Operator approach to quantum electrodynamics in an external field. Electron loops

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The operator diagram technique for the analysis of processes in a homogeneous (constant with respect to space and time) external electromagnetic field, developed by the authors earlier,[1] is extended to the case of charged-particle loops. The contribution of a loop with n-photon lines is represented as an n-fold integral of an expression that contains no operators. Explicit representations of the contributions to the photon polarization operator of scalar and spinor particles are obtained and analyzed. It is shown that particle-pair production by a field can be described simply within the framework of the given approach.

1. INTRODUCTION

In an earlier paper[1] (henceforth cited as I) we formulated an operator diagram technique for the analysis of processes in a homogeneous external electromagnetic field ($F_{\mu\nu}$ = const), and obtained the mass operators of scalar and spinor particles, i.e., we considered diagrams in which the external lines are charged particles. In this paper we study another class of diagrams, the external lines of which are photons. In Sec. 2 we present a general representation of an electron loop with n photon ends, and in Sec. 3 we obtain an explicit expression for the contribution of the scalar and spinor particles to the polarization operator of the photon and discuss this expression. In Sec. 4 we describe particle pair production by a field (photonless electron loops) within the framework of the presented approach.

2. ELECTRON LOOPS WITH n PHOTONS

In I we presented an operator form for writing down the amplitude of photon scattering by an external field (electron loop with two photon lines), expressed in terms of the polarization tensor $\Pi_{\mu\nu}(k_1, k_2)$ (formulas (1.15)–(1.17) of I). We can write down analogously an operator expression for the tensor $\Pi_{\mu_1\ldots\mu_n}(k_1, \ldots, k_n)$, in terms of which are expressed the amplitudes for the transformation of one number of photons into another number (see the figure): $T = i(2\pi)^4 e^{\mu_1}\ldots e^{\mu_n}\Pi_{\mu_1\ldots\mu_n}$. These amplitudes describe such processes as the splitting of a photon into two photons or the coalescence of two photons into one, the scattering of light by light, etc. The form assumed by the contribution of the diagram shown in the figure is the same as for free particles (it is necessary to add to the contribution of this diagram also the contribution of the diagrams with all the permutations of the photon lines); for spin-1/2 particles we have

$$\Pi_{\mu_1\ldots\mu_n}(k_1,\ldots,k_n)=\frac{ie^n}{(2\pi)^4}\mathrm{Sp}\left\langle 0\left|\prod_{j=1}^{n}\frac{1}{\hat P-\hat l_{j-1}-m+i\varepsilon}\gamma_{\mu_j}\right|0\right\rangle, \qquad (2.1)$$

where

$$P_\mu=i\partial_\mu-eA_\mu,\quad l_j=\sum_{m=1}^{j}k_m,\quad l_0=l_n=0.$$

Within the framework of the developed approach, the main problem is to calculate the mean value over the states x = 0: $\langle 0|\ldots|0\rangle$, which contains in it an aggregate of non-commuting operators P_μ. It is convenient to parametrize the electron propagator:

$$\frac{1}{\hat P-\hat l_{j-1}-m+i\varepsilon}=-i(\hat P-\hat l_{j-1}+m)\int_0^\infty ds_j\exp\{is_j[(P-l_{j-1})^2+{}^1/_2 e\sigma F-m^2]\}, \qquad (2.2)$$

where $\sigma F = \sigma^{\mu\nu}F_{\nu\mu}$. It will be useful in what follows to shift all the exponential factor to the right. We use here the relations

$$P_\mu(s)\equiv e^{isP^2}P_\mu e^{-isP^2}=(e^{-2eFs}P)_\mu,$$
$$\gamma_\mu(s)\equiv e^{ie\sigma Fs/2}\gamma_\mu e^{-ie\sigma Fs/2}=(\gamma e^{2eFs})_\mu, \qquad (2.3)$$
$$k_\mu(s)\equiv(e^{-2eFs}k)_\mu.$$

The first two of these relations were derived in I (see (2.9) and (3.4) in that reference), while the third is the definition of $k_\mu(s)$; it appears if we consider an expression of the type

$$e^{i(P-k)^2s}(P_\mu-k_\mu)e^{-i(P-k)^2s}=P_\mu(s)-k_\mu(s), \qquad (2.4)$$

where k_μ is a c-number. Commutation of a typical pair of operators yields, with allowance for (2.3),

$$\exp\{i(\hat P-\hat l_{j-1})^2s_j\}(\hat P-\hat l_j)=[\hat P-\hat l_{j-1}-k_j^\mu\gamma_\mu(s_j)]\exp\{i(\hat P-\hat l_{j-1})^2s_j\}$$
$$=[\hat P-\hat l_{j-1}-k_j^\mu(-s_j)\gamma_\mu]\exp[is_j(\hat P-\hat l_{j-1})^2]. \qquad (2.5)$$

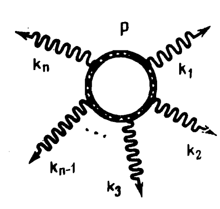
Substituting the representation (2.2) in (2.1) and repeating in succession the operation (2.5) with allowance for (2.3), we obtain

$$\Pi_{\mu_1\mu_2\ldots\mu_n}(k_1,k_2,\ldots,k_n)=\frac{ie^n}{(2\pi)^4}(-i)^n\left(\prod_{k=1}^{n}\int_0^\infty ds_k\right)$$
$$\times\mathrm{Sp}\left\{\langle 0|\prod_{j=1}^{n}(\hat P-\hat h_{j-1}+m)\gamma_{\mu_j}(t_j)\Theta|0\rangle\exp[it_n(e\sigma F/2-m^2)]\right\}, \qquad (2.6)$$

where

$$t_k=\sum_{m=1}^{k}s_m,\quad \hat h_m=\sum_{l=1}^{m}k_l^\mu\gamma_\mu(t_l)=\sum_{l=1}^{m}k_l^\mu(-t_l)\gamma_\mu.$$
$$\Theta=\prod_{j=1}^{n}\exp[is_j(P-l_{j-1})^2]. \qquad (2.7)$$

In (2.6), the mean values of the operators which we must calculate are aggregates of expressions



$$N^{(n)}_{\mu_1\mu_2\ldots\mu_m} \equiv \langle 0|P_{\mu_1}P_{\mu_2}\ldots P_{\mu_m}\Theta|0\rangle \quad (2.8)$$

together with the quantity

$$N^{(n)} = \langle 0|\Theta|0\rangle. \quad (2.9)$$

We consider the operator[1)] of the coordinates:

$$X(s) = \Theta X \Theta^{-1}, \quad \Theta^{-1} = \prod_{j=n}^{1} \exp[-is_j(P-l_{j-1})^2]. \quad (2.10)$$

