Relaxation of the electron distribution in a parametrically unstable plasma located in a strong electromagnetic field

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It is shown that under conditions of resonance between the frequency of a high-intensity ($E_0^2 \gg 4\pi n_e T_e$) external radiation and the electron Langmuir frequency an anisotropy arises in the velocity distribution of the electrons. In a definite interval of time the anisotropy is characterized by an excess of the energy of the transverse (to the electric vector of the pump field) motion of the electrons over the energy of their longitudinal motion, with allowance for the oscillations in the pump wave. Allowance for the influence of the redistribution of the electrons on the development of parametric instability permits the determination of the law of variation of the longitudinal and transverse electron energies and the revelation of the causes leading, after a sufficiently long period of time, to the formation of the inverse anisotropy. The time-dependent electron distribution function is found in the limit of a strongly pronounced excess of the longitudinal energy over the transverse energy of the electrons, a situation which corresponds to one-dimensional relaxation.

INTRODUCTION AND A BRIEF SUMMARY

One of the most important problems arising in the investigation of the parametric action of powerful radiation on a plasma is the problem of the determination of the laws characterizing the distribution of the particles in the plasma during the development of parametric instability[1]. It must be noted at the same time that in spite of the rapid accumulation of experimental data, little theoretical investigation on this problem is being performed at present. Particularly little attention has been given to the subject of the present article—the investigation of plasma-particle distributions under conditions when the plasma is under the influence of a strong pump-wave field in which the oscillation velocity of the electrons is high compared to their thermal velocity. The principles of the theory of plasma-electron distributions under such conditions have been formulated in[2] (see also the book[3]). For a concrete analysis of the particle distributions, it is important that in the definite time interval during which the parametric instability develops the motion of the particles differ little from the oscillatory motion in the pump field and the phase velocities of the growing high-frequency plasma fluctuations be comparatively high[1,4]. This allows us, as has been shown in[5] (see also[3]), to make definite qualitative assertions about the velocity redistribution of the particles under conditions when parametric turbulence develops.

At the same time, as is shown in the present communication, allowance for the influence of the plasma fluctuations on the electron motion leads to the important effect wherein an anisotropy appears in the electron distribution. It then turns out to be possible to follow the variation of such an anisotropy, which, in a definite time interval, is characterized by an excess of the energy of the motion of the electrons across the electric vector of the pump field over the value of the energy of their longitudinal motion with allowance for the oscillations in the pump wave. Causes that lead to the inverse anisotropy appear after a sufficiently long interval of time. The time-dependent electron distribution function describing the reversible—in time—pulsations in velocity space with irreversible increase of the amplitude of such pulsations is found in the limit of a strongly pronounced excess of the longitudinal energy over the transverse energy of the electrons, a situation which corresponds to one-dimensional relaxation. The found quantitative laws of time variation of the distribution of the main bulk of the electrons of a parametrically unstable plasma indicate that an experimental investigation of the time evolution of parametric turbulence is particularly urgent.

1. THE BASIC EQUATIONS

It is well known that a plasma, when acted on by sufficiently powerful electromagnetic radiation, becomes parametrically unstable with respect to the excitation of electric-field potential perturbations. In this case the investigation of the instability is possible under a wide range of conditions in the framework of the model of a homogeneous unbounded plasma located in a spatially-homogeneous external electric field $\mathbf{E}(t)$[1,3].

To study the evolution of the distribution function $F_a(\mathbf{p}_a, t)$ of the type-a particles under the action of parametrically excited—in the plasma—fields (described by the potential $\varphi_{\mathbf{k}}(t)$), we can use the following system of equations[2]:

$$\frac{\partial F_a(\mathbf{p}_a,t)}{\partial t} = e_a^2 \frac{\partial}{\partial p_{ai}} \int \frac{d\mathbf{k}}{(2\pi)^3} k_i k_j \varphi_{-\mathbf{k}}(t) \times \int_{-\infty}^{0} d\tau \frac{\partial F_a(\mathbf{p}_a, t+\tau)}{\partial p_{aj}} \varphi_{\mathbf{k}}(t+\tau) e^{i\psi_a}, \tag{1.1}$$

