Determination of the fixed point and critical indices

S. L. Ginzburg

Leningrad Nuclear Physics Institute, USSR Academy of Sciences
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The critical indices, invariant charge, susceptibility, and other quantities in theory of phase transitions are calculated using the renormalization-group method. In particular, the transition from perturbation theory to the scaling region is traced.

1. INTRODUCTION

There has recently arisen a new approach to the theory of phase transitions, associated with the expansion of the different quantities in the parameter $\epsilon = 4 - d$, where d is the dimensionality of space [1,2]. It has been found that, although the series in ϵ are asymptotic, for certain quantities (e.g., for the susceptibility index γ) the first terms of the series fall off rather rapidly. In spite of this however, it would be desirable to construct a theory directly for three-dimensional space. The first attempt to construct such a theory was made in the papers of Migdal and Polyakov [3,4]. Later di Castro and Jona-Lasinio applied the renormalization-group method [5-7] to this problem, in the formulation described in detail in [8] (a review of some of these papers, and also a number of others, is given by di Castro [9]). In these papers, however, the vertices and indices were not calculated. After the ϵ-expansion was proposed, its principal results were obtained by a renormalization-group method similar to the Callan-Simanzyk method [10,11] in a number of articles (cf., e.g., [12,13]), and, in particular, the Gell-Mann—Low function (referred to as the GMLF below) was calculated in lowest order in ϵ.

An attempt of a somewhat different kind is made in this paper. A differential equation for the so-called invariant charge is derived directly for the three-dimensional case, using the Ward identity for the vertex function; the equation is the analog of the renormalization-group equation (apropos of the renormalization-group equation, see also the review by Wilson [14]). The GMLF appearing in this equation is calculated in the form of a Taylor series to fourth order in the invariant charge (a certain difference between our GMLF and the GMLF of [5-14] is discussed in detail below). It then turns out that there exists a fixed point (FP) corresponding to a scale-invariant (scaling) solution; moreover, the position of the FP obtained from the GMLF calculated to second order is very little different from its position obtained from the GMLF calculated to fourth order. This surprising phenomenon arose because of the fact that the third- and fourth-order terms cancel each other almost exactly near the FP; this evidently indicates that the calculated position of the FP is fairly close to the true position, and that the GMLF itself, approximated in this way, differs little from the true GMLF in the interval between its first two zeros. Since for the description of most of the properties of the system it is sufficient to know the GMLF in this interval, this means that these properties can be described rather well using our approximation. We then calculate explicit expressions for the vertices and the susceptibility, valid both in the perturbation-theory region and in the scaling region; in addition, the Green function and critical indices are calculated.

2. RENORMALIZATION OF THE GREEN FUNCTION AND VERTEX

We shall consider the theory of an n-component field φ_α, the Hamiltonian of which is equal to [2]:

$$H/T = \int d^3r\{{}^1/_2\varkappa_0^2\varphi^2(\mathbf{r}) + {}^1/_2[\nabla\varphi(\mathbf{r})]^2 + {}^1/_8\Lambda_1[\varphi^2(\mathbf{r})]^2\},$$

$$\varphi^2 = \sum_\alpha \varphi_\alpha^2(\mathbf{r}), \quad (\nabla\varphi)^2 = \sum_\alpha (\nabla\varphi_\alpha)^2. \tag{1}$$

We shall determine the Green function

$$G_{\alpha\beta}(r) = \delta_{\alpha\beta}G(r) = \langle\varphi_\alpha(\mathbf{r})\varphi_\beta(0)\rangle, \tag{2}$$

where the averaging is performed with weight $\exp(-H/T)$. We note that in the construction of the perturbation theory a factor ½ must be associated with each closed loop. The zeroth Green function in the momentum representation equals

$$G_0(k) = 1/(k^2+\varkappa_0^2). \tag{3}$$

As is well-known, when the perturbation theory is constructed a series in the parameter Λ_1/κ_0 is obtained, and for $\Lambda_1 \gg \kappa_0$ such an expansion is completely inapplicable.

The way out of this situation is suggested by field theory (from a mathematical point of view, the theory of phase transitions is equivalent to field theory). It is necessary to carry out the renormalization of the interaction constant and κ_0 analogously to the renormalization of the charge and mass in field theory [15]. After this renormalization the entire dependence on the large parameter Λ_1/κ_0 goes over into the renormalized coupling constant u_R, which we shall call the invariant charge, and into the renormalized inverse correlation length κ. By means of Ward identities, for u_R and κ we obtain differential equations whose solution gives the explicit dependence of u_R and κ on Λ_1. The exact Green function equals

$$G(k) = \frac{1}{k^2+\varkappa_0^2-\Sigma(k)}, \tag{4}$$

where $\Sigma(k)$ is the self-energy part. The structure of the singularities of the Green function in the complex k^2-plane is well-known [3,4]. All the singularities lie at real negative values of k^2, the nearest singularity to the physical region being the pole at the point $k^2 = -\kappa^2$, the position of which is determined by the equation

$$k^2+\varkappa_0^2-\Sigma(k) = 0. \tag{5}$$

Next come branch points at $k^2 = -(2n+1)^2\kappa^2$, where n are integers. As in field theory, it is convenient to construct the theory by using the true, and not the bare pole as the zeroth approximation, i.e., by starting from the Green function

$$G=z/(k^2+\varkappa^2), \tag{6}$$

where z is the residue at the pole, equal to

$$z=\left[1-\left(\frac{\partial\Sigma_1}{\partial k^2}\right)_{k^2=-\varkappa^2}\right]^{-1}, \quad \Sigma_1(k)=\Sigma(k)-\Sigma(k^2=-\varkappa^2). \tag{7}$$

We represent the exact Green function in the form

$$G=zG_R=\frac{z}{k^2+\varkappa^2-\Sigma_R(k)}. \tag{8}$$

Here Σ_R is the regularized self-energy part, equal to

$$\Sigma_R=z\left\{\Sigma_1(k)-(k^2+\varkappa^2)\left(\frac{\partial\Sigma_1}{\partial k^2}\right)_{k^2=-\varkappa^2}\right\}. \tag{9}$$

That (8) is correct can be verified by direct substitution of (9) into (8).

