Phase transition of an isotropic system to a nonuniform state

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A phase transition with a small-amplitude order parameter, of the type that occurs in the crystallization of a liquid, is considered. The character of the transition and the properties of the different ordered phases are investigated for the cases of a scalar and a vector order parameter.

1. INTRODUCTION

We consider an isotropic or almost isotropic system in which the fluctuation spectrum $\omega(p)$ has a low absolute minimum, attained at nonzero momenta. If the system is strictly isotropic, then near the minimum $\omega(p)$ can be written in the form

$$\omega(p) = \Delta + (p - p_c)^2/m.$$

Since the minimum of $\omega(p)$ is attained on a surface $|p| = p_c$ of finite dimensions, the phase volume of fluctuations with $\omega(p) \approx \Delta$ is large. This leads to the result that for $\Delta \ll p_c^2/m$ the characteristic field amplitudes of the fluctuations $\varphi(r)$ increase rapidly with decreasing Δ. In fact, in the harmonic approximation the mean square of the fluctuations $\varphi(r)$ is equal to

$$\langle \varphi^2(r) \rangle = \Omega \int \frac{dp}{(2\pi)^3} \frac{T}{\omega(p)} = \frac{\Omega p_c^2 T}{2\pi (m\Delta)^{1/2}}, \qquad (1)$$

i.e., $\langle \varphi^2 \rangle$ increases like $\Delta^{-1/2}$.

A considerable increase of $\langle \varphi^2 \rangle$ can also be observed in weakly anisotropic systems, such as in the magnets with large-period superstructures considered in the papers of Dzyaloshinskiĭ[1,2]. In this case, the minimum of $\omega(p)$ is attained, generally speaking, at isolated points lying on lines of high symmetry, and, consequently, near these points the phase volume of the fluctuations is small, as in isotropic systems with $p_c = 0$. However, if the anisotropy is small, there exists an energy boundary ω_0 such that the surfaces $\omega(p) = \omega = $ const, which enclose the isolated extremal points at $\omega < \omega_0$, merge at $\omega > \omega_0$ into one surface with area of the order of $4\pi p_c^2$, where p_c is the distance of the extremal points from the center of the Brillouin zone. Then two cases are possible. If the anisotropy is small by virtue of the smallness of p_c^2/m compared with the atomic (or exchange) energy, as in, antiferromagnets of the type MnO, then $\omega_0 \approx \omega(0) \gg \Delta$ and, consequently, the fluctuations with energy $\omega \approx \omega_0$ can be neglected. But if the parameters p_c^2/m and ω_0 are independent, as in the rare-earth metals, it becomes possible to fulfill the inequality $\omega_0 \ll p_c^2/m$. In this case, in the region $\omega_0 \ll \Delta \ll p_c^2/m$ the characteristic amplitudes of the fluctuations increase with decreasing Δ, as in an isotropic system. In this region the behavior of the system can be described by an isotropic model. Accordingly, in the following we consider only isotropic systems.

The only known type of isotropic system in which considerable fluctuations of the order parameter are observed in the region of a transition to a nonuniform state is a cholesteric liquid crystal. Liquid ^{3}He in the region of the expected transition to the antiferromagnetic state at a very low temperature should also be such a system. Because of the strong interaction, the crystallization of any liquid, including ^{4}He, evidently occurs before the effect of the fluctuations (the rotons, in ^{4}He) on the parameters of the system is manifested.

We consider systems in which the interaction between the fluctuations is weak, and this enables us to consider the region of sufficiently low energies Δ. It will be shown that at sufficiently small Δ the uniform state becomes thermodynamically unfavorable and the system experiences a first-order transition to a nonuniform state with a univariate distribution of the order parameter. Here the discontinuous character of the transition is not connected with the influence of third-order terms in the expansion of the thermodynamic potential in the mean values $\langle \varphi \rangle$. The results explain qualitatively the character of the transition in cholesteric liquid crystals. In a separate paper, specific allowance will be made for the tensor order parameter and for the presence of two correlation lengths in these substances, and a discussion of the experimental data will also be given. Here we confine ourselves to the simpler cases of a scalar and a vector order parameter, corresponding to the phase transitions in liquids and isotropic magnets.

2. PROPERTIES OF THE UNIFORM PHASE

1. We start from the phenomenological expansion of the free-energy functional $\mathscr{F}\{\varphi(r)\}$ of a weakly perturbed state of the system, in powers of the dimensionless field $\varphi(r)$. This imposes the obvious restriction $\langle \varphi^2(r) \rangle \ll \varphi_0^2$, where φ_0 is the scale of the expansion of the functional. Without this restriction the behavior of the system would depend essentially on the character of the interaction at short distances. Besides this, we assume that $\Delta \ll p_c^2/m$. This condition makes it possible to confine ourselves throughout to the region of momenta close to the sphere $|p| = p_c$, and to perform all the calculations analytically. Combining the above inequalities, we obtain, taking (1) into account,

$$(Tp_c^2\Omega/2\pi\varphi_0)^2 \ll \Delta \ll p_c^2/m.$$

Consequently, the period $L = 2\pi/p_c$ of the structure that arises should be sufficiently large:

$$L/a \gg \varphi_0^{-1} a^2 p_T^2,$$

where $a = \Omega^{1/3}$ is the interatomic distance and $p_T = (mT)^{1/2}$ is the thermal momentum.