$$\varphi_{\mathbf{k}}(t) = \sum_a \frac{4\pi e_a^2}{k^2} \int_{-\infty}^{0} d\tau\, \varphi_{\mathbf{k}}(t+\tau) \int d\mathbf{p}_a i\mathbf{k} \frac{\partial F_a(\mathbf{p}_a, t+\tau)}{\partial \mathbf{p}_a} e^{i\psi_a}, \tag{1.2}$$

where

$$F_a(\mathbf{p}_a, t) = f_a(\mathbf{p}_a + e_a \int_{-\infty}^{t} \mathbf{E}(\tau)\, d\tau, t)$$

is the distribution function in an oscillating coordinate system that eliminates the motion of the particles under the action of the external pump field $\mathbf{E}(t)$ ($f_a(\mathbf{p}_a, t)$ is the distribution function of the type-a particles in the laboratory coordinate system);

$$\psi_a = (\mathbf{k}, \mathbf{v}_a\tau + \frac{e_a}{m_a}\tau \int_{-\infty}^{t} \mathbf{E}(t')\, dt' + \frac{e_a}{m_a} \int_{t+\tau}^{t} (t'-t-\tau)\mathbf{E}(t')\, dt'),$$

e_a, m_a, and $\mathbf{v}_a = \mathbf{p}_a/m_a$ are the charge, mass, and velocity of the type-a particles. In deriving Eqs. (1.1) and (1.2), the external field $\mathbf{E}(t)$ was assumed to be much stronger than the fields excited in the plasma, whose nonlinear interaction has been neglected. Notice, however, that Eqs. (1.1) and (1.2) are suitable for the description of that state of the plasma which, from the point of view of the linear theory, is unstable and, consequently, is characterized by the excitation of large-amplitude oscillations significantly exceeding the thermal fluctuations in intensity.

The time dependence of the external periodic pump field of frequency ω_0 is given by

$$\mathbf{E}(t)=\mathbf{E}_0\sin\omega_0 t.$$

It is then expedient for the investigation of Eqs. (1.1) and (1.2) to expand the distribution functions $F_a(\mathbf{p}_a, t)$ and the spatial Fourier harmonics of the potential $\varphi_{\mathbf{k}}(t)$ of the field excited in the plasma in terms of the harmonics of the external-field frequency:

$$F_a(\mathbf{p}_a,t)=\sum_{n=-\infty}^{\infty}F_a^{(n)}(\mathbf{p}_a,t)e^{-in\omega_0 t}$$
$$\varphi_{\mathbf{k}}(t)=\sum_{n=-\infty}^{\infty}\varphi^{(n)}(\mathbf{k},t)e^{-in\omega_0 t}, \qquad (1.3)$$

where the amplitudes $F_a^{(n)}(\mathbf{p}_a, t)$ and $\varphi^{(n)}(\mathbf{k}, t)$ of the harmonics vary little over the period of the external field.

It is convenient to represent the weak time dependence of the amplitudes of the potential in the following form:

$$\varphi^{(n)}(\mathbf{k},t)=\sum_r\varphi_n(\mathbf{k},\omega_r)\exp\left[-i\int_0^t\omega_r(\mathbf{k},t')dt'+\int_0^t\gamma_r(\mathbf{k},t')dt'\right]$$
$$\varphi_n(\mathbf{k},\omega_r)=\sum_{m=-\infty}^{\infty}J_{m-n}(z)\Phi_m^{(r)}, \qquad (1.4)$$

where the sum over r is extended to all the roots of the dispersion equation with a given wave vector $\mathbf{k}$; $J_m(z)$ is a Bessel function of order m and of argument $z = (\mathbf{k}, \mathbf{r}_E)$ characterized by the displacement $\mathbf{r}_E = e\mathbf{E}_0/m_e\omega_0^2$ of the electrons under the action of the external field.