To find (2.10) it is necessary to calculate the quantity

$$X(s_m) = \exp[is_m(P-l_{m-1})^2] X \exp[-is_m(P-l_{m-1})^2]. \quad (2.11)$$

Differentiating (2.11) with respect to s_m, we obtain, taking (2.3) into account,

$$\frac{dX(s_m)}{ds_m} = -2[P(s_m) - l_{m-1}(s_m)] = -2e^{-2s_m eF}(P-l_{m-1}). \quad (2.12)$$

The solution of this equation with the boundary condition $X(0) = X$ is

$$X(s_m) = U(s_m)(P-l_{m-1}) + X, \quad (2.13)$$

where

$$U(s_m) = \frac{1}{eF}(e^{-2eFs_m} - 1). \quad (2.14)$$

Using in succession expressions of the type (2.13), we can calculate $X(s)$ in (2.10):

$$X(s) = \sum_{m=1}^{n} U(s_m)P(t_{m-1}) - \sum_{m=1}^{n}\sum_{i=1}^{m-1} U(s_m)k_i(t_{m-1}-t_i) + X. \quad (2.15)$$

Transforming this expression, we obtain

$$X(s) = UP - K + X, \quad (2.16)$$

where

$$U \equiv U(t_n), \quad K \equiv K(s) = \sum_{i=1}^{n-1} U(t_n - t_i)k_i. \quad (2.17)$$

We consider now the commutator

$$\langle 0|[X_{\mu_1}, [X_{\mu_2}, \ldots, [X_{\mu_m}, \Theta]\ldots]]|0\rangle = 0, \quad (2.18)$$

since $X_{\mu k}|0\rangle = 0$. According to the definition (2.10) we have the commutator

$$[X_\mu, \Theta] = (X_\mu - X_\mu(s))\Theta = [-UP+K]_\mu\Theta. \quad (2.19)$$

Since

$$[X_\mu, P_\nu] = -ig_{\mu\nu}, \quad (2.20)$$

all the commutators in (2.18) can be calculated directly. At m = 1 we have

$$\langle 0|(-UP+K)\Theta|0\rangle = 0, \quad (2.21)$$

and at m = 2

$$\langle 0|[(-UP+K)_{\mu_2}(-UP+K)_{\mu_1} + iU_{\mu_2\mu_1}]\Theta|0\rangle = 0. \quad (2.22)$$

Continuing these operations, we easily verify that for odd m we get the equation

$$\langle 0|\prod_{k=1}^{2l-1}(-UP+K)_{\mu_k}\Theta|0\rangle = 0, \quad (2.23)$$

and for even m

$$\langle 0|\Big[\prod_{k=1}^{2l}(-UP+K)_{\mu_k} - (-i)^l \sum_{i<j}\prod^{l} U_{\mu_i\mu_j}\Big]\Theta|0\rangle = 0, \quad (2.24)$$

where the summation is carried out over all the commutations $i < j$. The last two equations can be transformed into

$$\langle 0|\Big[\prod_{k=1}^{2l-1}(-P+Q)_{\mu_k}\Big]\Theta|0\rangle = 0,$$

$$\langle 0|\Big[\prod_{k=1}^{2l}(-P+Q)_{\mu_k} - (-i)^l \sum_{i<j}\prod^{l}(U^{-1})^T_{\mu_i\mu_j}\Big]\Theta|0\rangle = 0, \quad (2.25)$$

where we have for the vector $Q = U^{-1}K$. It follows therefore that the mean values $\langle 0|\ldots\Theta|0\rangle$ which contain in them the operators P_μ can be expressed in terms of the mean values of the operator Θ. In particular, for $l = 1$ we have

$$N^{(n)}_{\lambda_1} = \langle 0|P_{\lambda_1}\Theta|0\rangle = Q_{\lambda_1}\langle 0|\Theta|0\rangle = Q_{\lambda_1}N^{(n)},$$
$$N^{(n)}_{\lambda_1\lambda_2} = \langle 0|P_{\lambda_1}P_{\lambda_2}\Theta|0\rangle = [Q_{\lambda_1}Q_{\lambda_2} - i(U^{-1})^T_{\lambda_1\lambda_2}]N^{(n)}. \quad (2.26)$$

We present the convolution of the two operators P_{λ_1} and P_{λ_2}:

$$\overline{P_{\lambda_1}P_{\lambda_2}} = -i(U^{-1})^T_{\lambda_1\lambda_2}. \quad (2.27)$$

We can now formulate a statement that is analogous in a certain sense to the Wick theorem.

The mean value $N^{(n)}_{\mu_1\mu_2\ldots\mu_m}$ (2.8) is a sum of terms in which the operators P_λ are replaced by Q_λ with all the possible convolutions (2.27):

$$N^{(n)}_{\mu_1\mu_2\ldots\mu_m} = [Q_{\mu_1}Q_{\mu_2}\ldots Q_{\mu_m} + \overline{P_{\mu_1}P_{\mu_2}}Q_{\mu_3}\ldots Q_{\mu_m} + \ldots]N^{(n)}. \quad (2.28)$$

For m = 1 and m = 2 this theorem is obvious (see (2.26)). For arbitrary m, the theorem can be proved without difficulty by starting from (2.25) and using mathematical induction.

The result (2.28) reduces to calculation of expressions (2.8) to the problem of finding $N^{(n)} = \langle 0|\Theta|0\rangle$, to which we now proceed. We introduce the quantity $N^{(n)}(\alpha)$, in which all $s_j \to \alpha s_j$ and $N^{(n)} \equiv N^{(n)}(1)$, and differentiate this quantity with respect to α:

$$dN^{(n)}(\alpha)/d\alpha = i\langle 0|\sum_{m=1}^{n} s_m(\tilde{P}_m - l_{m-1})^2\Theta(\alpha)|0\rangle, \quad (2.29)$$

where

$$\Theta(\alpha) = \prod_{j=1}^{n}\exp\{i\alpha s_j(P-l_{j-1})^2\},$$

and the operator

$$\tilde{P}_m - l_{m-1} = P(\alpha t_{m-1}) - k_1(\alpha(t_{m-1}-t_1)) - k_2(\alpha(t_{m-1}-t_2)) - \ldots - k_{m-1} \quad (2.30)$$

has appeared when the exponential factors were shifted to the right in accordance with the relations (2.3)–(2.5). Taking the same relations into account, we have

$$(\tilde{P}_m - l_{m-1})^2 = P^2 - 2P\sum_{l=1}^{m-1} k_l(-\alpha t_l) + \sum_{l=1}^{m-1} k_l^2$$
$$+ 2\sum_{p>l}^{m-1} k_p k_l(\alpha t_p - \alpha t_l) \equiv P^2 - 2PI_m + M_m. \quad (2.31)$$

Substituting this expression into (2.29) we obtain

$$dN^{(n)}(\alpha)/d\alpha = i\langle 0|(P^2 t_n - 2PI + M)\Theta(\alpha)|0\rangle. \quad (2.32)$$

Here

$$I = \sum_{m=1}^{n} s_m I_m = \sum_{l=1}^{n-1} k_l(-\alpha t_l)(t_n - t_l), \quad M = \sum_{m=1}^{n} s_m M_m. \quad (2.33)$$