In the following we shall use dimensionless quantities: $\tau = \Delta/T$, $p_0 = p_c/p_T$. The momenta and coordinates will also be measured in units of p_T and p_T^{-1}. The dimensionless interaction constant λ of the fluctuations is connected with the dimensional energy U by the relation $U = \lambda T$. In systems with short-range interaction, usually $\varphi_0 \approx 1$ T $\approx$ U and $p_T \approx a^{-1}$, i.e., $L \gg a$ and $\lambda \approx 1$.

In the present work we assume that the system is described by a vector order parameter

$$\varphi^\alpha(r) = N^{-1/2} \sum_p \varphi_p^\alpha e^{ipr},$$

where $\alpha = 1, 2, \ldots, $ n is the vector index and N is the number of degrees of freedom, and that it is completely isotropic both in coordinate space and in the vector space. Then, when the conditions indicated above are

fulfilled, the expansion of the functional $\mathscr{F}\{\varphi\}$ has the form $\mathscr{F}\{\varphi\} = \mathscr{F}_0 + T\mathscr{H}\{\varphi\}$, where

$$\mathscr{H}(\varphi) = \frac{1}{2}\sum_p [\tau + (p - p_0)^2]\varphi_p{}^{\alpha}\varphi_{-p}{}^{\alpha} + \frac{\lambda}{4!}\sum_{p_1+p_2+p_3+p_4=0}\varphi_{p_1}{}^{\alpha}\varphi_{p_2}{}^{\alpha}\varphi_{p_3}{}^{\beta}\varphi_{p_4}{}^{\beta}. \quad (2)$$

2. We consider certain properties of the isotropic phase using the example of a scalar field: $n = 1$, $\varphi_p{}^{\alpha} = \varphi_p$. We shall be interested in the region in which corrections to the correlation function $g(p) = \langle \varphi_p\varphi_{-p}\rangle$ and vertex part $\Gamma(p_1,p_2,p_3,p_4)$ first become important. It is clear that the effect of the large phase volume of the fluctuations should be greatest in those diagrams in which the arguments of the correlation functions $g(p)$ being integrated simultaneously coincide. This occurs only in the diagrams of Fig. 1a for the self-energy part and in ladder diagrams for the vertex part with zero total momentum. These diagrams have the relative order of magnitude $\alpha\lambda\tau^{-3/2}$, where $\alpha = p_0^2/4\pi$. In the following we shall always assume that $\alpha\lambda|\tau|^{-3/2} \approx 1$.

More complicated skeleton diagrams, e.g., Fig. 1b or diagrams with insertions in the ladder loops, have relative order $\alpha\lambda|\tau|^{-3/2}(p_0\lambda|\tau|^{-1})^m$ with $m \geq 1$. If $\alpha\lambda|\tau|^{-3/2} \approx 1$, then $\lambda p_0|\tau|^{-1} \approx |\tau|^{1/2}/p_0 \ll 1$. Consequently, these diagrams can be neglected. Inasmuch as the diagram taken into account (fig. 1a) does not depend on the momentum, we can write

$$g^{-1}(p) = r + (p - p_0)^2,$$

where r satisfies the equation

$$r = \tau + \alpha\lambda r^{-1/2}. \quad (3)$$

On further decrease of τ, according to (2), r decreases, so that $r \to 0$ only as $\tau \to -\infty$, i.e., as $T \to 0$. It follows from what has been said that Eq. (3) becomes inapplicable when $\lambda p_0 r^{-1} \approx 1$, when all the skeleton diagrams for the self-energy part become of the same order, i.e., when $-\tau \approx \alpha\lambda r^{-1/2} \approx \lambda^{1/2}p_0^{3/2}$. In this region it is possible in principle to reach the point of absolute instability $r = 0$. This question, however, will not be considered in the present paper.

3. We now consider the vertex part $\Gamma(p,-p,p',-p') = \Gamma(p,p')$. Since, in all the expressions containing $\Gamma(p,p')$ (e.g., in the expansion of the thermodynamic potential of the ordered phase), the momenta lying in the layer $|p - p_0| \lesssim \sqrt{r}$ are important, we shall be interested only in the angular dependence of this quantity for $|p| = |p'| = p_0$. For arbitrary angles between p and p', when $|p'\pm p| \gg \sqrt{r}$, the vertex $\Gamma(p,p')$ has one channel with zero total momentum. In this case it is necessary to sum the ladder of diagrams of the type in Fig. 2a, and we obtain

$$\Gamma(p, p') = \lambda/(1 + \lambda\Pi),$$
$$\Pi = \frac{1}{2}\int\frac{dp}{(2\pi)^3}g^2(p) = \frac{\alpha\lambda}{2r^{3/2}}.$$

