^2/2m. \tag{4}$$

2. PROTON RETARDATION WHEN $\hbar\omega_H > mv^2/2$ ($H > H_0$)

It can be seen from (3) that at $\theta \ll 1$ and $\hbar\omega > mv^2/2$ the whole number $n \equiv 0$, i.e., a proton moving in a direction inclined at a small angle to a magnetic field of intensity $H > H_0 = mv^2/4\mu_0 = 2.2 \times 10^{13}(v/c)^2$ G cannot increase the energy of the transverse motion of an electron of the medium (here $\mu_0 = e\hbar/2mc$ is the Bohr magneton). If $\theta = 0$, then the collisions of a proton with electrons of the medium can be divided into two types: a) the proton flies past a plasma electron (the sign

"−" in front of the radical sign in (3)); b) an electron is scattered along the direction of motion of the proton (it is "reflected" from the proton in its rest frame; the sign "+" in front of the radical sign in (3)). In the case a) the energy lost by the proton in one collision,

$$\varepsilon=\frac{p^2}{2m}=\frac{mv^2}{2}\left(\frac{1}{2}\frac{M}{m}\frac{P_\perp^2}{P^2}\right)^2\leqslant\frac{1}{8}\left(\frac{m}{M}\right)^2 mv^2, \tag{5}$$

is negligibly small, and such collisions can be neglected. In the case b) this energy is $\epsilon = 2mv^2$. The cross section for scattering of the type b) has been computed in[9]. It depends on the intensity H of the magnetic field: for $H \gg H_0$ it decreases in proportion to H^{-1}, while for $H \ll H_0$ it assumes a constant value. A detailed analysis shows that at no value of the field can the collisions of the type b) become the decisive mechanism by which the protons are slowed down, since the mean free path corresponding to these collisions alone, even when $H = H_0$, exceeds by a factor of two the mean free path (11) due to the angular diffusion process described below.

Thus, the principal mechanism by which a proton initially moving along a magnetic field of intensity $H > H_0$ is slowed down amounts to the following.

At first the proton is deflected in each scattering by an electron or a proton of the plasma through a small angle

$$\Delta\theta=P_\perp/P\leqslant m/M, \tag{6}$$

practically without losing energy. In successive scattering events the deflections θ build up from zero. The energy transferable to an electron in each scattering event increases rapidly with increasing θ, and is given, when $1 \gg \theta \gg m/M$, by the expression

$$\varepsilon=\frac{p^2}{2m}=\frac{P_\perp^2}{2m}\theta^2=\frac{2e^4}{mv^2r^2}\theta^2, \tag{7}$$

where $P_\perp = 2e^2/rv$ is computed in the usual manner with the aid of first-order perturbation theory[8] and r is the classical impact parameter. A proton begins to effectively lose energy only after it has been deflected through an angle $\theta \gtrsim (m/M)^{1/4}$ (cf. the expressions (8) and (9)). We allowed above for the fact that the transverse proton recoil momentum $P_\perp \leq mv$, and the upper sign in front of the radical sign in the expression (3) was retained since the electron motion is one-dimensional at n = 0.

The above-discussed qualitative picture of the retardation of protons moving parallel to the field is described by the two equations:

$$\frac{d\theta^2}{dx}=(N_e+N_p)\int\limits_{r_{min}}^{r_{max}}\left(\frac{P_\perp}{P}\right)^2\cdot 2\pi r\,dr=2N_e\frac{8\pi e^4\ln\Lambda}{M^2v^4}, \tag{8}$$

$$\frac{dE_p}{dx}=-N_e\int\limits_{r_{min}}^{r_{max}}\varepsilon 2\pi r\,dr=-N_e\frac{4\pi e^4\ln\Lambda}{mv^2}\theta^2. \tag{9}$$

The equation (8) of diffusion in angle space was derived for a hydrogen plasma with allowance for the fact that the deflections of a proton in successive scattering events are not correlated and that it is the quantity $(\Delta\theta)^2$, and not $\Delta\theta$, that builds up. In this case we neglect the difference between the values of the Coulomb logarithm lnΛ for proton-proton and proton-electron collisions, since the resultant error is small.

The system of equations (8) and (9) is equivalent to the single second-order equation

$$\frac{d^2E_p}{dx^2}=-\left(N_e\frac{8\pi e^4\ln\Lambda}{mv^2}\right)^2\frac{m}{M}, \tag{10}$$

which can be integrated. We then find the mean free path

$$l=\sqrt{\frac{\pi}{2}\frac{m}{M}}E_p^2/4N_e\pi e^4\ln\Lambda.$$

Equation (16) for the rate of energy loss by protons in a plasma without a magnetic field can also be easily integrated. The ratio of the corresponding mean free paths is (the difference between the values of the Coulomb logarithms is again neglected):

$$\frac{l_{H>H_0}}{l_{H=0}}=\sqrt{\frac{\pi}{2}\frac{M}{m}}\approx 54. \tag{11}$$

Notice that allowance for the above-described scattering of the plasma electrons in the direction of motion of the protons in the case when $H \sim H_0$ will decrease somewhat the mean free path (11).