The terms with P^2 and P which enter in the right-hand side, can be reduced with the aid of (2.26) to expressions that do not contain an operator in front of $\Theta(\alpha)$:

$$dN^{(n)}(\alpha)/d\alpha = i\tilde{B}(\alpha)N^{(n)}(\alpha), \quad (2.34)$$

where

$$\tilde{B}(\alpha) = t_n Q^2(\alpha) - 2Q(\alpha)I + M - it_n\,\mathrm{Sp}[U^{-1}(\alpha t_n)]. \quad (2.35)$$

Solving the differential equation (2.34), we have

$$N^{(n)}(\alpha)=C'\exp\left[i\int^{\alpha}B(\alpha)\,d\alpha\right]. \tag{2.36}$$

The integral of the last term (2.35) can be taken in explicit form and we then obtain

$$N^{(n)}(\alpha)=C\{\det[-{}^1/_2U(\alpha t_n)]\}^{-1/2}\exp\left[i\int_0^{\alpha}B(\alpha)\,d\alpha\right], \tag{2.37}$$

where

$$B(\alpha)=t_nQ^2(\alpha)-2Q(\alpha)I+M. \tag{2.38}$$

The constant C can be obtained for the limiting case $F_{\mu\nu}\to 0$ and $\alpha\to 0$. As $\alpha\to 0$, formula (2.37) yields

$$N^{(n)}(\alpha)=C/\alpha^2t_n^2. \tag{2.39}$$

On the other hand, as $F_{\mu\nu}\to 0$ we can go over in (2.9) (where $s_j\to\alpha s_j$) to the momentum representation and use the completeness theorem

$$\langle x|R(p)|x\rangle=\int d^4p\,\langle x|R(p)|p\rangle\langle p|x\rangle=\int d^4p\,R(p), \tag{2.40}$$

where R is an arbitrary operator. Then formula (2.9) reduces as $\alpha\to 0$ to an integral of the type of (2.31) of I. As a result we have

$$C=-i\pi^2. \tag{2.41}$$

Substituting this expression in (2.37) and putting $\alpha = 1$, we obtain the sought quantity $N^{(n)} = N^{(n)}(1)$. The factor in front of the exponential in (2.37) is universal and does not depend on the number of the external photon lines. Its explicit form can be determined with the aid of the procedure described in I (see the Appendix there). It must be borne in mind that det $[-U(t_n)]$ has an infinite number of zeros, so that it is necessary to see to it that the branches of the pre-exponential factor are correctly chosen. The final result takes the form

$$N^{(n)}\equiv N^{(n)}(1)=-i\pi^2\Phi(t_n)\exp\left[i\int_0^1 B(\alpha)\,d\alpha\right], \tag{2.42}$$

where $E, H = [(\mathscr{F}^2+\mathscr{G}^2)^{1/2}\pm\mathscr{F}]^{1/2}$; $\mathscr{F}$, and $\mathscr{G}$ are field invariants:

$$\Phi(t)=\frac{e^2EH}{\sin(|e|Ht)\,\mathrm{sh}(|e|Et)}. \tag{2.43}$$

Knowing the explicit form of $N^{(n)}$ and substituting it in (2.28), we obtain the aggregate of the expression $N^{(n)}_{\mu_1\mu_2\ldots\mu_m}$, the use of which yields the matrix element (2.6), which is an n-fold integral (with respect to s_1, s_2, . . . , s_n) that contains no operators. From the point of view of the γ-matrix structure, it takes the form (with (2.3) taken into account)

$$\gamma^{\lambda_1}\ldots\gamma^{\lambda_k}e^{ie\sigma Ft_n/2}. \tag{2.44}$$

When calculating the traces of expressions of this type, it is convenient to use the formula (see the Appendix)

$$\gamma^{\mu}e^{ie\sigma Fs/2}=i\gamma^5[e^{eFs}\sin(eF^*s)\gamma]^{\mu}+[e^{Fs}\cos(eF^*s)\gamma]^{\mu},$$
$$F^*_{\alpha\beta}=\varepsilon_{\alpha\beta\gamma\delta}F^{\gamma\delta}/2, \tag{2.45}$$

which reduces the problem to a determination of the traces of an ordinary aggregate of γ matrices.

The solution of the considered problem in scalar electrodynamics is perfectly analogous, since the problem reduces there to a consideration of expressions (2.8) and (2.9).

The obtained formulas enable us to obtain the explicit form of expressions for any given n. The case n = 2 (polarization operator) will be considered in detail in the next section. We therefore consider here, by way of illustration, the quantity $N^{(3)}$ [Eq. (2.42)] for n = 3 (splitting of a photon in an external field). In this case we have

$$Q(\alpha)=U^{-1}(\alpha t_3)[U(\alpha t_3-\alpha t_1)k_1+U(\alpha t_3-\alpha t_2)k_2],$$
$$I=(t_3-t_1)e^{2eF\alpha t_1}k_1+(t_3-t_2)e^{2eF\alpha t_2}k_2, \tag{2.46}$$
$$M=2(t_3-t_2)k_2e^{-2eF\alpha(t_2-t_1)}k_1+(t_3-t_1)k_1{}^2+(t_3-t_2)k_2{}^2.$$

The matrix expressions contained here can be expanded with the aid of the technique described in the Appendix of I, as a result of which we obtain the explicit form of $B(\alpha)$ [Eq. (2.38)]. The integral with respect to α, which is of interest to us, can then be evaluated directly[2)]

$$\int_0^1 B(\alpha)\,d\alpha=\frac{1}{2eE\,\mathrm{sh}(eEt_3)}\{(k_2C^2k_2)[\mathrm{ch}(eEt_3)-\mathrm{ch}(eE(t_3-2t_2))]$$
$$+(k_1C^2k_1)[\mathrm{ch}(eEt_3)-\mathrm{ch}(eE(t_3-2t_1))]+(k_1C^2k_2)[\mathrm{ch}(eEt_3)$$
$$+\mathrm{ch}(eE(2t_2-2t_1-t_3))-\mathrm{ch}(eE(t_3-2t_2))-\mathrm{ch}(eE(t_3-2t_1))]$$
$$+(k_1Ck_2)[\mathrm{sh}(eEt_3)+\mathrm{sh}(eE(2t_2-2t_1-t_3))+\mathrm{sh}(eE(t_3-2t_2))$$
$$-\mathrm{sh}(eE(t_3-2t_1))]\}$$
$$+\text{ terms in which } E\to iH,\quad C\to -iB.$$

The expression obtained for the phase in (2.42) agrees with the expressions obtained in particular cases by Adler [2] for E = 0 and $k_1^2 = k_2^2 = k_2^2 = 0$ and by Papanyan and Ritus [3] for $\mathscr{F} = \mathscr{G} = 0$.