We shall now renormalize the vertex part $\Gamma_{\alpha\beta\mu\nu}$. The purpose of the renormalization of the vertex part is to expand Γ not in Λ_1 but in the renormalized interaction constant u_R. We put

$$\Gamma_{\alpha\beta\mu\nu}(0,0,0,0)=\gamma_R I_{\alpha\beta\mu\nu}, \quad I_{\alpha\beta\mu\nu}=\delta_{\alpha\beta}\delta_{\mu\nu}+\delta_{\alpha\mu}\delta_{\beta\nu}+\delta_{\alpha\nu}\delta_{\beta\mu}, \tag{10}$$

where u_R is connected with γ_R by the following relation:

$$u_R=z^2\gamma_R. \tag{11}$$

For the following it is convenient to introduce also the quantity z_1:

$$\gamma_R=\Lambda_1/z_1. \tag{12}$$

This notation corresponds to the usual notation in field theory[15]. We put

$$\begin{gathered}\Gamma_{\alpha\beta\mu\nu}=\Lambda_1 I_{\alpha\beta\mu\nu}+L_{\alpha\beta\mu\nu},\\ \Gamma_{R\alpha\beta\mu\nu}=z_1\Gamma_{\alpha\beta\mu\nu}, \quad L_{1\alpha\beta\mu\nu}=z_1 L_{\alpha\beta\mu\nu},\\ L_{R\alpha\beta\mu\nu}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4)=L_{1\alpha\beta\mu\nu}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4)-L_{1\alpha\beta\mu\nu}(0,0,0,0).\end{gathered} \tag{13}$$

The function $\Gamma_{R\alpha\beta\mu\nu}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4)$ is called the renormalized vertex part. From (13) we obtain with the aid of simple algebra:

$$\Gamma_{R\alpha\beta\mu\nu}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4)=\Lambda_1 I_{\alpha\beta\mu\nu}+L_{R\alpha\beta\mu\nu}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4). \tag{14}$$

From (12)—(14) we have the required expression for the vertex part:

$$\Gamma_{\alpha\beta\mu\nu}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4)=\gamma_R I_{\alpha\beta\mu\nu}+z_1^{-1}L_{R\alpha\beta\mu\nu}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4). \tag{15}$$

We now show how, with each graph for $L_{\alpha\beta\mu\nu}$, we can associate a corresponding expression for $L_{R\alpha\beta\mu\nu}$. We shall consider the simplest graph for $L_{\alpha\beta\mu\nu}$, illustrated in Fig. 1a. The rectangle in Fig. 1 corresponds to the zeroth vertex $\Gamma_{0\alpha\beta\mu\nu}$. Just as for the Green function, for the vertex it is natural to choose not Λ_1 but γ_R as the zeroth approximation; we then obtain

$$L_{\alpha\beta\mu\nu}=-z^2\gamma_R^2 I_{\alpha\beta\sigma\rho}I_{\sigma\rho\mu\nu}\int\frac{d^3p}{(2\pi)^3}G_R(p)G_R(\mathbf{p}+\mathbf{k}). \tag{16}$$

From (13), (15) and (16) we have

$$\Gamma_{\alpha\beta\mu\nu}=\gamma_R\left\{I_{\alpha\beta\mu\nu}-u_R I_{\alpha\beta\sigma\rho}I_{\sigma\rho\mu\nu}\int\frac{d^3p}{(2\pi)^3}G_R(p)\left[G_R(\mathbf{p}+\mathbf{k})-G_R(p)\right]\right\}, \tag{17}$$

where u_R is defined in (11). We introduce the so-called invariant vertex

$$U_{\alpha\beta\mu\nu}=z^2\Gamma_{\alpha\beta\mu\nu}. \tag{18}$$

It can be expanded in the invariant charge u_R. For example, from (17) we have

$$U_{\alpha\beta\mu\nu}=u_R I_{\alpha\beta\mu\nu}-u_R^2 I_{\alpha\beta\sigma\rho}I_{\sigma\rho\mu\nu}\int\frac{d^3p}{(2\pi)^3}G_R(p)\left[G_R(\mathbf{p}+\mathbf{k})-G_R(p)\right]. \tag{19}$$

Formulas (17) and (19) give renormalized expressions for the graph 1a. The renormalization of the more complicated graphs proceeds in stages[15]. For example, in the graph 1b we must first renormalize the fragment corresponding to the right-hand part of the graph, and then the remaining part. As a result, we obtain the following expression:

$$\begin{gathered}\Delta U_{\alpha\beta\mu\nu}=u_R^3 I_{\alpha\beta\sigma\rho}I_{\sigma\mu\xi\lambda}I_{\rho\nu\xi\lambda}\int\frac{d^3p_1\,d^3p_2}{(2\pi)^6}\{G_R(p_1)G_R(p_2)G_R(\mathbf{p}_1+\mathbf{k})\\ \times[G_R(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p})-G_R(p_2)]-G_R^2(p_1)G_R(p_2)[G_R(\mathbf{p}_1+\mathbf{p}_2)-G_R(p_2)]\}\end{gathered} \tag{20}$$