3. PROTON RETARDATION IN THE CASE WHEN $\hbar\omega_H < mv^2/2$ ($H < H_0$)

When $H < H_0$, it is possible for an electron interacting with a fast proton to jump from one Landau level to another. Below, in estimating the rate of proton retardation, we set $\theta = 0$, neglecting the diffusion of the protons in angle space and the contribution of the transitions with n = 0 considered above. These assumptions are entirely justified, since the proton energy decreases to a value at which $\hbar\omega_H = mv^2/2$ over the distance given by (14) and (16), which is much shorter than the mean free path that is obtained from the solution of Eq. (10).

In computing the proton-energy losses, the cross section for each individual scattering event is estimated from the Rutherford formula. To determine the range of angles at which scattering can occur, we use the quantum conservation laws in a magnetic field. The application of the Rutherford formula in the proton-velocity region where $\hbar v/e^2 \gg 1$ and which is of interest to us here, is entirely justified, since in the angle region where scattering is allowed the transit time turns out to be small compared to ω_H^{-1}, the gyrorevolution time. Indeed, in the region of allowed scattering angles the energy transferred to an electron should exceed $\hbar\omega_H$:

$$\varepsilon=2mv^2\sin^2\frac{\vartheta}{2}=\frac{2mv^2}{1+m^2v^4r^2/e^4}>\hbar\omega_H. \tag{12}$$

Here ϑ is the scattering angle for the electron in the rest frame of the proton. It follows from (12) and the conditions $\hbar v/e^2 \gg 1$ and $\hbar\omega_H \lesssim mv^2$ that for all the allowed impact parameters $r/v \ll \omega_H^{-1}$.

A. The Case when $mv^2/4 \leqslant \hbar\omega_H \leqslant mv^2/2$ ($H_0/2 \leqslant H \leqslant H_0$)

In this case electron transitions to a neighboring Landau level are possible: n = 0 or 1. Let us estimate the contribution of the transitions with n = 1. It follows from the expressions (3) and (4) that the energy lost by a proton in each scattering event for $\theta = 0$ is given by

$$\varepsilon=mv^2\left[1\mp\left(1-\frac{2\hbar\omega_H}{mv^2}\right)^{1/2}\right]. \tag{13}$$

The sought rate of energy loss is equal to

$$\frac{dE_p}{dx}=-N_e\int\limits_{\vartheta_1}^{\vartheta_2}\varepsilon\,d\sigma=-N_e\frac{4\pi e^4}{mv^2}\left(1-\frac{2\hbar\omega_H}{mv^2}\right)^{1/2}, \tag{14}$$

where $d\sigma = (e^2/2mv^2)^2 \sin^{-4}(\vartheta/2)d\Omega$ is the differential cross section for scattering in the Coulomb field and the limiting angles $0 < \vartheta_1 \leq \pi/2 \leq \vartheta_2 < \pi$ are determined from the equation

$$\frac{1}{2}mv^2 \sin^2\vartheta_{1,2} = \hbar\omega_H. \tag{15}$$

Equation (15) for the angles ϑ_1 and ϑ_2 at which the integration should be cut off is obtained from the condition that the energy $\epsilon_\perp = 1/2mv^2 \sin^2\vartheta$ of the transverse electron motion change by the smallest possible amount $\hbar\omega_H$. In evaluating the integral in (14) we took into account the fact that in the angle interval $\vartheta_1 < \vartheta < \pi/2$ the expression (13) should be taken with the upper sign in front of the radical sign, while in the interval $\pi/2 < \vartheta < \vartheta_2$ it should be taken with the lower sign.

A proton discards its energy according to (14) from the value $Mv^2/2$ to $\hbar\omega_H M/m$, after which this expression becomes inapplicable (see the preceding section). The corresponding mean free path decreases with increasing H (this can be seen from (14)): it has its maximum value at $\hbar\omega_H = mv^2/4$, this value being $\approx 1.25 \ln\Lambda\sqrt{m/M} \approx 1/4$ the mean free path computed from the formula (10) (in astrophysical situations $\ln\Lambda \approx 10$; see, for example,[3]).