3. POLARIZATION OPERATOR

If n = 2, then the quantity $\Pi_{\mu_1\mu_2}$ is a polarization operator. In this case

$$Q(\alpha)=\frac{U(\alpha t_2-\alpha t_1)}{U(\alpha t_2)}k,\quad I=(t_2-t_1)e^{2eF\alpha t_1}k,\quad M=(t_2-t_1)k^2 \tag{3.1}$$

where $k\equiv k_1$. Substituting these expressions in (2.38) and (2.42) we obtain

$$N^{(2)}=-i\pi^2\Phi(s)e^{i(\psi+m^2s)}, \tag{3.2}$$

where $\Phi(s)$ is given by formula (2.43),

$$\psi=-\frac{1}{2}\left[\frac{kB^2k}{eH}\zeta_2+\frac{kC^2k}{eE}\zeta_1\right]-sm^2,$$
$$\zeta_2=\frac{\cos(eHs)-\cos(eHvs)}{\sin(eHs)},\quad \zeta_1=\frac{\mathrm{ch}(eEs)-\mathrm{ch}(eEvs)}{\mathrm{sh}(eEs)}. \tag{3.3}$$

We have changed over here to the variables

$$s\equiv t_2=s_1+s_2,\quad v=\frac{2t_1-t_2}{t_2}=\frac{s_1-s_2}{s_1+s_2}. \tag{3.4}$$

Knowing $N^{(2)}$, we obtain from (2.26) the values of $N^{(2)}_{\mu_1}$ and $N^{(2)}_{\mu_1\mu_2}$, where $Q_\mu\equiv Q_\mu(1)$). These expressions can be used to describe both electron loops and scalar-particle loops.

Let us calculate the contributions of the scalar parbiels to the polarization operator of the photon. It is necessary here to consider two diagrams (see Fig. 2 of I). The contribution of the first will be represented in the form

$$\Pi^{(a)}_{\mu_1\mu_2}=\frac{ie^2}{(2\pi)^4}\int_0^\infty ds_1\int_0^\infty ds_2\langle 0|(2P_{\mu_1}-k_{\mu_1})$$
$$\times(2(e^{-2eFs_1}P)_{\mu_2}-k_{\mu_2})e^{is_1P^2}\exp[is_2(P-k)^2]|0\rangle\exp[-i(s_1+s_2)m^2], \tag{3.5}$$

where a parametrization of the type (2.2) was carried out and relations (2.3) was used. To find the explicit form of $\Pi_{\mu_1\mu_2}$ it is necessary to use the expressions given above for $N^{(2)}$, $N^{(2)}_{\mu}$, and $N^{(2)}_{\mu_1\mu_2}$. The calculation then reduces to a number of algebraic operations. The result is

$$\Pi_{\mu\nu}=\frac{\alpha}{8\pi}\int_{-1}^{1}dv\int_0^\infty ds\,s\Phi(s)\left\{(\rho k)_\mu(\rho k)_\nu-(\lambda k)_\mu(\lambda k)_\nu+\frac{2i}{s}\frac{\partial\rho_{\mu\nu}}{\partial v}\right\}e^{i\psi}, \tag{3.6}$$

where we have changed over to the variables s and v [Eq. (3.4)], ψ is given by formula (3.3), and we have introduced the tensors

$$\rho_{\mu\nu}=C_{\mu\nu}{}^2\zeta_3-B_{\mu\nu}{}^2\zeta_4,\quad \lambda_{\mu\nu}=C_{\mu\nu}\zeta_1-B_{\mu\nu}\zeta_2;\qquad \zeta_3=\frac{\mathrm{sh}(eEvs)}{\mathrm{sh}(eEs)},\quad \zeta_4=\frac{\sin(eHvs)}{\sin(eHs)}. \tag{3.7}$$

Integrating the last term with respect to v by parts, we obtain

$$\Pi_{\mu\nu}^{(a)}=\frac{\alpha}{8\pi}\int_{-1}^{1}dv\int_0^\infty ds\, s\Phi(s)\{(\rho k)_\mu(\rho k)_\nu-\rho_{\mu\nu}(k\rho k)-(\lambda k)_\mu(\lambda k)_\nu\}e^{i\psi}+\frac{i\alpha}{2\pi}g_{\mu\nu}\int_0^\infty ds\,\Phi(s)\,e^{-ism^2}. \tag{3.8}$$

The contribution of the second diagram $\Pi_{\mu\nu}^{(b)}$ [(1.17) of I] is the mean value of the propagator. We can calculate it by using formula (3.2) with v = 1 (or as k → 0):

$$\Pi_{\mu\nu}^{(b)}=-\frac{i\alpha}{2\pi}g_{\mu\nu}\int_0^\infty ds\,\Phi(s)\,e^{-ism^2}. \tag{3.9}$$

The sum of (3.8) and (3.9)

$$\Pi_{\mu\nu}^{(0)}=\Pi_{\mu\nu}^{(a)}+\Pi_{\mu\nu}^{(b)} \tag{3.10}$$

gives the total contribution of the scalar particles to the polarization operator of the photon. In this sum, the terms containing $ig^{\mu\nu}$ cancel each other, after which we are left with a gauge-invariant expression (the first term of (3.8)). The polarization operator $\Pi_{\mu\nu}^{(0)}$ should be renormalized. To this end, we represent it in the form

$$\Pi_{\mu\nu}^{(0)}=[\Pi_{\mu\nu}^{(0)}-\Pi_{\mu\nu}^{(0)}(F=0)]+\Pi_{\mu\nu}^{(0)}(F=0). \tag{3.11}$$

The first term vanishes at a field F = 0, while the second term (which does not depend on the field) should be renormalized in standard fashion. As a result we have a final expression for the renormalized polarization operator:

$$\Pi_{\mu\nu}^{(0)R}=\frac{\alpha}{8\pi}\int_{-1}^{1}dv\int_0^\infty ds\, s\Phi(s)[(\rho k)_\mu(\rho k)_\nu-\rho_{\mu\nu}(k\rho k)-(\lambda k)_\mu(\lambda k)_\nu]e^{i\psi}+(g^{\mu\nu}k^2-k_\mu k_\nu)\Omega^{(0)}, \tag{3.12}$$

where

$$\Omega^{(0)}=\frac{\alpha}{8\pi}\int_{-1}^{1}dv\,\zeta^{(0)}\left\{\int_0^\infty\frac{ds}{s}\exp\left\{-is\left[m^2-\frac{k^2}{4}(1-v^2)\right]\right\}+\ln\left[1-\frac{k^2}{4m^2}(1-v^2)\right]\right\},\quad \zeta^{(0)}=v^2. \tag{3.13}$$

The obtained expression (3.12) enables us to analyze a large number of problems. The imaginary part of the polarization operator determines the probability of the production of a pair of scalar particles by a photon having a definite polarization:

$$W^{(0)}=\frac{1}{k_0}e^\mu e^{\nu*}\,\mathrm{Im}\,\Pi_{\mu\nu}^{(0)}(k), \tag{3.14}$$

where k_0 is the photon frequency.