In exactly the same way we can also obtain expressions for Σ_R. For example, renormalization of the graph 1c gives the following expression for Σ_R:

$$\begin{gathered}\Sigma_R(k)\delta_{\alpha\beta}=u_R^2 I_{\alpha\mu\nu\lambda}I_{\beta\mu\nu\lambda}\left\{\varphi(k)-\varphi(i\varkappa)-(k^2+\varkappa^2)\left(\frac{\partial\varphi}{\partial k^2}\right)_{k^2=-\varkappa^2}\right\},\\ \varphi(k)=\int\frac{d^3p_1\,d^3p_2}{(2\pi)^6}G_R(p_1)G_R(p_2)G_R(\mathbf{p}_1+\mathbf{p}_2+\mathbf{k}).\end{gathered} \tag{21}$$

The expressions (19)—(21) correspond to the well-known subtraction procedure of field theory.

3. EQUATION FOR THE GREEN FUNCTION

We shall write the Ward identity for the Green function. Let κ_{0c} be the value of κ_0 at which the phase transition occurs. We denote

$$\tau=\varkappa_0^2-\varkappa_{0c}^2. \tag{22}$$

Then the Ward identity for the Green function has the form[3,4]:

$$\partial G^{-1}/\partial\tau=T(p), \tag{23}$$

where T(p) is a sum of graphs; some of these are illustrated (before renormalization) in Fig. 2. The point in these graphs corresponds to $\Lambda_1 I_{\alpha\beta\mu\nu}$, and the wavy line represents differentiation of the given Green function. The function T(p), just like the vertex parts Γ and U, can be renormalized, i.e., represented in the form of a series in u_R. We denote

$$T(0)=t_R. \tag{24}$$

Then, in exactly the same way as in the preceding section, we expand T(p) in t_R and u_R. For example, we shall renormalize the graphs in Fig. 2. The renormalization of the graph 2b gives zero when the subtractions are made, and the sum of the graphs 2a and 2c is equal to

$$\begin{gathered}T(p)\delta_{\alpha\beta}=t_R\delta_{\alpha\beta}+\frac{1}{2}t_R u_R^2\delta_{\mu\nu}I_{\alpha\sigma\mu\rho}I_{\sigma\rho\beta\nu}\int\frac{d^3p_1\,d^3p_2}{(2\pi)^6}G_R^2(p_1)G_R(p_2)\\ \times[G_R(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p})-G_R(\mathbf{p}_1+\mathbf{p}_2)].\end{gathered} \tag{25}$$

It is easy to see from (25), and also from the more complicated graphs, that

$$T(p)=t_R\alpha(p^2/\varkappa^2). \tag{26}$$

The fact that α depends on the ratio p^2/κ^2 follows from the fact that, as we shall see below, $u_R\sim\kappa$. Solving Eq. (23) near $p^2=-\kappa^2$, we obtain an equation for κ^2:

$$\frac{\partial\varkappa^2}{\partial\tau}=zt_R(\varkappa)\alpha(-1). \tag{27}$$

We note that analogous equations were considered in[3,4].

From (23) and (27) we obtain an equation for G:

$$z\frac{\partial G^{-1}}{\partial\varkappa^2}=\frac{\alpha(p^2/\varkappa^2)}{\alpha(-1)}. \tag{28}$$

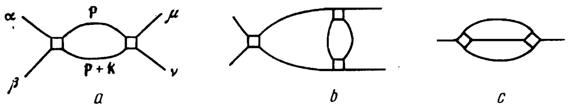

FIG. 1

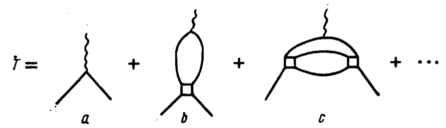

FIG. 2

Taking account of (8) and the fact that $\kappa^2 G_R$ depends only on p^2/κ^2, we obtain after simple transformations:

$$\frac{\partial \ln z}{\partial \ln \varkappa^2}=\frac{\eta}{2}=1-\frac{\partial \ln g}{\partial \ln \varkappa^2}-\frac{\alpha(p^2/\varkappa^2)}{\alpha(-1)}g,$$

$$g=\varkappa^2 G_R=\frac{\varkappa^2}{z}G. \tag{29}$$

In formula (29) $\eta/2$ is simply the parameter associated with the separation of the variables. Since on the left we have an expression depending on κ, and on the right an expression depending on p^2/κ^2, (29) represents, in essence, two different equations. From the left-hand part of the equation we obtain

$$z\sim\varkappa^{\eta}, \tag{30}$$

i.e., η is the Fisher parameter. From the right-hand part of the equation we have

$$1+\frac{\partial \ln g(x)}{\partial \ln x}-g\frac{\alpha(x)}{\alpha(-1)}=\frac{\eta}{2}, \quad x=p^2/\varkappa^2. \tag{31}$$

First we shall consider Eq. (31) for $x \gg 1$. It is easy to show that $g(x)\alpha(x) \to 0$ when $x \to \infty$. Then,

$$g(x)=x^{-1+\eta/2}, \quad G_R=\frac{\varkappa^{-\eta}}{p^{2-\eta}}, \quad G\sim\frac{1}{p^{2-\eta}}. \tag{32}$$

This is the only possible solution, since for $p \gg \kappa$, $G(p, \kappa)$ cannot depend on κ.