If, however, $p' \to \pm p$, a second channel with zero total momentum $q = p' \mp p \to 0$ appears. In this case it is necessary to sum diagrams of the type in Fig. 2b also, and we obtain
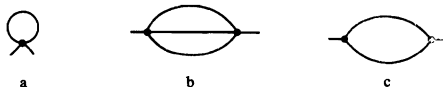
$$\Gamma(p, \pm p) = \lambda + 2\left(\frac{\lambda}{1+\lambda\Pi} - \lambda\right) = \lambda\frac{1-\lambda\Pi}{1+\lambda\Pi}.$$

It can be seen that for $\lambda\Pi > 1$ this vertex becomes negative, i.e., the possibility appears of inflections and, consequently, minima in the thermodynamic potential of the nonuniform phase. We arrive at the conclusion that in the region $\alpha\lambda|\tau|^{-3/2} \approx 1$ it is necessary to consider the possibility of the existence of stable nonuniform phases with $\langle\varphi\rangle \neq 0$, which can coexist with the uniform phase. The following sections of the article will be devoted to investigating the nonuniform states.

4. To conclude this section, we shall consider the effect of third-order terms of the type

$$\mathscr{H}_3 = \frac{\mu}{3!}\int\varphi^3(\mathbf{r})\,d\mathbf{r}, \quad (4)$$

which should be present in the expansion of $F\{\varphi\}$ for a scalar field, and also for the tensor field in liquid crystals. According to the Landau theory[3], these terms lead, for sufficiently small r, to a first-order transition to a nonuniform state such that the vectors p for which $\langle\varphi_p\rangle \neq 0$ can form closed triangles. The structures that arise will consequently be at least two-dimensional.

As will be shown below, the instability we are considering leads to the appearance of one-dimensional structures. It sets in earlier than the instability considered in[3], if the coefficient μ is sufficiently small.

The transition that was considered by Landau occurs for $\mu^2 \approx \lambda r$. Consequently, the Hamiltonian (4) can be neglected if $\mu^2 \ll \lambda|\tau|$. But if $\mu^2 \approx \lambda|\tau|$, the Hamiltonian (4) must be taken into account in the derivation of the equation of state of the nonuniform phase. However, it does not affect the calculation of the correlation function. For example, the diagram in Fig. 1c for $p \approx p_0$ is of the order of

$$\Sigma_c \approx \mu^2 p_0/r \approx \lambda p_0 \ll r,$$

i.e., has the same relative smallness $\lambda p_0/r \approx \lambda p_0/|\tau|$, associated with the narrow range of integration over the angles, as the other discarded terms. In the following, as a rule, we shall put $\mu = 0$.

3. PROPERTIES OF THE NONUNIFORM PHASES

1. We now derive the equation of state of a nonuniform phase. (Different variants of the derivation of the equation of state for a classical continuous field have been considered in[4,5].) In a nonuniform phase the averages $\langle\varphi_p{}^{\alpha}\rangle = \bar{\varphi}_p{}^{\alpha}$ are nonzero, and it can be assumed that $|\mathbf{p}| = p_0$. Generally speaking, higher harmonics of the function $\varphi^{\alpha}(\mathbf{r})$ with $|\mathbf{p}| \neq p_0$ are always present, but are small in relation to $|\tau|^{1/2}/p_0$, since a large nonuniformity energy is associated with them. In the approximation under consideration they need not be taken into account.

We introduce the field $\psi_i = \varphi_i - \bar{\varphi}_i$ (here and below, i, j, ... = $(p_i\alpha_i)$, $(p_j\alpha_j)$,....), so that $\langle\psi_i\rangle = 0$. The Hamiltonian of this field $\tilde{\mathscr{H}}\{\bar{\varphi}_i, \psi_i\} = \mathscr{H}\{\bar{\varphi}_i + \psi_i\}$ is of the form

FIG. 1

FIG. 2

$$\mathcal{H}\{\varphi_i, \psi_i\} = \mathcal{H}\{\psi_i\} + \sum_i \psi_i \left[\sum_j \tau_{ij}\varphi_j + \frac{1}{6}\sum_{jkl}\lambda_{ijkl}\varphi_j\varphi_k\varphi_l \right]$$

$$+ \frac{1}{2}\sum_{ij}\psi_i\psi_j \sum_{kl}\frac{1}{2}\lambda_{ijkl}\varphi_k\varphi_l + \frac{1}{6}\sum_{ijk}\psi_i\psi_j\psi_k\sum_l\lambda_{ijkl}\varphi_l$$

$$+ \frac{1}{2}\sum_{ij}\tau_{ij}\varphi_i\varphi_j + \frac{1}{24}\sum_{ijkl}\lambda_{ijkl}\varphi_i\varphi_j\varphi_k\varphi_l, \qquad (5)$$

where

$$\tau_{ij} = [\tau + (p_i - p_0)^2]\delta_{\mathbf{p}_i + \mathbf{p}_j, 0}\delta_{\alpha_i\alpha_j},$$

$$\lambda_{ijkl} = \frac{1}{3}\lambda\delta_{\mathbf{p}_i+\mathbf{p}_j+\mathbf{p}_k+\mathbf{p}_l, 0}(\delta_{\alpha_i\alpha_j}\delta_{\alpha_k\alpha_l}+\delta_{\alpha_i\alpha_k}\delta_{\alpha_j\alpha_l}+\delta_{\alpha_i\alpha_l}\delta_{\alpha_j\alpha_k}).$$