Substituting $\Pi_{\mu\nu}^{(0)R}$ (3.12) in the Dyson equation $(k^2-\Pi)D=1$, we obtain the propagator of the photon in the external field. The eigenvalues $\kappa_i^{(0)}$ corresponding to the eigenvectors $b_{(i)\mu}^{(0)}$ of the tensor (3.12) play here the role of the square of the photon mass in the field;

$$\Pi_{\mu\nu}^{(0)R}\,b_{(i)}^{(0)\nu}=\varkappa_i^{(0)}\,b_{(i)\mu}^{(0)}\qquad (i=1,2,3,4). \tag{3.15}$$

Solving this equation[3], we obtain four mutually orthogonal vectors $b_{(i)\mu}^{(0)}$:

$$b_{(1)\mu}^{(0)}=(C^2k)_\mu(kB^2k)-(B^2k)_\mu(kC^2k),$$

$$b_{(2)\mu}^{(0)}=(Bk)_\mu+\frac{2}{\Lambda^{(0)}}(kB^2k)\Omega_4^{(0)}(Ck)_\mu,\qquad b_{(3)\mu}^{(0)}=(Ck)_\mu-\frac{2}{\Lambda^{(0)}}(kC^2k)\Omega_4^{(0)}(Bk)_\mu,\qquad b_{(4)\mu}^{(0)}=k_\mu. \tag{3.16}$$

For the eigenvalues $\kappa_i^{(0)}$ we have

$$\varkappa_1^{(0)}=(\Omega^{(0)}+\Omega_1^{(0)})k^2,\quad \varkappa_2^{(0)}=\varkappa_1^{(0)}-\Lambda_2^{(0)}-\Lambda_1^{(0)}/2\Lambda^{(0)},\quad \varkappa_3^{(0)}=\varkappa_1^{(0)}+\Lambda_3^{(0)}+\Lambda_1^{(0)}/2\Lambda^{(0)},\quad \varkappa_4^{(0)}=0. \tag{3.17}$$

In formulas (3.16) and (3.17) we have introduced the notation

$$\Lambda_2^{(0)}=(kB^2k)\Omega_2^{(0)},\quad \Lambda_3^{(0)}=(kC^2k)\Omega_3^{(0)},\quad \Lambda_1^{(0)}=4(kC^2k)(kB^2k)(\Omega_4^{(0)})^2,$$
$$\Lambda^{(0)}=\Lambda_2^{(0)}+\Lambda_3^{(0)}+[(\Lambda_2^{(0)}+\Lambda_3^{(0)})^2+\Lambda_1^{(0)}]^{1/2},$$
$$\Omega_i^{(0)}=-\frac{\alpha}{8\pi}\int_{-1}^{1}dv\int_0^\infty ds\, s\Phi(s)e^{i\psi}\omega_i^{(0)}; \tag{3.18}$$
$$\omega_1^{(0)}=\zeta_3\zeta_4,\quad \omega_4^{(0)}=-\zeta_1\zeta_2,\quad \omega_2^{(0)}=\zeta_4^2-\zeta_3\zeta_4+\zeta_2^2,\quad \omega_3^{(0)}=\zeta_3^2-\zeta_3\zeta_4-\zeta_1^2,$$

where we used the quantities ζ_k introduced in (3.3), (3.7), and (3.13).

The obtained expressions enable us to represent the polarization operator in diagonal form:

$$\Pi_{\mu\nu}^{(0)R}=\sum_{(i)}\frac{b_{(i)\mu}^{(0)}b_{(i)\nu}^{(0)}}{b_{(i)}^{(0)2}}\varkappa_i^{(0)}, \tag{3.19}$$

which is convenient for applications. Using the representation (3.19), we can write the photon Green's function in the field in the form

$$D_{\mu\nu}=\sum_i\frac{b_{(i)\mu}^{(0)}b_{(i)\nu}^{(0)}}{b_{(i)}^{(0)2}}\frac{1}{k^2-\varkappa_i^{(0)}}, \tag{3.20}$$

so that the properties of the functions $\kappa_i^{(0)}$ determine the character of the propagation of the electromagnetic waves[4] (with definite polarization) in the external field.

The analytic properties of the functions $\kappa_i^{(0)}$ are determined by the specifics of the interaction of the photon with the charged particles in the field, and differ significantly from the properties of the same functions in vacuum. In particular, they represent the spectrum of the states of a scalar particle in a field. Let us illustrate the last circumstance using as an example the function $\kappa_2^{(0)}$ in a purely magnetic field (E = 0)

$$\varkappa_2^{(0)}=\Omega^{(0)}k^2+\frac{\alpha}{2\pi}m^2\int_{-1}^{1}dv\int_0^\infty\frac{dx}{\sin x}\left\{-rv\frac{\sin vx}{\sin x}+q\left[\left(\frac{\sin vx}{\sin x}\right)^2+\left(\frac{\cos x-\cos vx}{\sin x}\right)^2\right]\right\}e^{i\psi_H}; \tag{3.21}$$

here

$$\psi_H=\frac{1}{\mu}\left[2q\left(\frac{\cos x-\cos vx}{\sin x}\right)-x(1-r(1-v^2))\right], \tag{3.22}$$
$$x=|e|Hs,\quad \mu=\frac{H}{H_0},\quad r=\frac{kC^2k}{4m^2}=\frac{k_0^2-k_3^2}{4m^2},\quad q=\frac{kB^2k}{4m^2}=\frac{k_\perp^2}{4m^2}$$

(the field H is directed along the 3 axis, and $k_\perp^2=k_1^2+k_2^2$).

We shall show that the function $\kappa_2^{(0)}$ has singularities at the points

$$r=\left(\frac{\mathscr{E}^{(0)}(l)+\mathscr{E}^{(0)}(l')}{2}\right)^2=\frac{1}{2}[1+(l+l'+1)\mu+\{[1+(l+l'+1)\mu]^2-(l-l')^2\mu^2\}^{1/2}], \tag{3.23}$$

where $^{(0)}(l)=\sqrt{1+(2l+1)\mu}$; $l,\ l'=0,1,\ldots$, corresponding to the quantum character of the transverse motion of the scalar particles in the field. When considering this question, we can leave out of $\kappa_2^{(0)}$ the terms that do not depend on the field, and bear in mind the fact that the singularities are produced by the region of large values of the variable x. We confine ourselves to calculation of the term

$$\frac{\alpha}{2\pi} m^2 \int_{-1} dv \int_{x_0}^{\infty} \frac{dx}{\sin x} q \exp\left\{ i \left[u(\cos x - \cos vx) - \frac{x}{\mu}(1 - r(1-v^2)) \right] \right\}, \tag{3.24}$$

where $u = 2q/\mu \sin x$. Calculation of the remaining terms (as well as, incidentally, of other functions $\kappa_i^{(0)}$) is analogous. We use the expansion (see [5], p. 987)