We now consider Eq. (31) for small x. Inasmuch as (31) is a complicated nonlinear equation, we shall simplify it by making use of the following fact. It is well-known that the Fisher parameter $\eta \ll 1$ (this will also be clear from the explicit expressions for η in Sec. 7 of this paper). Below we shall show that, for $x \sim 1$,

$$|\alpha(x)-1|\sim\eta, \quad \left|g(x)-\frac{1}{x+1}\right|\sim\eta. \tag{33}$$

Therefore, we can linearize (31). First of all, we note that from the condition that $g(x)$ is finite as $x \to 0$ we obtain from (31):

$$g(0)=(1-\eta/2)\alpha(-1)\approx\alpha(-1)-\eta/2. \tag{34}$$

We next put

$$g(x)=\frac{g(0)}{x+1}\frac{1}{1-\varphi(x)}, \quad \varphi(0)=0. \tag{35}$$

Substituting (35) into (31) and linearizing the equation, we obtain

$$x(x+1)\varphi'(x)-\varphi(x)=\alpha(x)-1+x\eta/2. \tag{36}$$

Equation (36) has two singular points, $x = 0$ and $x = -1$. We shall be interested in the solution that is regular at these points. Such a solution exists only under the condition

$$\eta=-2\alpha'(0) \tag{37}$$

and has the form

$$\varphi(x)=\frac{x}{x+1}\int_{-1}^{x}dx'\left\{\frac{\alpha(x')-1}{x'^2}+\frac{\eta}{2x'}\right\}. \tag{38}$$

We recall that $\alpha(0) = 1$, and, therefore, when (37) is fulfilled, $\varphi(x)$ is regular at zero. Finally, we obtain from (34), (35) and (38):

$$g(x)=[\alpha(-1)-\eta/2]\left[x+1-x\int_{-1}^{x}dx'\left\{\frac{\alpha(x')-1}{x'^2}+\frac{\eta}{2x'}\right\}\right]^{-1}. \tag{39}$$

Obviously, $g(-1) = (x + 1)^{-1}$, as it should, inasmuch as the entire residue at the pole is taken into account in the expression for z.

The formula (37) determines the Fisher parameter. However, to calculate $\alpha(x)$ we need to know u_R. (33) follows from (37) and (38). We note that (33) implies that, to within η for $p \sim \kappa$,

$$T(p)=t_R, \quad G_R(p)=1/(p^2+\varkappa^2). \tag{40}$$

4. THE EQUATION FOR u_R

We proceed now to the derivation of the equation for u_R. For this we again make use of a Ward identity. We shall calculate the quantity $\partial\gamma_R/\partial\tau$. This derivative is expressed by a sum of graphs, of which the simplest are illustrated (before renormalization) in Fig. 3; all momenta emerging from the graphs are equal to zero. It is clear that the whole set of graphs for $\partial\gamma_R/\partial\tau$ can be expressed in terms of the exact vertices Γ and T, i.e., in the form of skeleton diagrams containing the exact Green functions and exact vertices. The simplest skeleton diagrams are illustrated (before renormalization) in Figs. 4 and 5, where the shaded triangle represents the vertex T and the shaded circles represent the vertex Γ.

We now renormalize the vertices appearing in the skeleton graphs, i.e., we expand them in u_R. We first consider the lowest order in u_R. For this we replace Γ by $\gamma_R I_{\alpha\beta\mu\nu}$ and T by $t_R\delta_{\alpha\beta}$ and combine the expression thus obtained with the corresponding expressions from the two other channels obtained by the interchange of indices $\beta \rightleftarrows \mu$ and $\beta \rightleftarrows \nu$. As a result, we obtain

$$\frac{\partial\gamma_R}{\partial\tau}=t_R\gamma_R^2(n+8)\int\frac{d^3p}{(2\pi)^3}G^3(p). \tag{41}$$

Taking (8), (11) and (27) into account, we have

$$-\frac{2}{z}\frac{\partial z}{\partial r}\xi(-1)u_R+\xi(-1)\frac{\partial u_R}{\partial r}=u_R^2(n+8)\int\frac{d^3p}{(2\pi)^3}G_R^3(p). \tag{42}$$

In (42) we have introduced the convenient new notation

$$r=\varkappa^2. \tag{43}$$

Everywhere in the following we shall neglect corrections of order η, and therefore the results obtained below will contain errors of order η. Taking (40) into account, for G_R we shall everywhere confine ourselves to its pole term, and replace $T(p)$ by t_R. Inasmuch as $\partial \ln z/\partial \ln r = \eta/2$ and $|\alpha(-1) - 1| \sim \eta$, we obtain in our approximation the following equation for u_R:

$$\frac{\partial u_R}{\partial r}=\frac{u_R^2(n+8)}{32\pi r^{3/2}}. \tag{44}$$

The principal contribution to $\partial u_R/\partial r$ arises from taking skeleton graphs of higher order in Γ into account, and from the renormalization of the vertices Γ appearing in the graphs; we shall neglect everything that is associated with the renormalization of G_R and $T(p)$.