Below we shall focus our attention on the study of the relaxation of the distribution function of the electrons of a plasma located in a strong electromagnetic field (whose energy is substantially greater than the thermal energy of the electrons) under conditions of resonance between the external-radiation frequency ω_0 and the electron Langmuir frequency $\omega_{Le} = (4\pi e^2 n_e/m_e)^{1/2}$. Under these conditions, as is well known[1,4], the phase velocity of the most intensely excitable oscillations is high compared to the thermal electron velocity, which allows us, in investigating Eq. (1.2), to neglect the Cerenkov absorption of the growing oscillations and to take their spatial dispersion into account with the aid of perturbation theory (restricting ourselves to the second moment of the electron distribution function). Then for a plasma with one type of ions ($m_i \gg m_e$), assuming that the distribution function of the electrons over velocity is an even function, and taking the slowness of the variation in time of the frequency $\omega_r(\mathbf{k}, t)$ and the increment $\gamma_r(\mathbf{k}, t)$ into account, we can derive from Eq. (1.2) the following system of equations for the amplitudes $\Phi_m^{(r)}$ of the harmonics:

$$R_m\Phi_m^{(r)}+\sum_{\substack{l=-\infty\\ l\neq 0}}^{\infty}\mathscr{E}_m^{(l)}\Phi_{m-l}^{(r)}=-\delta\varepsilon_i(\omega_r+i\gamma_r)J_m(z)\varphi_0(\mathbf{k},\omega_r), \qquad (1.5)$$

where

$$R_m\equiv R(m\omega_0+\omega_r+i\gamma_r),\ R(\omega)=1-\omega_{Le}^2/\omega^2+\mathscr{E}^{(0)}(\omega),$$
$$\mathscr{E}_m^{(l)}\equiv\mathscr{E}^{(l)}(m\omega_0+\omega_r+i\gamma_r),\ \mathscr{E}^{(l)}(\omega)=-3\frac{\omega_{Le}^2}{\omega^2}\frac{k_ik_jM_{ij}^{(l)}(t)}{\omega^2}$$
$$M_{ij}^{(l)}(t)=\int d\mathbf{p}_e v_i v_j F_e^{(l)}(\mathbf{p}_e,t) \qquad (1.6)$$

being the second moment of the l-th harmonic of the electron distribution function; $\delta\epsilon_i(\omega) = -\omega_{Li}^2/\omega^2$ is the contribution of the ions to the ordinary linear permittivity of the plasma ($\omega_{Li}^2 = 4\pi e_i^2 n_i/m_i$).

With the aid of these equations it is easy to establish that in the case being discussed here, of resonance between the external-radiation frequency and the electron Langmuir frequency, when

$$|\omega_0-\omega_{Le}|\ll\omega_0 \qquad (1.7)$$

(i.e., when $|R_{\pm1}| \ll 1$ and $|R_n| \approx 1$ for $n \neq \pm1$), only the two amplitudes $\Phi_{\pm1}^{(r)}$, which are expressible in terms of the zeroth-order harmonic of the perturbation-field potential in the following manner

$$\Phi_{\pm1}^{(r)}=\mp J_1(z)\delta\varepsilon_i(\omega_r+i\gamma_r)\frac{R_{\mp1}+\mathscr{E}_{\pm1}^{(\pm2)}}{R_{-1}R_1-\mathscr{E}_{-1}^{(-2)}\mathscr{E}_1^{(2)}}\varphi_0(\mathbf{k},\omega_r), \qquad (1.8)$$

are large.

Substituting these expressions into the right-hand side of the formula (1.4) for $n = 0$, we obtain a dispersion equation for the spectrum and growth rate of the perturbations:

$$\frac{1}{\delta\varepsilon_i(\omega_r+i\gamma_r)}+J_1^2(z)\frac{R_1+R_{-1}+\mathscr{E}_1^{(2)}+\mathscr{E}_{-1}^{(-2)}}{R_1R_{-1}-\mathscr{E}_1^{(2)}\mathscr{E}_{-1}^{(-2)}}=0. \qquad (1.9)$$