The external field h_i conjugate to $\bar\varphi_i$ is determined from the thermodynamic relation

$$h_i = -\frac{\delta\Phi\{\varphi\}}{\delta\varphi_i} = \left\langle \frac{\delta\mathcal{H}\{\varphi, \psi\}}{\delta\varphi_i} \Big|_{\psi=\mathrm{const}} \right\rangle,$$

where $i = (-p_i, \alpha_i)$. Varying (5), we obtain

$$h_i = \frac{1}{6}\sum_{jkl}\lambda_{ijkl}\bar\varphi_j\bar\varphi_k\bar\varphi_l + \sum_j\tau_{ij}\bar\varphi_j + \frac{1}{2}\sum_{jkl}\lambda_{ijkl}\langle\psi_k\psi_l\rangle\bar\varphi_j$$

$$+ \frac{1}{6}\sum_{jkl}\lambda_{ijkl}\langle\psi_j\psi_k\psi_l\rangle. \qquad (6)$$

We shall see later that $\bar\varphi^2 \approx |\tau|/\lambda$. Then the first three terms in Eq. (6) have the same order of magnitude. The last term is determined by diagrams of the type of Fig. 3a, in which a line with a blob denotes $\bar\varphi_i$. This term is small compared with the preceding terms, for the same reasons that the correction of Fig. 1b was small compared with Fig. 1a. Consequently, in the approximation under consideration the equation of state (6) is determined entirely by the pair correlation function $g_{ij} = \langle\psi_i\psi_j\rangle$, or, in other words, by $g_{\alpha\beta}(\mathbf{p},\mathbf{p}') = \langle\psi_{\mathbf{p}}^\alpha\psi_{\mathbf{p}'}^\beta\rangle$.

The self-energy part Σ_{ij} is determined by the diagrams of Figs. 1a, 3b and 3c, which are of the order of $|\tau|$. The diagram in Fig. 3c is important only in a narrow range of directions of the momenta of the external lines, when the total momentum $\mathbf{q} = \mathbf{p} + \mathbf{p}_i$ of the lines in the loops is close to zero ($q \lesssim |\tau|^{1/2}$). These directions correspond to external-line momenta close to the reciprocal-lattice vectors $\mathbf{p}_i$. In a broad range of directions this diagram has relative smallness of order $|\tau|^{1/2}/p_0$.

As can be seen from the diagrams of Figs. 3b and 3c, the matrix elements Σ_{ij} off-diagonal in the momenta are, generally speaking, of the same order as the diagonal elements. However, the off-diagonal matrix elements $g_{\alpha\beta}(\mathbf{p},\mathbf{p}')$ ($\mathbf{p} \neq -\mathbf{p}'$) have the same order τ^{-1} as the diagonal elements in only a narrow range of directions, and in a broad range we order p_0^{-2}. For example, in first order in Σ_{ij},

$$g^{(1)}(\mathbf{p},\mathbf{p}') = \sum_{ij} g^{(0)}(\mathbf{p})\Sigma(\mathbf{p},\mathbf{p}')g^{(0)}(\mathbf{p}')\delta(\mathbf{p}+\mathbf{p}'+\mathbf{p}_i+\mathbf{p}_j)$$

and we see that, since $\mathbf{p}_i + \mathbf{p}_j \neq 0$, the conditions $|\mathbf{p}| = p_0$ and $|\mathbf{p}'| = p_0$ can be fulfilled simultaneously only when $\mathbf{p} = -\mathbf{p}_i$ and $\mathbf{p}' = -\mathbf{p}_j$ (or vice versa). Consequently, in the calculation of the third term of Eq. (6) and also of the diagram Fig. 1a, which contain an integral of the correlation function over the momenta, we can neglect the matrix elements that are off-diagonal in the momenta, and also the correction of Fig. 3c to the diagonal elements. The case of a one-dimensional structure, when

the integral over the indicated narrow range of directions diverges logarithmically[3], may be an exception. In cases of physical interest, however, this divergence is cut off. The correlation function in the region indicated will be calculated in the Appendix.