Before proceeding to calculate the contribution of

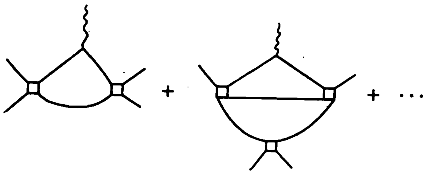

FIG. 3

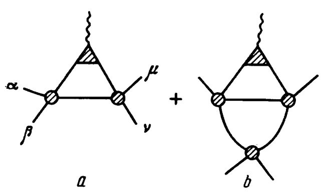

FIG. 4

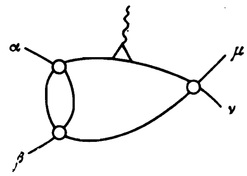

FIG. 5

more complicated graphs, we shall consider Eq. (44) in more detail. Putting

$$u_R = 16\pi g\sqrt{r}, \tag{45}$$

we obtain

$$\frac{\partial g}{\partial t} = -\frac{1}{2}g + \frac{n+8}{2}g^2, \quad t = \ln r. \tag{46}$$

As will be seen from the following, allowance for the next powers in the expansion of $\partial u_R/\partial r$ in u_R leads to the result that Eq. (46) takes the form

$$\partial g/\partial t = \Psi(g), \tag{47}$$

where $\Psi(g)$ is expanded in a series in g, and we have written out the first two terms of the expansion in (46). We note that allowance for terms of order η will not change the form of Eq. (47), but only leads to a correction to $\Psi(g)$. Equation (47) is analogous to the renormalization-group equation [14], and $\Psi(g)$ is the GMLF. It is necessary to emphasize the difference between $\Psi(g)$ and the usual GMLF. The point is that, in field theory, the analogous equation is usually considered in the region of large momenta ($p \gg \kappa$). Inasmuch as $\Psi = \Psi(\kappa/p, g)$, one usually considers $\Psi(0, g)$, whereas we are studying $\Psi(\infty, g)$. We note that for the four-dimensional case $\Psi(0, g)$ has been calculated by a series expansion in g up to g^4 by Avdeeva and Belavin [16].

We return now to Eq. (47). The function appearing in the right-hand side in (46) vanishes when $g_0 = (n + 8)^{-1}$ (we shall call the representation of $\Psi(g)$ by the first two terms of the series the quadratic approximation (QA)). On the other hand, it is well-known [14] that the zeros of $\Psi(g)$ at which $\Psi'(g) > 0$ are fixed points of Eq. (47), and the question of the presence or absence of a FP is the question of whether or not a scale-invariant solution exists. The position of the FP determines the renormalized interaction constant u_R. Of course, the presence of a zero of $\Psi(g)$ in the QA tells us nothing. Below we shall calculate the third- and fourth-order terms. It will then turn out that not only is this zero conserved but its position is almost unchanged. Clearly, this is evidence that a FP does indeed exist, and that our calculated value of its position is close to the true one.

We shall now discuss where the FP has come from. It follows from the calculations of Avdeeva and Belavin [16] that in the four-dimensional case $\Psi(g)$ has a zero only at g = 0. We shall examine how this zero appears when $\epsilon \neq 0$ ($\epsilon = 4 - d$). From an equation analogous to (44), but with $\epsilon \ll 1$ (such an equation was considered in [17,18]), we obtain

$$u_R(r) = 16\pi^2 g r^{\varepsilon/2}, \quad \frac{\partial g}{\partial t} = -\frac{\varepsilon}{2}g + \frac{n+8}{2}g^2 \tag{48}$$

It can be seen from (48) that for $g_0 = \epsilon/(n + 8)$ a FP appears (this FP was discussed in [2]), and the corrections to g_0 are proportional to higher powers of ϵ. Our problem is to elucidate whether this zero is conserved at $\epsilon = 1$, and where it falls.

We shall calculate the next terms of the expansion of $\Psi(g)$. For this it is necessary, firstly, to replace Γ by γ_R in the higher-order irreducible skeleton graphs (e.g., replacement of Γ by γ_R in the graph 4c gives a third-order contribution), and, secondly, to take into account the next terms of the expansion of Γ in u_R. It is convenient to combine the renormalization of Γ and the differentiation into single graphs, one of which is illustrated in Fig. 5. The unshaded circles correspond to u_R, and the triangle to t_R. Figure 5 is the "deciphered" Fig. 4a, in which the left-hand vertex is expanded up to u_R^2. Here it must be remembered that in the part of the graph which originated from the expansion of Γ we must make subtractions. Taking all this into account, and also discarding all corrections of order η (as in the derivation of (44)), we obtain, combining the contributions from three channels in third order in u_R:

$$\frac{\partial u_R}{\partial r} = -\frac{n+8}{2}\frac{\partial K}{\partial r}u_R^2 + \frac{5n+22}{2}u_R^3\frac{\partial}{\partial r}(2J - K^2),$$
$$J = \int \frac{d^3p\, d^3q}{(2\pi)^3} G^2(p)G(q)G(\mathbf{p}+\mathbf{q}),$$
$$K = \int \frac{d^3p}{(2\pi)^3} G^2(p). \tag{49}$$

Hence we have

$$\Psi(g) = -\frac{1}{2}g + \frac{n+8}{2}g^2 - \frac{2}{3}(5n+22)g^3. \tag{50}$$

It is not difficult to see that in $\Psi(g)$, as defined in (50), there is no FP. The fact that $\Psi(g)$ calculated to order g^3 has no FP tells us nothing. The point is that $\Psi(g)$ is an alternating series, and the zeros of such series arise because of the mutual cancellation of the even and odd terms. For example, if in the expansion of sinx in x we confine ourselves to three terms of the series there will be no zero at $x \sim \pi$, while four terms of the series give a zero at x = 3.09. Therefore, we shall calculate the fourth term of the expansion. The contribution of all fourth-order graphs (apart from the non-parquet graph, whose contribution is $\sim (5n + 22)/(n + 8)^4 \ll 1$) equals

$$\left(\frac{\partial u_R}{\partial r}\right)_4 = -u_R^4\{(2n^2+21n+58)\Phi_1(r) + 2(n^2+20n+60)\Phi_2(r) + {}^1/_4(3n^2+22n+56)\Phi_3(r)\},$$