The relations (1.3) and (1.4) allow us to write in place of the Eqs. (1.1) the following system of equations for the harmonics $F^{(n)}(\mathbf{v}, t) \equiv F_e^{(n)}(m_e\mathbf{v}, t)$ of the electron distribution function:

$$\left(\frac{\partial}{\partial t}-in\omega_0\right)F^{(n)}(\mathbf{v},t)=\sum_{l=-\infty}^{\infty}\frac{\partial}{\partial v_i}D_{ij}^{(n)}(l)\frac{\partial}{\partial v_j}F^{(l)}(\mathbf{v},t), \qquad (1.10)$$

where

$$D_{ij}^{(n)}(l)=\frac{e^2}{m_e^2}\int\frac{d\mathbf{k}}{(2\pi)^3}k_ik_j\sum_{u,r}\exp\left\{-i\int_0\left[\omega_u(\mathbf{k},t')+\omega_r(-\mathbf{k},t')\right]dt'\right.$$
$$\left.+\int_0^t[\gamma_u(\mathbf{k},t')+\gamma_r(\mathbf{k},t')]dt'\right\}\sum_{s=-\infty}^{\infty}\Phi_{s-l}^{(u)}[\Phi_{s-n}^{(r)}]^*L(s),$$
$$L(s)=\left(1+i\frac{\partial}{\partial\omega_u}\frac{\partial}{\partial t}\right)\frac{i}{\omega_u+i\gamma_u+s\omega_0-\mathbf{kv}}. \qquad (1.11)$$

Notice that in the case of interest to us, when high-intensity oscillations are excitable in the plasma, the slow (in comparison with the time scale $2\pi/\omega_0$) variation of the harmonics of the distribution functions is wholly determined by exponentially (with increment γ) growing perturbations in the electric field. Therefore, in Eqs. (1.10), which are correct to the first time derivatives of the slowly varying quantities, the time derivatives are as important as the terms proportional to the increment γ (see (1.10)), and are decisive for, for example, the zeroth-order harmonic ($n = 0$).

Further, we shall consider the consequences of Eqs. (1.10) under conditions when $\omega_0 \lesssim \omega_{Le}$ and the parametrically excited low-frequency perturbations grow aperiodically in time: $\omega_u = 0$ and $\gamma_u(\mathbf{k}, t) = \gamma(\mathbf{k}, t)$. As was shown above, under the conditions of the resonance (1.7), only the two amplitudes (1.8) are large. When only these two amplitudes are taken into account, the system (1.10) splits up into two systems of equations respectively for the even and odd harmonics:

$$\left(\frac{\partial}{\partial t}-in\omega_0\right)F^{(n)}(\mathbf{v},t)=\frac{e^2}{m_e^2}\frac{\partial}{\partial v_i}\int\frac{d\mathbf{k}}{(2\pi)^3}k_ik_j\exp\left[2\int_0^t\gamma(\mathbf{k},t')\,dt'\right]$$
$$\times\left\{\left[|\Phi_1|^2L(n+1)+|\Phi_{-1}|^2L(n-1)\right]\frac{\partial F^{(n)}(\mathbf{v},t)}{\partial v_j}\right.$$
$$\left.+\Phi_1\Phi_{-1}^*L(n-1)\frac{\partial F^{(n-2)}(\mathbf{v},t)}{\partial v_j}+\Phi_{-1}\Phi_1^*L(n+1)\frac{\partial F^{(n+2)}(\mathbf{v},t)}{\partial v_j}\right\}. \quad (1.12)$$

The parameter that allows the splitting up of the system of equations (1.10) into the two systems (1.12) is the parameter indicating the smallness of the amplitudes Φ_m, with $m \neq \pm 1$, in comparison with the amplitudes (1.8). As is easy to see from the Eqs. (1.5), such a parameter under the conditions (1.7) is the ratio of the maximum increment γ_{max} of the instability to the external-radiation frequency ω_0:

$$x_0\equiv\frac{\gamma_{max}}{\omega_0}\approx\left(\frac{m_e}{m_i}\right)^{1/3}\ll 1.$$

Since at the initial moment of time the higher harmonics ($F^{(n)}(\mathbf{v}, t = 0)$, $n \neq 0$) of the distribution function are negligibly small[3], the odd harmonics are smaller than the corresponding even harmonics by factors characterized by the same smallness parameter. We therefore investigate below the even harmonics of the electron distribution function. Limiting ourselves to the region of electron velocities lower than the phase velocity of the excited high-frequency oscillations (i.e., to the region where $v < \omega_0/k$), we use for the operator (1.11) the following approximate expression:

$$L(s)=\frac{i}{s\omega_0}+\frac{1}{s^2\omega_0^2}\left(\gamma+\frac{\partial}{\partial t}\right).$$