Thus, denoting $\bar g_{\alpha\beta}(\mathbf{p}) = g_{\alpha\beta}(\mathbf{p}, -\mathbf{p})$, we obtain

$$h_{\mathbf{p}}^\alpha = \tau\varphi_{\mathbf{p}}^\alpha + \frac{\lambda}{6}\varphi_{\mathbf{p}}^\nu[\langle\bar g_{\mu\mu}\rangle\delta_{\alpha\nu}+2\langle\bar g_{\alpha\nu}\rangle] + \frac{\lambda}{6}\sum_{\mathbf{p}_1\mathbf{p}_2}\varphi_{\mathbf{p}_1}^\nu\varphi_{\mathbf{p}_2}^\nu\varphi_{\mathbf{p}-\mathbf{p}_1-\mathbf{p}_2}^\alpha, \qquad (7)$$

$$\bar g_{\alpha\beta} = [\tau + (p-p_0)^2]\delta_{\alpha\beta} + \frac{\lambda}{6}[\langle\bar g_{\mu\mu}\rangle\delta_{\alpha\beta}+2\langle\bar g_{\alpha\beta}\rangle]$$

$$+ \frac{\lambda}{6}[\varphi_{\mathbf{p}}^\nu\varphi_{-\mathbf{p}}^\nu\delta_{\alpha\beta}+2\varphi_{\mathbf{p}}^\alpha\varphi_{-\mathbf{p}}^\beta], \qquad (8)$$

where

$$\langle\bar g_{\alpha\beta}\rangle = \int\frac{d\mathbf{p}}{(2\pi)^3}\bar g_{\alpha\beta}(\mathbf{p}).$$

Taking (8) into account, we can rewrite Eq. (7) in the form

$$h_{\mathbf{p}}^\alpha = \bar g_{\alpha\nu}^{(-1)}(p_0)\varphi_{\mathbf{p}}^\nu + \frac{\lambda}{6}\sum_{\mathbf{p}_1\mathbf{p}_2}\varphi_{\mathbf{p}_1}^\nu\varphi_{\mathbf{p}_2}^\nu\varphi_{\mathbf{p}-\mathbf{p}_1-\mathbf{p}_2}^\alpha -$$

$$- \frac{\lambda}{6}\sum_{\mathbf{p}_1}[\varphi_{\mathbf{p}_1}^\nu\varphi_{-\mathbf{p}_1}^\nu\varphi_{\mathbf{p}}^\alpha+2\varphi_{\mathbf{p}_1}^\alpha\varphi_{\mathbf{p}_1}^\nu\varphi_{\mathbf{p}}^\nu]. \qquad (9)$$

Equations (8) and (9) determine φ_i as a function of τ and h_i.

2. We consider below different structures for a scalar field ($n = 1$) and one-dimensional structures for a vector field.

We shall begin with the one-dimensional structure for $n = 1$. In this case $\varphi(\mathbf{r}) = 2a\cos(\mathbf{p}_0\cdot\mathbf{r})$, where $|\mathbf{p}_0| = p_0$, i.e., $\varphi_{\mathbf{p}_0} = \varphi_{-\mathbf{p}_0} = a$ and, correspondingly $h_{\mathbf{p}_0} = h_{-\mathbf{p}_0} = h$ are nonzero. Equations (9) and (8) take the form

$$h = ra - \frac{1}{2}\lambda a^3, \qquad (10)$$

$$r = \tau + \frac{\alpha\lambda}{\sqrt{r}} + \lambda a^2 \qquad (11)$$

where $r = \bar g^{(-1)}(p_0)$ and $\alpha = p_0^2/4\pi$. For $h = 0$ we obtain

$$-\tau = r + \alpha\lambda/\sqrt{r}. \qquad (12)$$

Equation (12) has real solutions for r when $-\tau \geq \tau_c = 3(\alpha\lambda/4)^{2/3}$, and of the two roots it is necessary to choose the larger, corresponding to the minimum of the thermodynamic potential. When $-\tau = \tau_c$, $r = \tau_c$, i.e., a nonuniform state arises with a finite amplitude $a_c = (2\tau_c/\lambda)^{1/2}$ and is metastable in a certain region $\tau_c' > -\tau > \tau_c$. To verify the existence of a τ_c' such that for $-\tau > \tau_c'$ a nonuniform state is more favorable, it is necessary to calculate the difference $\Delta\Phi$ of the thermodynamic potentials of the two phases. For this we assume that the external field h is not equal to zero, but varies gradually, starting from $h = 0$ for $a = 0$ and finishing at the value $h = 0$ again at a value $a \neq 0$ corresponding to the minimum of Φ. Then,

$$\frac{\partial\Phi}{\partial a} = \sum_{\mathbf{p}}\frac{\delta\Phi}{\delta\varphi_{\mathbf{p}}}\frac{\partial\varphi_{\mathbf{p}}}{\partial a} = 2h, \qquad (13a)$$

whence

$$\Delta\Phi = \int_0^a 2h\, da = \int_{r_0}^r 2h\frac{da}{dr}dr, \qquad (13b)$$
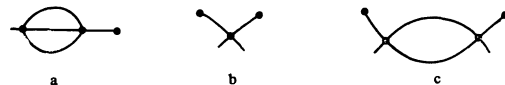

FIG. 3

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where r and r_0 are determined by Eqs. (12) and (3) respectively, and the derivative da/dr is determined from Eq. (11). From (13b) we obtain

$$\Delta\Phi = -\frac{r_0^2}{2\lambda} - \frac{\alpha}{2}r_0^{\gamma_h} - \frac{r^2}{2\lambda} + \frac{\alpha}{2}r^{\gamma_h}. \tag{14}$$

Analyzing formula (14) qualitatively together with Eqs. (12) and (3), it is not difficult to verify that there indeed exists a $\tau'_c \approx (\alpha\lambda)^{2/3}$, $\tau'_c > \tau_c$, such that for $-\tau > \tau'_c$ we shall have $\Delta\Phi < 0$.