$$\Phi_1(r) = \int \frac{d^3p\, d^3p_1\, d^3p_2}{(2\pi)^9}\Big\{\frac{\partial G^2(p)}{\partial r}G(p_1)[G(\mathbf{p}+\mathbf{p}_1) - G(p_1)]G(p_2)$$
$$\times[G(\mathbf{p}+\mathbf{p}_2) - G(p_2)] + 2G^2(p)\frac{\partial}{\partial r}[G(p_1)G(\mathbf{p}+\mathbf{p}_1)]G(p_2)[G(\mathbf{p}+\mathbf{p}_2) - G(p_2)]\Big\},$$

$$\Phi_2(r) = \int \frac{d^3p\, d^3p_1\, d^3p_2}{(2\pi)^9}\Big\{\frac{\partial G^2(p)}{\partial r}G(p_1)[G(\mathbf{p}+\mathbf{p}_1) - G(p_1)]G(p_2)$$
$$\times[G(\mathbf{p}_1+\mathbf{p}_2) - G(p_2)] + G^2(p)\frac{\partial}{\partial r}[G(p_1)G(\mathbf{p}+\mathbf{p}_1)]G(p_2)[G(\mathbf{p}_1+\mathbf{p}_2) - G(p_2)]$$
$$+ G^2(p)G(p_1)G(\mathbf{p}+\mathbf{p}_1)\frac{\partial}{\partial r}[G(p_2)G(\mathbf{p}_1+\mathbf{p}_2)]\Big\},$$

$$\Phi_3(r) = \int \frac{d^3p\, d^3p_1\, d^3p_2}{(2\pi)^9}\Big\{2\frac{\partial G^2(p)}{\partial r}G^2(p_1)G(p_2)[G(\mathbf{p}+\mathbf{p}_1+\mathbf{p}_2) - G(\mathbf{p}_1+\mathbf{p}_2)]$$
$$+ G^2(p)G^2(p_1)\frac{\partial}{\partial r}[G(p_2)G(\mathbf{p}+\mathbf{p}_1+\mathbf{p}_2)]\Big\}. \tag{51}$$

Calculating these integrals, we obtain the following expression for Ψ:

$$\Psi(g)=-\frac{1}{2}g+\frac{n+8}{2}g^2-\frac{2}{3}(5n+22)g^3+Dg^4,$$
$$D=(2n^2+21n+58)\rho_1+2(n^2+20n+60)\rho_2+{}^1/_4(3n^2+22n+56)\rho_3,$$
$$\rho_1=64(3a/\pi-5/48)\approx 0{,}33,$$
$$\rho_2=16\left(\ln\frac{4}{3}-\frac{1}{4}\right)\approx 0{,}56,\quad \rho_3=\frac{2}{3}\approx 0{,}67,$$
$$a=\int_0^\infty\frac{dx}{(x^2+1)^2}\left(\operatorname{arctg}\frac{x}{2}\right)^2\approx 0{,}1147. \tag{52}$$

For the subsequent analysis it is convenient to make a change of variables. We put

$$g=\frac{\lambda}{n+8},\quad \varphi(\lambda)=(n+8)\Psi\left(\frac{\lambda}{n+8}\right), \tag{53}$$
$$\frac{\partial\lambda}{\partial t}=\varphi(\lambda)=-\frac{\lambda}{2}+\frac{\lambda^2}{2}-\frac{1}{2}b\lambda^3+\frac{1}{2}a\lambda^4,$$
$$a(n)=\frac{2D}{(n+8)^3},\quad b(n)=\frac{4}{3}\frac{5n+22}{(n+8)^2},$$
$$a(1)\approx 0.36,\quad b(1)\approx 0.44.$$

In the new variables all the coefficients are of order unity. The function $\varphi(\lambda)$ is drawn in Fig. 6. It can be seen from (53) that when four terms of the expansion are taken into account there is a fixed point at

$$\lambda_0\approx 1.06. \tag{54}$$

Inasmuch as we had $\lambda_0 = 1$ in the QA, we see that the fourth-order term almost completely cancels the third-order term, and, as can be seen without difficulty, this occurs over almost the whole interval $[0, \lambda_0]$. This means that the QA is good enough for the GMLF in the entire interval between its first two zeros. All this taken together makes it possible to hope that formula (52) approximates the GMLF well.

5. THE EQUATION FOR t_R

Before proceeding to the analysis of the Gell-Mann-Low equations, we derive the analogous equation for the function t_R determining the dependence of r on $\tau = \kappa_0^2 - \kappa_{0c}^2$, where κ_{0c} is the critical value of κ_0. From the Ward identity we have, to within terms of order η,

$$dr/d\tau=t_R,\quad T_{\alpha\beta}(0)=t_R\delta_{\alpha\beta}. \tag{55}$$

By exactly the same method as for $\partial u_R/\partial r$, we can also obtain an equation for $\partial t_R/\partial r$. The simplest graph for $\partial t_R/\partial r$ is drawn in Fig. 7a, where the wavy line drawn to the side denotes the differentiation $\partial G/\partial r$, the triangle denotes t_R, and the circle denotes u_R. This graph is proportional to $t_R u_R$. The nonzero third-order graphs are illustrated in Figs. 7b and 7c. They are proportional to $t_R u_R^2$ (we recall that it is necessary to make a subtraction in the internal loop in the graph 7b). Since we have calculated $\partial u_R/\partial r$ up to u_R^4, we must calculate $\partial t_R/\partial r$ to $t_R u^3$. As a result, we obtain

$$\partial\ln t_R/\partial t=\xi(g), \tag{56}$$
$$\xi(g)=\frac{n+2}{2}g-(n+2)g^2+D_1g^3,$$
$$D_1=(n+2)[{}^1/_4(n+8)\rho_1+(n+8)\rho_2+{}^3/_4(n+2)\rho_3]. \tag{57}$$