Under these assumptions, we obtain from the system of equations (1.12) for the even harmonics the following system describing the relation of the electron distribution function ($n = 0, \pm 1, \pm 2, \ldots$);

$$\left\{\frac{\partial}{\partial t}-i2n\omega_0-\left[\frac{4n^2+1}{(4n^2-1)^2}\left(\frac{d\Delta_{ij}^{(0)}}{dt}+2\Delta_{ij}^{(0)}\frac{\partial}{\partial t}\right)\right.\right.$$
$$\left.\left.+i\frac{4n\omega_0}{4n^2-1}\Delta_{ij}^{(0)}\right]\frac{\partial^2}{\partial v_i\partial v_j}\right\}F^{(2n)}(\mathbf{v},t)=$$
$$=\left[\frac{-\omega_0}{2n-1}\Delta_{ij}+\frac{i}{(2n-1)^2}\left(\frac{1}{2}\frac{d\Delta_{ij}}{dt}+\Delta_{ij}\frac{\partial}{\partial t}\right)\right]\frac{\partial^2F^{(2n-2)}(\mathbf{v},t)}{\partial v_i\partial v_j}$$
$$+\left[\frac{\omega_0}{2n+1}\Delta_{ij}^*-\frac{i}{(2n+1)^2}\left(\frac{1}{2}\frac{d\Delta_{ij}^*}{dt}+\Delta_{ij}^*\frac{\partial}{\partial t}\right)\right]\frac{\partial^2F^{(2n+2)}(\mathbf{v},t)}{\partial v_i\partial v_j}. \quad (1.13)$$

The general expressions for the integral diffusion coefficients $\Delta_{ij}^{(0)}$ and Δ_{ij}, which, according to (1.12), (1.8), and (1.6), depend on the second moments of the distribution function, are rather unwieldy. However, in the case under discussion here of resonance aperiodic instability[3], when

$$\omega_0-\omega_{Le}\approx-\gamma_{max}, \quad (1.14)$$

taking into account the fact that the dominant contribution to the integral over the wave numbers of the perturbations (see (1.12)) is made by the region corresponding to the maximum value of the increment, we can use for the diffusion coefficient the following approximate expression:

$$\Delta_{ij}^{(0)}=\Delta_{ij}=\frac{1}{8}\frac{e^2}{m_e^2}\frac{|\delta\varepsilon_i(i\gamma_{max})|^2}{\gamma_{max}^2}$$
$$\times\int\frac{d\mathbf{k}}{(2\pi)^3}k_ik_jJ_1^2(z)|\varphi_0(\mathbf{k})|^2\exp\left[2\int_0^t\gamma(\mathbf{k},t')\,dt'\right], \quad (1.15)$$

in which the dependence of the increment on the wave numbers (and, consequently, on the moments of the distribution function) is taken into account only in the most rapidly varying exponential factors.

Equations (1.13) and (1.15) together with the dispersion equation (1.9) that determines the increment $\gamma(\mathbf{k}, t)$ constitute a closed system of equations describing the quasilinear relaxation of the electron distribution in the considered region of velocity space under conditions of resonance aperiodic parametric instability.

2. MOMENTS OF THE DISTRIBUTION FUNCTION

Being interested in the asymptotic expressions that arise when the fluctuation level greatly exceeds the initial (e.g., thermal) level, we shall neglect the initial values of the amplitudes of the higher harmonics of the distribution function and of the diffusion coefficients (1.15):

$$F^{(2n)}(\mathbf{v},t)=0,\ t=0,\ n\neq 0,$$
$$\Delta_{ij}(t=0)=0. \quad (2.1)$$

If in the spectrum of the initial perturbations $|\varphi_0(\mathbf{k})|^2$ there are no preferred directions not coinciding with the direction of the external pump field $\mathbf{E}_0$, then in the coordinate system with $Oz \parallel \mathbf{E}_0$, the nonzero components of the tensor (1.15) are the diagonal ones ($\Delta_i(t) \equiv \Delta_{ii}(t)$):

$$\Delta_1(t)=\Delta_2(t)\equiv\Delta_\perp(t),\ \Delta_3(t)\equiv\Delta_\parallel(t). \quad (2.2)$$