$$e^{-iu\cos vx} = \sum_{n=-\infty} (-i)^n J_n(u) e^{invx}, \tag{3.25}$$

where $J_n(u)$ is a Bessel function, and take the integral with respect to v, retaining the highest-order terms in the expansion in 1/x:

$$\int_{-1}^{1} dv \exp\left\{ ix \left(nv - \frac{r}{x} v^2 \right) \right\} \approx e^{-i\pi/4} \sqrt{\frac{\pi\mu}{rx}} \exp\left(\frac{ixn^2\mu}{4r} \right) \tag{3.26}$$

this formula is valid if $|n\mu/2r| < 1$. Substituting it in (3.24), we get

$$\frac{\alpha}{2\pi} m^2 e^{-i\pi/4} q \left(\frac{\pi\mu}{r} \right)^{1/2} \sum_{l=l_0}^{\infty} \sum_{n=-\infty}^{\infty} \frac{(-i)^n (-1)^{l(n+1)}}{\sqrt{l\pi}}$$
$$\times \int_{-\pi/2}^{\pi/2} \frac{dx}{\sin x} \exp\left\{ iu\cos x + i(l\pi + x) \left[\frac{n^2\mu}{4r} - \frac{1-r}{\mu} \right] \right\} J_n(u), \tag{3.27}$$

where we took into account the fact that $x_0 \gg 1$ ($l_0 \gg 1$), subdivided the integral into a sum of integrals, and made the change of variable $x \to x + l\pi$. At the point

$$\frac{n^2\mu}{4r} - \frac{1-r}{\mu} = m \quad \text{(m is an integer)} \tag{3.28}$$

the sum over l in (3.27) diverges if n + m is an odd number. Equation (3.28) is satisfied at points where

$$r = r_{mn} = {}^1/_2 \{ 1 + m\mu + [(1+m\mu)^2 - (n\mu)^2]^{1/2} \}. \tag{3.29}$$

in the vicinity of these points $r = r_{mn} + \delta$ we can sum over l in (3.27) and retain the higher-order terms:

$$\sum_{l=l_0} \frac{(-1)^{l(n+1)}}{\sqrt{l\pi}} \exp\left\{ il\pi \left[\frac{n^2\mu}{4r} - \frac{(1-r)}{\mu} \right] \right\}$$
$$= \frac{1}{\pi} \left\{ \pi\mu/|\delta| \left[1 - \left(\frac{n\mu}{2r_{mn}} \right)^2 \right] \right\}^{1/2} \exp\left(i\frac{\pi}{4} \operatorname{sign}\delta \right). \tag{3.30}$$

Substituting this expansion in (3.27), we obtain

$$\frac{\alpha}{2\pi} |e| H e^{-i\pi\theta(-\delta)/2} q \sum_{n=-\infty}^{\infty} (-i)^n \{ \delta^2 [(1+m\mu)^2 - (n\mu)^2] \}^{-1/4} \cdot$$
$$\int_{-\pi/2}^{\pi/2} e^{i(mx + u\cos x)} J_n(u) \frac{dx}{\sin x}. \tag{3.31}$$

Proceeding analogously, we obtain an expression for the entire quantity $\kappa_2^{(0)}$. In the case n = 0 and m = 1 we have

$$\varkappa_2^{(0)} = -\frac{i\alpha k_\perp^2 |e| H}{4m^2(1+\mu)} \left[\frac{k_0^2 - k_3^2}{4m^2(1+\mu)} - 1 \right]^{-1/2} \exp\left(-\frac{k_\perp^2}{2|e|H} \right). \tag{3.32}$$

It is easy to see that the condition (3.29) coincides with the condition (3.23) if we put $m = l + l' + 1$, $n = l - l'$), because m and n have opposite parity.

We have thus shown that at the points (3.23) $k_0^2 = k_3^2 + m^2(\mathscr{E}^{(0)}(l) + \mathscr{E}^{(0)}(l'))^2$ the function $\kappa_2^{(0)}$ has a root singularity. The reason why this singularity is not a pole but a branch point is that motion along the field is not quantized. The appearance of the root singularity can be understood from an analysis of the imaginary part for $\kappa_2^{(0)}$, which in accordance with (3.14) is connected with the probability of the production of a pair of particles by a photon of even polarization. As is well known (see [6]), for spinor particles this probability has a root singularity at the points (3.23), the origin of which is due to the properties of the phase volume in the given order of perturbation theory (infinitesimally narrow levels[5]).

The contribution of the spinor particles to the polarization operator of the photon can be obtained from the general formulas[6] of Sec. 2, in which it is necessary to substitute the explicit form of the quantities (3.1)–(3.4). The analysis is perfectly analogous to that presented above for scalar particles (the new algebraic detail, namely the calculation of the traces of the γ matrices, is carried out without difficulty with the aid of formulas (2.3) and (2.45); see the appendix). The result can be finally represented in the form (3.19), where

$$b_{(1)\mu}^{(1/2)} = b_{(1)\mu}^{(0)}, \quad b_{(4)\mu}^{(1/2)} = b_{(4)\mu}^{(0)}, \quad b_{(2)\mu}^{(1/2)} = (Bk)_\mu + 2(Ck)_\mu \Omega_4^{(1/2)} (kB^2k)/\Lambda^{(1/2)},$$
$$b_{(3)\mu}^{(1/2)} = (Ck)_\mu - 2(Bk)_\mu \Omega_4^{(1/2)} (kC^2k)/\Lambda^{(1/2)} \cdot$$
$$\varkappa_1^{(1/2)} = (\Omega^{(1/2)} + \Omega_1^{(1/2)}) k^2, \quad \varkappa_2^{(1/2)} = \varkappa_1^{(1/2)} - \Lambda_2^{(1/2)} - \Lambda_1^{(1/2)}/2\Lambda^{(1/2)},$$
$$\varkappa_3^{(1/2)} = \varkappa_1^{(1/2)} + \Lambda_3^{(1/2)} + \Lambda_1^{(1/2)}/2\Lambda^{(1/2)}, \quad \varkappa_4^{(1/2)} = 0. \tag{3.33}$$

Here

$$\Lambda_1^{(1/2)} = 4(\Omega_4^{(1/2)})^2 (kB^2k)(kC^2k),$$
$$\Lambda_2^{(1/2)} = \Omega_2^{(1/2)} (kB^2k), \quad \Lambda_3^{(1/2)} = \Omega_3^{(1/2)} (kC^2k), \tag{3.34}$$
$$\Lambda^{(1/2)} = \Lambda_2^{(1/2)} + \Lambda_3^{(1/2)} + [(\Lambda_2^{(1/2)} + \Lambda_3^{(1/2)})^2 + \Lambda_1^{(1/2)}]^{1/2}.$$