From Eq. (57) it is easy to obtain the index γ, defined by the condition

$$r\sim\tau^\gamma. \tag{58}$$

From (55)—(57) it follows that

$$\gamma=\frac{1}{1-\xi(g_0)}, \tag{59}$$

where g_0 is the FP of the GMLF. If we confine ourselves to the QA, which corresponds to taking only the first term for $\xi(g)$ into account in (57), we obtain

$$\gamma=2(n+8)/(n+14). \tag{60}$$

If in $\Psi(g)$ and $\xi(g)$ we take into account all the terms written out in (52) and (57), the corrections to $\xi(g_0)$ and to γ for n = 1 are respectively equal to:

$$\Delta\xi(g_0)/\xi(g_0)\approx 0.03,\quad \Delta\gamma/\gamma\approx 0.006. \tag{61}$$

The small size of these corrections is connected with the fact that the corrections from $\Psi(g)$ and $\xi(g)$ cancel each other. Evidently, this is precisely why the series in ϵ for γ converges well.

We now calculate the specific-heat index α in the QA. The specific heat is related to the polarization operator Π. In the QA, $\Pi(r)$ satisfies the equation represented in Fig. 7d, i.e.,

$$\frac{\partial\Pi}{\partial r}=\frac{n}{2}t_R^2(r)K'(r). \tag{62}$$

Inasmuch as, in the scaling region, it follows from (56) that

$$t_R\sim r^{\xi(g_0)}\sim r^{(\gamma-1)/\gamma}, \tag{63}$$

we have, in the QA,

$$\Pi(r)=\tau^{-\alpha},\quad \alpha=(4-n)/(n+14). \tag{64}$$

6. SOLUTION OF THE EQUATION FOR u_R AND t_R

We turn now to the solution of Eqs. (47), (53) and (56). We shall consider Eq. (53). The boundary condition on it is determined from the condition that $u_R = \Lambda_1$ when $r = \Lambda^2$ (Λ is the cutoff momentum). Then from (45) and (53) we have

$$\frac{\partial\lambda}{\partial t}=\varphi(\lambda),\quad \lambda|_{r=\Lambda^2}=(n+8)\Lambda_1/16\pi\Lambda=\lambda_1. \tag{65}$$

The general solution of (65) has the form

$$\ln\frac{r}{r'}=\int_{\lambda'}^{\lambda}\frac{dy}{\varphi(y)}. \tag{66}$$

Let λ and λ' lie in the interval $[0, \lambda_0]$. Inasmuch as $\varphi(\lambda)$ has zeros at $\lambda = 0$ and $\lambda = \lambda_0$, it is convenient to represent $\varphi^{-1}(\lambda)$ in the form ($\varphi'(0) = -{}^1/_2$):

$$\frac{1}{\varphi(\lambda)}=-\frac{2}{\lambda}+\frac{1}{\varphi'(\lambda_0)(\lambda-\lambda_0)}+\rho(\lambda), \tag{67}$$

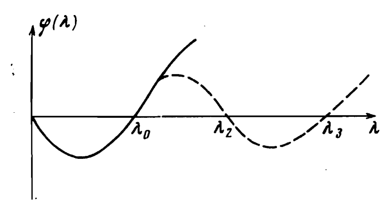

FIG. 6

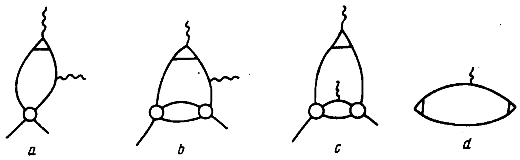

FIG. 7

where $\rho(\lambda)$ is a function with no poles. Substituting (67) into (66), we obtain, putting $r' = \Lambda^2$, $\lambda' = \lambda_1$,

$$\frac{\lambda}{\lambda_1}\frac{\sqrt{r}}{\Lambda}=\left(\frac{\lambda_0-\lambda}{\lambda_0-\lambda_1}\right)^{1/2\varphi'(\lambda_0)}\exp\left(\frac{1}{2}[f(\lambda)-f(\lambda')]\right),$$

$$f(\lambda)=\int_0^\lambda \rho(y)\,dy, \quad \lambda_0\neq\lambda_1, \tag{68}$$

with $\varphi'(\lambda_0) > 0$. It can be seen from (68) that the equation has a solution for any relationship between $\sqrt{r}$ and Λ_1, provided that $\lambda_1 \neq \lambda_0$. But if $\lambda_1 = \lambda_0$, the only solution of (68) is $\lambda \equiv \lambda_0$. This can be seen easily by rewriting (68) in the form

$$\lambda=\lambda_0+(\lambda_1-\lambda_0)\left\{\frac{\lambda}{\lambda_1}\frac{\sqrt{r}}{\Lambda}\exp\left(-\frac{1}{2}[f(\lambda)-f(\lambda_1)]\right)\right\}^{2\varphi'(\lambda_0)}. \tag{69}$$

The solution corresponding to $\lambda = \lambda_0$ is singular; in particular, for this solution u_R does not depend on Λ or Λ_1. The existence of the singular solution and its properties were used by Wilson[2] to obtain the ϵ-expansion.