In this case it is easy to derive for the moments of the electron distribution function

$$F_e(\mathbf{v},t)=F^{(0)}(\mathbf{v},t)+2\sum_{n=1}^{\infty}\{\operatorname{Re}F^{(2n)}(\mathbf{v},t)\cos 2n\omega_0t$$
$$+\operatorname{Im}F^{(2n)}(\mathbf{v},t)\sin 2n\omega_0t\}$$

with the aid of the Eqs. (1.13) the following expressions, which are valid up to small terms of order $x_0 = \gamma_{max}/\omega_0$:

$$\int F_e(\mathbf{v},t)\,d\mathbf{v}=\int F^{(0)}(\mathbf{v},t)\,d\mathbf{v}=n_e=\text{const},$$
$$\int v_i^2F_e(\mathbf{v},t)\,d\mathbf{v}=\mu_i+2n_e\Delta_i(t)\,[1-\sin 2\omega_0t],$$
$$\int v_i^2v_j^2F_e(\mathbf{v},t)\,d\mathbf{v}=\mu_{ij}+2[\Delta_i(t)\,\mu_j(1+4\delta_{ij})+\Delta_j(t)\,\mu_i]$$
$$\times[1-\sin 2\omega_0t]+{}^4/_3n_e\Delta_i(t)\,\Delta_j(t)\,(1+2\delta_{ij})\,[1-\sin 2\omega_0t]^2, \quad (2.3)$$

where

$$\mu_i=\int v_i^2F_e(\mathbf{v},t=0)\,d\mathbf{v},\quad \mu_{ij}=\int v_i^2v_j^2F_e(\mathbf{v},t=0)\,d\mathbf{v}.$$

Since the contribution of the spontaneous emission is neglected in the basic equations (1.1) and (1.2) and in the initial conditions (2.1), the small corrections of the order of the ratio of the energy of the initial fluctuations in the field to the thermal energy of the plasma electrons are also not taken into account in the formulas (2.3).

It follows from these formulas that a rapidly oscillating (with the frequencies of the even harmonics of the external pump field) velocity distribution of the electrons appears in the hydrodynamic phase of the development of the resonance parametric instability (when the coefficients $\Delta_i(t)$ exponentially increase in time). Such oscillations are reversible: the moments of the distribution function (and, consequently, the function itself) assume their initial values twice in the period of the external field (when $2\omega_0t = \pi/2 + 2\pi k$). The growth of the average (over the period of the external field) electron energy is then determined by the energy of the growing longitudinal field of the plasma perturbations. Indeed, according to (2.3) and (1.15)

$$\int \frac{m_e \mathbf{v}^2}{2} [\overline{F_e(\mathbf{v},t)}^{2\pi/\omega_0} - F_e(\mathbf{v}, t=0)] d\mathbf{v} = m_e n_e \operatorname{Sp} \Delta(t)$$
$$\approx \frac{3}{2} \int \frac{d\mathbf{k}}{(2\pi)^3} \frac{k^2 |\varphi_0(\mathbf{k})|^2}{4\pi} \exp\left[2\int_0^t \gamma(\mathbf{k}, t') dt' \right]. \tag{2.4}$$

Thus, in order to determine the law according to which the mean energy increases and the degree of anisotropy in the electron distribution arising in the plasma, it is necessary to find the longitudinal and transverse diffusion coefficients (2.2), which, according to (1.15) and (1.9), are determined by the following system of integral equations:

$$\Delta_\parallel(t) = \frac{3}{2} \frac{1}{n_e m_e} \int \frac{d\mathbf{k}}{(2\pi)^3} \frac{k_\parallel^2 |\varphi_0(\mathbf{k})|^2}{4\pi} \exp\left[2\int_0^t \gamma(\mathbf{k}, t') dt' \right], \tag{2.5}$$