The function $\Omega^{(1/2)}$ is given by formula (3.13), where it is necessary to make the substitution $\zeta^{(0)} \to \zeta^{(1/2)} = 2(1 - v^2)$; the functions $\Omega_k^{(1/2)}$ are given by formula (3.18), in which the following quantities must be substituted:

$$\omega_1^{(1/2)} = 2[\cos(eHvs)\operatorname{ch}(eEvs) - \operatorname{ctg}(eHs)\operatorname{cth}(eEs)\sin(eHvs)\operatorname{sh}(eEvs)],$$
$$\omega_2^{(1/2)} = -4\operatorname{ch}(eEs)\frac{\cos(eHs) - \cos(eHvs)}{\sin^2(eHs)} - \omega_1^{(1/2)}$$
$$\omega_3^{(1/2)} = 4\cos(eHs)\frac{\operatorname{ch}(eHs) - \operatorname{ch}(eEvs)}{\operatorname{sh}^2(eEs)} - \omega_1^{(1/2)}, \tag{3.35}$$
$$\omega_4^{(1/2)} = 2\left[\frac{(\cos(eHs)\cos(eHvs) - 1)(\operatorname{ch}(eEs)\operatorname{ch}(eEvs) - 1)}{\sin(eHs)\operatorname{sh}(eEs)} + \operatorname{sh}(eEvs)\sin(eHvs) \right].$$

This result coincides with [7], but the notation is different here.

The functions $\kappa^{(1/2)}$ (in a pure magnetic field E = 0) also have root singularities due to the properties of the phase volume at the point (3.29). The study of these singularities is perfectly analogous to that carried out above. The only difference lies in the fact that n + m is even. In the calculation this difference arises because when the integral with respect to x is replaced by a sum of integrals (cf. (3.27)) we obtain $(-1)^{ln}$ in place of the factor $(-1)^{l(n+1)}$. The reason is that the condition (cf. (3.23))

$$r = \left(\frac{\mathscr{E}^{(1/2)}(l) + \mathscr{E}^{(1/2)}(l')}{2} \right)^2 \tag{3.36}$$

contains $\mathscr{E}^{(1/2)}(l) = \sqrt{1 + 2l\mu}$. We present by way of example the function $\kappa_3^{(1/2)}$ (near r_{mn}):

$$\varkappa_3^{(1/2)} = \frac{\alpha}{\pi} |e| H e^{-i\pi\theta(-\delta)/2} \sum_{n=-\infty}^{\infty} (-i)^n \{ \delta^2 [(1+m\mu)^2 - (n\mu)^2] \}^{-1/4}$$
$$\times \int_{-\pi/2}^{\pi/2} \frac{dx}{\sin x} \left\{ \frac{iq}{2} [J_{n-1}(u) - J_{n+1}(u)] - r\cos x J_n(u) \right\} e^{i(mx + u\cos x)}. \tag{3.37}$$

In the particular case n = m = 0 ($r_{mn} = 1$) we have (for $\delta > 0$)

$$\varkappa_2^{(1/2)} = -\frac{2i\alpha|e|Hm}{(k_0^2-k_3^2-4m^2)^{1/2}} \exp\left(-\frac{k_\perp^2}{2|e|H}\right) \tag{3.38}$$

which agrees, with allowance for (3.14), with the results of Klepikov[6].

We note in conclusion that the transition to the particular case of a crossed field E → 0, H → 0 (the quasiclassical approximation) was demonstrated in detail in I with the mass operator of a scalar particle as an example. The transition is analogous for the polarization operator, and the results coincide with the known ones. (see, e.g., [9]).

4. VACUUM LOOPS

So far we have considered electron loops with $n \neq 0$, connected with processes in which there are real photons in the initial and final states. There is, however, a class of diagrams with n = 0, which determine the amplitude of the transition of the vacuum into a vacuum, i.e., connected with the very definition of the vacuum. In the presence of an external field, these diagrams describe processes that are in principle observable, for example pair production by an external electric field. It is therefore advantageous to discuss this group of questions within the framework of the developed approach[7)].

We consider, for the sake of argument, vacuum electron loops (loops of scalar particles are calculated in perfect analogy). Let $L^{(n)}$ be the amplitude describing a closed loop of free particles interacting with an external field n times. The explicit form of the amplitude in the coordinate representation was already discussed by Feynman[10]:

$$L^{(n)} = \frac{ie^n}{n(2\pi)^4} \int\int \ldots \int d^4x_1\, d^4x_2 \ldots d^4x_n\, \mathrm{Sp}[G(x_1, x_2) \times \hat{A}^F(x_2) G(x_2, x_3) \hat{A}^F(x_3) \ldots G(x_{n-1}, x_n) A^F(x_n) G(x_n, x_1) \hat{A}^F(x_1)]. \tag{4.1}$$

It is convenient to use the following representation of G:

$$G(x_i, x_j) = \left\langle x_i \left| \frac{1}{\hat{p}-m} \right| x_j \right\rangle, \quad p_\mu = i\partial_\mu$$

(cf. (1.12) of I). Using the self-adjoint character of the operator p_μ and the completeness of the system of states $|x_i\rangle$, we have $L^{(n)}$ in a form convenient for analysis:

$$L^{(n)} = \frac{i}{n(2\pi)^4} \int d^4x\, \mathrm{Sp} \left\langle x \left| \left(\frac{1}{\hat{p}-m+i\varepsilon} e\hat{A}^F\right)^n \right| x \right\rangle. \tag{4.2}$$

The contribution of the electron loop with any number of interactions with an external field $L = \sum_n L^{(n)}$ contains the sum

$$\sum_n \frac{1}{n} \left(\frac{e\hat{A}^F}{\hat{p}-m+i\varepsilon}\right)^n = -\ln\left(1 - \frac{e\hat{A}^F}{\hat{p}-m+i\varepsilon}\right) = -\ln\left(\frac{1}{\hat{p}-m}(\hat{P}-m)\right). \tag{4.3}$$

It is convenient to represent the logarithm of the last operator expression in the form of a Frullani integral:

$$\ln\left(\frac{\hat{P}-m}{\hat{p}-m}\right) = -\int_0^\infty \frac{ds}{s} \{\exp[is(\hat{P}-m+i\varepsilon)] - \exp[is(\hat{p}-m+i\varepsilon)]\}. \tag{4.4}$$

The last term of the expression in the right-hand side corresponds to subtraction at the point F = 0. We shall not write it out for the time being, and take it into account in the final expression. Thus, the problem has been reduced to a determination of the quantity

$$L = \int d^4x\, \mathscr{L}, \tag{4.5}$$

where

$$\mathscr{L} = \frac{i}{(2\pi)^4} \mathrm{Sp} \int_0^\infty \frac{ds}{s} \langle 0|\exp[is(\hat{P}-m+i\varepsilon)]|0\rangle. \tag{4.6}$$

We have used here the translational invariance in a homogeneous field (cf. (1.15), I). If we take into account the contribution of two, three, etc. electron loops, then the total amplitude of the vacuum-vacuum transition is e^{iL}.[10]