B. The Case when $\hbar\omega_H \ll mv^2/2$ ($H \ll H_0$)

In this case n runs through a set of integral values, each of which has its own region of allowed angles determined from the condition $n\hbar\omega_H = \epsilon_\perp = 1/2mv^2\sin^2\vartheta$. Using the expressions (4) and (3) for $\theta = 0$, we can easily find the total energy $\epsilon = 2mv^2\sin^2(\vartheta/2)$ transferable to an electron in a collision. As before, we have

$$\frac{dE_p}{dx} = -N_e\int_{\vartheta_1}^{\vartheta_2}\epsilon\, d\sigma = -N_e\frac{4\pi e^4}{mv^2}\ln\Lambda, \tag{16}$$

where

$$\Lambda = \left(\frac{1+(1-2\hbar\omega_H/mv^2)^{1/2}}{1-(1-2\hbar\omega_H/mv^2)^{1/2}}\right)^{1/2}. \tag{17}$$

Except for the expression for Λ, the formula (16) differs in no way from the well-known formula for the case of zero magnetic field. When $\hbar\omega_H = mv^2/2$ ($H = H_0$), we $\Lambda = 1$, and the expression (16) vanishes. In the limit of a weak magnetic field we easily find from (17) that

$$\Lambda = \sqrt{2mv^2/\hbar\omega_H} = r_H/\bar{\lambda}, \tag{18}$$

where $r_H = (2\hbar c/eH)^{1/2}$ is the radius of the zeroth Landau level and $\bar{\lambda} = \hbar/mv$ is the de Broglie wavelength of the electron. Notice that when $(1 - 2\hbar\omega_H/mv^2)^{1/2} \ll 1$ the expression (16) goes over into (14), and in the entire interval $H_0/2 < H < H_0$ these expressions differ by not more than 25%.

In the case of weak fields, when $r_H > r_D$, r_H in (18) should be replaced by the Debye radius r_D; the expression (16) then goes over into the well-known formula for fast-particle relaxation in a plasma[7,8].

4. CONCLUSIONS

The obtained results show that for $H > H_0$ the energy losses for protons moving along the magnetic lines of force as a result of Coulomb collisions with the electrons of the medium decrease by a factor of $(M/m)^{1/2}$. However, they are still much ($(M/m)^{1/2}$ times) larger than the Coulomb losses due to collisions with the protons of the medium. The principal retardation mechanism under the conditions of interest to us is nuclear collisions with the plasma protons. The strong-interaction cross section $\sigma_{pp} \approx 3 \times 10^{-26}$ cm^2, and the corresponding range is 50 g/cm^2, which is roughly 6–7 times greater than the Coulomb range for fast protons in a tenuous plasma without a magnetic field.

When $H < H_0$, $\hbar v/e^2 \gg 1$, and $r_H < r_D$, the magnetic field leads to a decrease in the Coulomb logarithm and to a corresponding increase in the proton range. The condition $r_H < r_D$ corresponds to

$$H > 16\pi\mu_0 N_e\frac{mc^2}{kT} = 2.8\cdot 10^5\left(\frac{N_e}{10^{22}\ cm^{-3}}\right)\left(\frac{10^8\,K}{T}\right)\ G,$$

from which it can be seen that the obtained formulas are applicable in the wide range of magnetic-field strengths $10^5 < H < 10^{16}$ G.

For $H < H_0$ and $\hbar v/e^2 \ll 1$, our treatment is inapplicable. This case has been considered in[1,2].

It is clear that the presence of a strong magnetic field also decreases the rate of collision relaxation of the electron and proton temperatures.

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[1]In this note we do not touch upon the question of the possible stopping of a proton beam in a collisionless shock wave [4,5], a question that arises in connection with the analysis of the collective processes, and essentially consider the problem of the retardation of individual fast protons.

1. T. Kihara, J. Phys. Soc. Japan 14, 1751 (1959).
2. V. P. Silin, Vvedenie v kineticheskuyu teoriyu gazov (Introduction to the Kinetic Theory of Gases), Nauka, 1971.
3. F. K. Lamb, C. J. Pethick, and D. Pines, Astrophys. J. 184, 271 (1973).
4. Yu. N. Gnedin and R. A. Sunyaev, Astron. Astrophys. 25, 233 (1973).
5. G. S. Bisnovatyĭ-Kogan, Astron. Zh. 50, 902 (1973) [Sov. Astron.-AJ 17, 574 (1974)].
6. M. M. Basko and R. A. Sunyaev, Astron. Astrophys., 1975 (in press); Preprint IKI, No. 208, 1974.
7. L. D. Landau, Zh. Eksp. Teor. Fiz. 7, 203 (1937).
8. E. Fermi, Nuclear Physics, University of Chicago Press, Chicago, 1950 (Russ. Transl., IIL, 1951), Chap. II, sect. 1.
9. J. Ventura, Phys. Rev. A8, 3021 (1973).

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