$$\Delta_\perp(t) = \frac{3}{2} \frac{1}{n_e m_e} \int \frac{d\mathbf{k}}{(2\pi)^3} \frac{k_\perp^2 |\varphi_0(\mathbf{k})|^2}{4\pi} \exp\left[2\int_0^t \gamma(\mathbf{k}, t') dt' \right], \tag{2.6}$$

$$\gamma(\mathbf{k}, t) = \frac{\omega_0}{\sqrt{2}} \{ \sqrt{[d^2-\xi^2]^2 - 4Ad} - [d^2-\xi^2] \}^{1/2}, \tag{2.7}$$

where

$$d = \Delta_0 - 2\xi_T, \quad \xi_T = \xi(t) + {}^3/_4 k^2 r_{D0}^2, \quad r_{D0} = v_{T0}/\omega_{Le}$$

is the Debye radius of the initial electron distribution ($v_{T0} = \sqrt{T_{e0}/m_e}$),

$$\xi = \xi(t) = \frac{3}{2} \frac{k_i k_j \Delta_{ij}(t)}{\omega_0^2} \equiv \frac{3}{2} \left[\frac{k_\parallel^2 \Delta_\parallel(t)}{\omega_0^2} + \frac{k_\perp^2 \Delta_\perp(t)}{\omega_0^2} \right], \tag{2.8}$$
$$\Delta_0 = (\omega_0 - \omega_{Le})/\omega_0, \quad |\Delta_0| \ll 1,$$
$$A = A(z) = J_1^2(z) \omega_{Li}^2/\omega_{Le}^2, \quad z = \mathbf{k}\mathbf{r}_E = k_\parallel v_E/\omega_0,$$

$v_E = eE_0/m_e\omega_0$ is the amplitude of the velocity of the electron oscillations in the pump-wave field.

Let us consider the solutions of the system (2.5)—(2.7) at large t, when the electron energy (2.4) due to the potential field excited in the plasma substantially exceeds the initial thermal energy:

$$t > t_T, \quad \xi_T(t_T) \approx \xi(t_T). \tag{2.9}$$

In this case for the detunings Δ_0 corresponding to the condition (1.14), the increment (2.7) can, in the vicinity of the maximum value γ_{max}, be written in the following form:

$$\gamma(\mathbf{k}, t) = \omega_0 \left\{ x_0 - \frac{1}{2x_0} [\xi(t) - \xi_0]^2 - \frac{x_0}{3} \left(1 - \frac{1}{z_0^2}\right) (z - z_0)^2 \right\}, \tag{2.10}$$

where

$$\gamma_{max} \equiv x_0 \omega_0 = \left[\frac{A(z_0)}{2} \right]^{1/3} \omega_0 \approx 0.55 \left[\frac{e_i}{e} \frac{m_e}{m_i} \right]^{1/3} \omega_0 \tag{2.11}$$

is the maximum value of the increment, which occurs at

$$z = z_0 \approx 1.84, \quad \xi(t) = \xi_0 = {}^2/_9 (x_0 + \Delta_0) \ll x_0. \tag{2.12}$$

Notice that the simultaneous satisfaction of the conditions (2.9) and (2.12) is possible only in a very strong external pump field whose energy substantially exceeds the thermal energy of the initial electron distribution:

$$\frac{E_0^2}{4\pi n_e T_{e0}} > \xi_0^{-1} > x_0^{-1} \approx \left(\frac{m_i}{m_e} \right)^{1/3}. \tag{2.13}$$

Restricting ourselves to precisely such a case of extremely strong external fields, we investigate the expressions (2.5) and (2.6) for the longitudinal and transverse diffusion coefficients at times not later than the period of the hydrodynamic phase of the development of the parametric instability, when the maximum increment does not depend on the higher moments of the electron distribution function and is determined by the expression (2.11). According to (2.10), the increment of the instability under consideration differs little from the hydrodynamic expression (2.11) when $\xi(t) < x_0$ and $z \approx z_0$, i.e., at times defined by the following condition:

$$t < t^*, \quad \Delta_\parallel(t^*) \approx \frac{2}{3} \frac{x_0}{z_0^2} v_E^2 \approx 0.1 \left(\frac{e_i}{e} \frac{m_e}{m_i} \right)^{1/3} v_E^2. \tag{2.14}$$