The fact that the trace of an odd number of γ matrices vanishes makes it possible to carry out the transformation

$$\mathrm{Sp}\left[\ln\frac{\hat{P}-m}{\hat{p}-m}\right] \to \frac{1}{2}\mathrm{Sp}\left[\ln\frac{\hat{P}^2-m^2}{\hat{p}^2-m^2}\right], \tag{4.7}$$

which yields a representation that is more convenient for the calculation

$$\mathscr{L} = \frac{i}{2(2\pi)^4} \mathrm{Sp} \int_0^\infty \frac{ds}{s} \langle 0|e^{i(\hat{P}^2-m^2)s}|0\rangle. \tag{4.8}$$

The mean value in (4.8) was calculated above (see (2.42)), where it is necessary to put $k_i = 0$ and $p_n = s$; then $B(\alpha) = 0$, i.e.,

$$\mathrm{Sp}\langle 0|e^{i\hat{P}^2 s}|0\rangle = \langle 0|e^{iP^2 s}|0\rangle \mathrm{Sp}\, e^{ie\sigma F s/2} = -4\pi^2 i\Phi(s)\cos(eHs)\,\mathrm{ch}(eEs) = -4\pi^2 ie^2 EH \,\mathrm{ctg}\,(eHs)\,\mathrm{cth}\,(eEs). \tag{4.9}$$

When substituting (4.9) in (4.8) it is necessary to subtract the first two terms of the expansion of the expression (4.9) as E, H → 0;

$$-\frac{4i\pi^2}{s^2}\left[1 + \frac{e^2 s^2}{3}(E^2-H^2)\right]. \tag{4.10}$$

The first of them corresponds to the subtraction term as F → 0 in (4.4). The second appears in the renormalization of the charge (and in the associated change of the scale of the fields)[8]. As a result we have

$$\mathscr{L}^{(1/2)} = \frac{1}{8\pi^2}\int_0^\infty \frac{ds}{s^3} e^{-is(m^2-i\varepsilon)} \cdot \left[s^2e^2EH\,\mathrm{ctg}(eHs)\,\mathrm{cth}(eEs) - 1 - \frac{s^2e^2}{3}(E^2-H^2)\right]. \tag{4.11}$$

A similar analysis for particles with zero spin yields

$$\mathscr{L}^{(0)} = -\frac{1}{16\pi^2}\int_0^\infty \frac{ds}{s^3} e^{-is(m^2-i\varepsilon)} \left[\frac{e^2s^2EH}{\sin(eHs)\,\mathrm{sh}(eEs)} - 1 + \frac{s^2e^2}{6}(E^2-H^2)\right]. \tag{4.12}$$

The functions $\mathscr{L}^{(0)}$ and $\mathscr{L}^{(1/2)}$ are the effective Lagrangians (corrections to the Lagrangian of the electromagnetic field as a result of the polarization of the vacuum). The result (4.11) was obtained by Heisenberg and Euler[11] with the aid of a different approach and by Schwinger[8], who also calculated (4.12) starting with a different initial formulation. In accordance with the meaning of the function $\mathscr{L}$, the probability of pair production in a unit 4-volume is 2 Im $\mathscr{L}$. The last quantity can be easily calculated by rotating the contour of integration in (4.11) in (4.12) through $-\pi/2$ and evaluating the integral with the aid of residue theory:

$$W^{(1/2)} = 2\,\mathrm{Im}\,\mathscr{L}^{(1/2)} = \frac{e^2EH}{4\pi^2}\sum_{n=1}\frac{1}{n}\mathrm{cth}\left(\frac{\pi nH}{E}\right)e^{-\pi nm^2/eE} \tag{4.13}$$

and

$$W^{(0)} = 2\,\mathrm{Im}\,\mathscr{L}^{(0)} = \frac{e^2EH}{8\pi^2}\sum_{n=1}^\infty\frac{(-1)^{n-1}}{n}e^{-\pi nm^2/eE}\Big/\mathrm{sh}\left(\frac{\pi nH}{E}\right). \tag{4.14}$$

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APPENDIX

Let us derive formula (2.45). To this end we rewrite its left-hand side in the form

$$\gamma^{\mu} e^{ie\sigma Fs/2} = (e^{eFs})_{\nu}{}^{\mu} m^{\nu}(s), \tag{A.1}$$

where

$$m^{\nu}(s) = e^{ie\sigma Fs/4} \gamma^{\nu} e^{ie\sigma Fs/4}.$$

We have used here relation (2.3). Differentiating $m^{\nu}(s)$ with respect to s, we obtain the equation

$$dm^{\nu}(s)/ds = ie\gamma^{5}(F^{*})_{\lambda}{}^{\nu} m^{\lambda}(s) \tag{A.2}$$

with the obvious initial condition $m^{\nu}(0) = \gamma^{\nu}$. The solution of (A.2) takes the form

$$m^{\nu}(s) = (e^{ie\gamma^{5}F^{*}s}\gamma)^{\nu}. \tag{A.3}$$

It can be rewritten also as

$$m^{\nu}(s) = i\gamma^{5}[\sin(eF^{*}s)\gamma]^{\nu} + [\cos(eF^{*}s)\gamma]^{\nu}. \tag{A.4}$$

Substituting (A.4) in (A.1), we obtain (2.45).

We calculate also one of the traces which appear in the analysis of the polarization operator of spinor particles:

$$\pi^{\nu\mu} = \frac{1}{4}\mathrm{Sp}(\gamma^{\nu}\gamma^{\mu}e^{ie\sigma Fs/2}) = \frac{1}{4}\mathrm{Sp}[\gamma^{\nu}(e^{eFs})_{\lambda}{}^{\mu} m^{\lambda}(s)]. \tag{A.5}$$

Substituting here the expression (A.4) for $m^{\lambda}(s)$, we get

$$\pi^{\nu\mu} = \frac{1}{4}\mathrm{Sp}\{\gamma^{\nu}(e^{eFs})_{\nu}{}^{\mu}[i\gamma^{5}(\sin(eF^{*}s)\gamma)^{\lambda} + (\cos(eF^{*}s)\gamma)^{\lambda}]\} = (e^{eFs})_{\lambda}{}^{\mu}(\cos(eF^{*}s))^{\lambda\nu}. \tag{A.6}$$

---

1)We use matrix notation.

2)Here and below we use the tensors B and C introduced in I (see (A.3) and (A.8)).

3)It is convenient to use an expansion of $b^{(0)}_{(i)\mu}$ in terms of four mutually orthogonal vectors: $(C^2k)_\mu$, $(Ck)_\mu$, $(B^2k)_\mu$, $(Bk)_\mu$.

4)For the case of particles with spin ½, this question was discussed by Shabad [4].

5)When the finite level widths are taken into account, the divergence of these points should vanish.

6)This problem was considered by Batalin and Shabad [7], who used the explicit form obtained by Schwinger [8] for the Green's function of an electron in a field.

7)A different treatment is contained in [8].

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