3. We now consider non-one-dimensional structures, characterizable by a set of vectors $\{\pm p_i\}$ ($i = 1, 2, \ldots, m$), $|p_i| = p_0$. In this case,

$$\varphi(r) = 2\sum_i a_i \cos p_i r.$$

In the approximation under consideration, when, in particular, the contribution of the off-diagonal matrix elements of the correlation function is not taken into account, all the equations are independent of the relative orientation of the vectors p_i and depend only on their number m. From the general formulas (8) and (9) we obtain

$$h_i = ra_i + \frac{1}{6}\lambda a_i^3 - \frac{2}{3}\lambda a_i\sum_j a_j^2,$$

$$r = \tau + \frac{\alpha\lambda}{r^{\gamma_h}} + \lambda\sum_j a_j^2,$$

whence, for $h_i = 0$, we find that all the amplitudes a_i are equal to

$$a_i^2 = \frac{6}{4m-1}\frac{r}{\lambda} = a^2.$$

The quantity r is now determined from the equation

$$\frac{\alpha\lambda}{r^{\gamma_h}} + C_m r + \tau = 0, \qquad C_m = \frac{2m+1}{4m-1}. \tag{15}$$

A metastable state of such a kind arises when $-\tau \geq \tau_m = 3C_m^{1/3}(\alpha\lambda/4)^{2/3}$. As m increases the coefficient C_m decreases from 1 at $m = 1$ to $1/2$ as $m \to \infty$. Consequently, the metastable states with large m are the first to appear. To determine the region of stability we shall calculate the change in the thermodynamic potential. Analogously to (13a),

$$\frac{\partial\Phi_m}{\partial a} = ma\left(2r + \frac{1-4m}{3}\lambda a^2\right).$$

Integrating, we obtain

$$\Delta\Phi_m = \frac{1}{2\lambda}\left(\frac{4m-7}{4m-1}r^2 - r_0^2\right) + \alpha(r^{\gamma_h} - r_0^{\gamma_h}). \tag{16}$$

It can be seen from Eq. (15) that for given τ the quantity r increases with increasing m. At the same time the coefficient of r^2 in formula (16) also increases. Consequently, $\Delta\Phi_m$ increases with increasing m, i.e., in the coexistence region the one-dimensional phase always has the lowest thermodynamic potential as compared with the other nonuniform phases. Moreover, using the obvious inequalities

$$r(\tau) \geq r(-\tau_m) = (\alpha\lambda/4C_m)^{\gamma_h},$$
$$r_0 = (\alpha\lambda/2(r_0-\tau))^2 < (\alpha\lambda/2\tau)^2,$$

we find that $\Delta\Phi_m$ can become negative only for $m = 1$ and 2. This is connected with the fact that the above-mentioned coefficient of r^2 changes sign from negative to positive as we pass from $m = 1$ to $m \geq 2$, and this greatly reduces the possibility of energy gain.

Naturally, this picture can change substantially in the presence of the Hamiltonian (4). The simplest struc-

ture on which this Hamiltonian has an effect when the condition $\mu^2 \approx \lambda|\tau|$ is fulfilled is a planar lattice constructed of equilateral triangles. In this case, $m = 3$ and $p_1 + p_2 + p_3 = 0$. The equation of state acquires the additional term

$$\Delta h_p = -\frac{\mu}{2}\sum_i [\langle\psi_{\pm p_i}\psi_{p\mp p_i}\rangle + \varphi_{\pm p_i}\varphi_{p\mp p_i}].$$

The first term in the brackets contains an integral of an off-diagonal correlation function over the angles and is, consequently, small. We obtain that $\Delta h_{\pm pi} = 1/2\mu a^2$ and $\Delta\Phi_3 = \mu a^3$.

The difference in the thermodynamic potentials of the triangular and one-dimensional structures has the form

$$\Delta\Phi_3 - \Delta\Phi_1 = \frac{1}{2\lambda}\left(\frac{5}{11}r_3^3 + r_1^2\right) + \alpha(r_3^{\gamma_h} - r_1^{\gamma_h}) + \mu a_3^3, \tag{17}$$

where r_1 is the solution of Eq. (12) and r_3 and a_3 are determined from the system of equations

$$r_3 - \frac{11}{6}\lambda a_3^2 + \frac{\mu}{2}a_3 = 0,$$

$$r_3 = \tau + \frac{\alpha\lambda}{r_3^{\gamma_h}} + 3\lambda a_3^2. \tag{18}$$

In the region $\alpha\lambda \approx |\tau|^{3/2}$, $\mu^2 \approx \lambda|\tau|$, all the terms in the right-hand side of (17) are of the same order. In this region the difference $\Phi_3 - \Phi_1$ changes sign. In fact, for $\mu^2 \ll \lambda|\tau|$, as pointed out above, $\Phi_3 - \Phi_1 < 0$. Now let $\mu^2 \gg \lambda|\tau|$. Then from (18) we obtain $a_3 \approx -(3/7)\mu\lambda^{-1}$, $r_3 \approx 3\lambda a_3^3$, and the principal terms of Eq. (17) have the form

$$\Phi_3 - \Phi_1 \approx \frac{5}{22}r_3^2/\lambda + \mu a_3^2 \approx -0.008\mu^4/\lambda^3,$$

i.e., $\Phi_3 - \Phi_1 < 0$ and, consequently, the one-dimensional phase becomes unfavorable.