In the QA, $\lambda_0 = 1$, $\varphi'(\lambda_0) = 1/2$, $\rho(\lambda) = 0$, and from (68) we have

$$\lambda=\left[1+\frac{\sqrt{r}}{\Lambda}\frac{1-\lambda_1}{\lambda_1}\right]^{-1}, \quad u_R=\Lambda_1\Big/\left\{1+\frac{n+8}{16\pi}\Lambda_1\left[\frac{1}{\sqrt{r}}-\frac{1}{\Lambda}\right]\right\}. \tag{70}$$

From formulas (68) and (70) it can be seen clearly how the large bare interaction is screened for $r \ll \Lambda_1$. We note that it is also easy to write a solution with $\lambda_1 \gtrsim \lambda_0$. But in the case $\lambda_1 \gg \lambda_0$, if $\varphi(\lambda)$ behaves as shown by the solid line in Fig. 6 it is known[14] that we have only one FP and, correspondingly, only one scale-invariant solution. But if $\varphi(\lambda)$ has the form shown by the dashed line, the FP $\lambda = \lambda_0$ corresponds to the bare values $\lambda_1 < \lambda_2$ only.

We turn now to Eq. (56). Inasmuch as $t_R(\Lambda) = 1$, we have

$$t_R=\exp\left\{-\int_r^{\Lambda^2}\xi[g(r')]\,dr'\right\}. \tag{71}$$

In the QA, $\xi(g) = 1/2(n + 2)g$ and

$$t_R=\left\{1+\frac{(n+8)\Lambda_1}{16\pi}\left[\frac{1}{\sqrt{r}}-\frac{1}{\Lambda}\right]\right\}^{-(n+2)/(n+8)}, \tag{72}$$

t_R is an experimentally observable quantity, and therefore formulas (71) and (72) can be checked directly.

We now calculate $r(\tau) = G^{-1}(\tau)$. From (55) we have, taking into account that $r \to 0$ as $\tau \to 0$,

$$\tau=\int_0^r\frac{dr'}{t_R(r')}=\int_0^r dr'\exp\left\{\int_{r'}^{\Lambda^2}dr''\,\xi(r'')\right\}. \tag{73}$$

We substitute (72) into (73). Then, assuming that $\Lambda_1 \ll \Lambda$, we obtain

$$\tau=2r\int_1^\infty\frac{dy}{y^3}\left[1+\frac{(n+8)\Lambda_1}{16\pi\sqrt{r}}y\right]^{(n+2)/(n+8)},$$

$$r=\tau, \quad r\gg\Lambda_1^2; \quad r\sim\tau^\gamma, \quad r\ll\Lambda_1^2. \tag{74}$$

7. CALCULATION OF THE FISHER PARAMETER

We shall calculate the Fisher parameter using formula (36). In the calculation of α we shall confine ourselves to lowest order in u_R (graph 4a). Calculating this graph, we obtain

$$\alpha\left(\frac{p^2}{\varkappa^2}\right)=1+\frac{3(n+2)}{64\pi^2\varkappa}u_R^2\left(\frac{1}{p}\operatorname{arctg}\frac{p}{3\varkappa}-\frac{1}{3\varkappa}\right),$$

$$\eta=-2\alpha'(0)=\frac{8}{27}\frac{n+2}{(n+8)^2}. \tag{75}$$

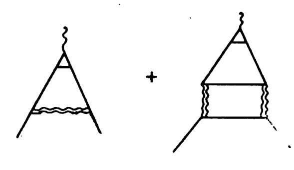

FIG. 8

From small ϵ we have $(u_R = 16\pi^2\epsilon/(n + 8))$

$$\alpha\left(\frac{p^2}{\varkappa^2}\right)=1+\frac{3u_R^2\varkappa}{64\pi^4 p}(n+2)\int_0^\infty dx\,K_1^2(x)K_0(x)\left\{J_1\left(\frac{p}{\varkappa}x\right)-\frac{px}{2\varkappa}\right\},$$

$$\eta=\frac{\varepsilon^2}{2}\frac{n+2}{(n+8)^2}, \tag{76}$$

where K_1, K_0 and J_1 are Bessel functions. The expression (76) coincides with the well-known result of Wilson[2]. We shall obtain another well-known result for $n \gg 1$ and compare it with the corresponding limit for (75). For $n \gg 1$ we must calculate, in lowest order in n^{-1}, the graphs for $T(p)$ given in Fig. 8, where the double wavy line represents the function

$$\Gamma(p)=\frac{\Lambda_1}{1+1/2\,n\Lambda_1\Phi(p)}\approx\frac{2}{n\Phi(p)},$$

$$\Phi(p)=\int\frac{d^3q}{(2\pi)^3}G_R(q)G_R(\mathbf{p}+\mathbf{q}). \tag{77}$$

As a result, we obtain

$$\alpha\left(\frac{p^2}{\varkappa^2}\right)=1-\frac{8\pi}{n}\int\frac{d^3p_1}{(2\pi)^3}\frac{p_1}{\operatorname{arctg}(p_1/2\varkappa)}\left\{\frac{1}{[(\mathbf{p}+\mathbf{p}_1)^2+\varkappa^2]^2}\right.$$
$$\left.-\frac{1}{(p_1^2+\varkappa^2)^2}-\frac{p_1}{\varkappa}\frac{1}{\operatorname{arctg}(p_1/2\varkappa)}\frac{1}{p_1^2+4\varkappa^2}\left[\frac{1}{(\mathbf{p}+\mathbf{p}_1)^2+\varkappa^2}-\frac{1}{p_1^2+\varkappa^2}\right]\right\},$$

$$\eta=8/3\pi^2 n. \tag{78}$$

This value of η coincides with that obtained by Ferrell and Scalapino[19]. Inasmuch as $8/3\pi^2 \approx 0.270$ and $8/27 \approx 0.296$, it can be seen that the limit of (75) as $n \to \infty$ coincides with (78) to within 10%. It may be hoped, therefore, that formula (75) approximates η well for all n.

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