4. We now consider one-dimensional structures describable by a vector field $\bar{\varphi}^\alpha(r)$. In this case,

$$\varphi^\alpha(r) = a_\alpha e^{ip_0 r} + a_\alpha^* e^{-ip_0 r}$$

and from Eqs. (8) and (9) we obtain

$$h_\alpha = \tilde{g}_{\alpha\nu}^{(-1)}a_\nu - \frac{1}{3}\lambda a_\alpha a_\nu a_\nu^* - \frac{1}{6}\lambda a_\alpha^* a_\nu a_\nu, \tag{19}$$

$$\tilde{g}_{\alpha\beta}^{(-1)}(p) = \left[\tau + (p-p_0)^2 + \frac{\lambda}{3}\langle g_{\nu\nu}\rangle + \frac{\lambda}{3}a_\nu a_\nu^*\right]\delta_{\alpha\beta}$$
$$+ \frac{2}{3}\lambda\langle g_{\alpha\beta}\rangle + \frac{\lambda}{3}(a_\alpha a_\beta^* + a_\beta a_\alpha^*). \tag{20}$$

It is clear that $\tilde{g}_{\alpha\beta}$ and $\tilde{g}_{\alpha\beta}^{(-1)}$ are real symmetric combinations of the tensors $\delta_{\alpha\beta}$, $a_\alpha a_\beta^*$ and $a_\beta a_\alpha^*$, and $a_\alpha a_\beta$ and $a_\alpha^* a_\beta^*$. Then from Eq. (19) for $h_\alpha = 0$ it follows that

$$A(\tau, \rho)a_\alpha + B(\tau, \rho)a_\nu a_\nu a_\alpha^* = 0,$$

where $\rho = a_\nu a_\nu^*$, and $A(\tau, \rho)$ and $B(\tau, \rho)$ are certain real functions. Combining this relation with its complex-conjugate relation we obtain that either $a_\nu a_\nu = 0$ or $a_\alpha^* = a_\alpha e^{i\delta}$. The first case corresponds to a spiral structure with $[\varphi^\nu(r)]^2 = 2\rho = \text{const}$, and the second to a sinusoidal structure with $[\varphi^\nu(r)]^2 = 4\rho\cos^2(p_0 \cdot r - \delta/2)$. In the case of the sinusoidal structure we choose the coordinate origin such that $\delta = 0$, i.e., $a_\alpha^* = a_\alpha$.

We introduce the tensor $\rho_{\alpha\beta}$, equal to $\rho_{\alpha\beta} = (a_\alpha a_\beta^* + a_\beta a_\alpha^*)\rho^{-1}$ for the spiral structure and $\rho_{\alpha\beta} = a_\alpha a_\beta\rho^{-1}$ for the sinusoidal structure. Separating the tensors $\tilde{g}_{\alpha\beta}$ and $\tilde{g}_{\alpha\beta}^{(-1)}$ into longitudinal and transverse parts:

$$\tilde{g}_{\alpha\beta}^{(-1)}(p) = a(p)[\delta_{\alpha\beta} - \rho_{\alpha\beta}] + a'(p)\rho_{\alpha\beta},$$
$$g_{\alpha\beta}(p) = a^{-1}(p)[\delta_{\alpha\beta} - \rho_{\alpha\beta}] + (a'(p))^{-1}\rho_{\alpha\beta}$$

and substituting into Eqs. (19) and (20), we obtain $a(p) = r + (p - p_0)^2$ and $a'(p) = r' + (p - p_0)^2$. The quantities r, r' and ρ are determined in each case from the corresponding system of equations. We write down these equations in a unified form by introducing the index $k = a_\nu a_\nu / a_\mu a_\mu^*$, which takes the values $k = 0$ for the spiral structure and $k = 1$ for the sinusoidal structure:

$$r_k = \tau + \frac{1}{3}\lambda\rho_k + \frac{n+k}{3}\frac{\alpha\lambda}{\sqrt{r_k}} + \frac{2-k}{3}\frac{\alpha\lambda}{\sqrt{r_k}},$$

$$r_k' = \tau + \frac{2+k}{3}\lambda\rho_k + \frac{n+k-2}{3}\frac{\alpha\lambda}{\sqrt{r_k}} + \frac{2+k}{3}\frac{\alpha\lambda}{\sqrt{r_k}},$$

$$h^\alpha = \left(r_k' - \frac{2+k}{6}\lambda\rho_k\right)a_\alpha. \qquad (21)$$

Investigating this system for $h_\alpha = 0$, one can convince oneself that, as before, there exists a $\tau_k \approx (\alpha\lambda)^{2/3}$ such that for $-\tau > \tau_k$ there arises a metastable state with r_k, $r_k' \approx \tau_k$ and $\rho_k = 6r'/(2 + k)\lambda$. We shall determine the change $\Delta\Phi$ in the thermodynamic potential by varying Φ with respect to ρ for given directions of the vectors a_α and a_α^*:

$$\frac{\partial\Phi}{\partial\sqrt{\rho}} = \frac{1}{\sqrt{\rho}}\sum_p h_p^\alpha \bar{\varphi}_{-p}^\alpha = 2\sqrt{\rho}\left(r_k' - \frac{2+k}{6}\lambda\rho_k\right),$$

whence

$$\Delta\Phi_k = \int_0^{\rho_k(\tau)}\left[r_k'(\rho, \tau) - \frac{2+k}{6}\lambda\rho_k\right]d\rho_k, \qquad (22)$$

where $\rho_k(\tau)$ is the solution of the system (21) for $h_\alpha = 0$, and $r_k'(\rho, \tau)$ is the solution of the first two equations of this system. The change of sign of $\Delta\Phi_k$ at sufficiently large $|\tau|$ can be traced directly from Eqs. (21) and (22). Let $-\tau \gg \tau_k$. Then in a broad range of $\rho(\rho\lambda \lesssim -\tau)$, the quantities r and r' are small ($r, r' \approx \tau_k^3/\tau^2 \ll \tau_k$) and, consequently, the integrand in (22) is negative. We find that for $-\tau \gg \tau_k$ we shall have

$$\Delta\Phi \approx \frac{3}{2+k}\frac{\tau^2}{\lambda}.$$

It can be seen that at a sufficiently low temperature the spiral structure is the most favorable.

In conclusion the author expresses his gratitude to I. E. Dzyaloshinskiĭ for suggesting the topic and for useful discussions.

APPENDIX

We shall calculate the correlation function $g(p, p')$ when the momenta p and p' are close to $\pm p_0$. In this case, as pointed out in Sec. 3, it is necessary to take into account both the off-diagonal matrix elements and diagrams of the type Fig. 3c. For definiteness, let $p \approx p_0$. Then from Figs. 3b and 3c we obtain

$$g^{(-1)}(p, -p+2p_0) = -\Sigma(p, -p+2p_0) = -\frac{1}{2}\Gamma(p-p_0)a^2 = a_2(p),$$

where

$$\Gamma(k) = \lambda[1 + \lambda\Pi(k)]^{-1}$$

and

$$g^{(-1)}(p, -p) = (|p| - p_0)^2 + r(p) = a_1(p),$$

where

$$r(p) = r[1 + \lambda\Pi(p - p_0)]^{-1}.$$

The correlation function satisfies the equation

$$\sum_{p_1} g^{(-1)}(p, p_1)g(p_1, p') = \delta_{p, p'}. \qquad (A.1)$$

It is not difficult to see that, to within terms of order τ/p_0^2, the function $g(p, p')$ for $p' \approx \pm p_0$ can be sought in the form

$$g(p, p') = g_1(p)\delta_{p', -p} + g_2(p)\delta_{p', -p+2p_0}.$$

Then from (A.1) we obtain

$$g_1(p) = \frac{a_1(-p+2p_0)}{a_1(-p)a_1(-p+2p_0) - a_2(-p)a_2(-p+2p_0)},$$

$$g_2(p) = \frac{-a_2(-p)}{a_1(-p)a_1(-p+2p_0) - a_2(-p)a_2(-p+2p_0)}.$$

If we denote $p = p_0 + k$, $|k| \ll p_0$, then for $|k| \ll \sqrt{|\tau|}$,

$$g_{1,2}(p_0+k) \approx \pm[(|p_0+k| - p_0)^2 + (|p_0-k| - p_0)^2]^{-1}$$
$$\approx \pm[(kp_0/p_0)^2 + (k^2/2p_0)^2]^{-1}. \qquad (A.2)$$

As can be seen from (A.2), the integral over the momenta of $g_{1,2}$ that appears in Eqs. (8) and (9) diverges logarithmically at small $|k|$ in accordance with the well-known theorem[3] about the impossibility of the existence of a one-dimensional periodic structure in an infinite isotropic system. But if the system is restricted to dimensions of the order of L, or the function $g_0^{-1}(p)$ has anisotropy of order $\epsilon \ll |\tau|$, this integral is found to be of the order of

$$\langle g_{1,2}\rangle \approx p_0 \ln(r/\max\{\epsilon, L^{-2}\}) = p_0 A.$$

Consequently, the replacement of the operator $g(p, p')$ by $\tilde{g}(p)\delta_{p, -p'}$ that was carried out above is valid under the condition

$$p_0 A \ll p_0^2/\sqrt{|\tau|}, \quad \text{i.e.} \quad A\sqrt{|\tau|} \ll p_0^2.$$

As has been pointed out by Lubensky[6], this condition is always fulfilled for the helical structures in cholesteric liquid crystals.

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