Let us first consider the solutions to the Eqs. (2.5), (2.6), and (2.10) in the case when the initial spectrum of the perturbations is isotropic and, for instance, thermal:

$$\frac{k^2 |\varphi_0(k)|^2}{4\pi} = T_{e0}. \tag{2.15}$$

Bearing in mind the conditions (2.9) and (2.14), we can easily evaluate the time and perturbation-wave-vector integrals on the right-hand sides of the formulas (2.5) and (2.6) if we assume the dependence on the time of the quantity $\xi(t)$, (2.8), to be exponential:

$$\frac{1}{\gamma_{max} \xi(t)} \frac{d\xi(t)}{dt} \approx 1.$$

It is easy to verify with the aid of the results obtained below that this equality is, in fact, asymptotically valid at times $t > t_T$, (2.9) ($t_T \gamma_{max} \gg 1$), of interest to us. The relations resulting in this case from (2.5) and (2.6) for the integral diffusion coefficients can be written in the following form:

$$\Delta_\parallel(t) = v_{T0}^2 B \frac{e^{2\gamma_{max} t}}{\sqrt{\gamma_{max} t}} \ln \left(\frac{x_0 v_E^2}{z_0^2 \Delta_\perp(t)} \right), \tag{2.16}$$

$$\Delta_\perp(t) = v_{T0}^2 B \frac{e^{2\gamma_{max} t}}{\sqrt{\gamma_{max} t}} \frac{x_0 v_E^2}{z_0^2 \Delta_\perp(t)}, \tag{2.17}$$

where

$$B = \frac{z_0^3}{16\pi} \sqrt{\frac{3}{z_0^2 - 1}} \frac{1}{n_e r_E^3} \approx \frac{1}{7} \frac{1}{n_e r_E^3}.$$

These expressions show that $x_0 v_E^2 > z_0^2 \Delta_\perp(t)$ in the time interval defined by the condition (2.14), and, consequently, in the case of an isotropic initial perturbation spectrum the transverse diffusion coefficient $\Delta_\perp(t)$ exceeds the longitudinal coefficient $\Delta_\parallel(t)$ in the hydrodynamic phase of the development of the instability.

Therefore, the relaxation of the electron distribution function at the hydrodynamic stage, (2.14), occurs in the following manner: Beginning from the moment of time $t \approx t_T$ (2.9):

$$t_T = \frac{1}{2\gamma_{max}} \ln \left(\frac{v_{T0}^2}{v_E^2 x_0} n_e r_E^3 \right), \tag{2.18}$$

when the energy of the electrons that is due to their motion under the action of the field of the parametrically excited oscillations in the plane perpendicular to $\mathbf{E}_0$ and that is determined by the transverse diffusion coefficient

$$\Delta_\perp(t) = \frac{1}{3} v_E v_{T0} \left[2\pi \left(\frac{e}{e_i} \frac{m_i}{m_e} \right)^{1/3} n_e r_E^3 \right]^{-1/2} \frac{e^{\gamma_{max} t}}{(\gamma_{max} t)^{1/4}} \tag{2.19}$$

exceeds the initial thermal energy T_{e0}, the initially isotropic electron distribution function becomes anisotropic. The anisotropy of the distribution function increases as the transverse diffusion coefficient (2.19) increases as long as the longitudinal coefficient (2.16) remains less than $v_{T0}^2 = T_{e0}/m_e$ and the energy of the electron motion along $\mathbf{E}_0$ is determined, as before, by the initial electron temperature T_{e0}. In this case the maximum anisotropy of the distribution is attained at $t \approx t_a$, where

$$t_a = \frac{1}{2\gamma_{max}} \ln(n_e r_E^3), \tag{2.20}$$

when the longitudinal diffusion coefficient

$$\Delta_\parallel(t)=v_{T0}^2\frac{2}{3\pi}\frac{1}{n_e r_E^3}\frac{e^{2\gamma_{max}t}}{\sqrt{\gamma_{max}t}}\ln\left(\frac{x_0 v_E^2}{z_0^2\Delta_\perp(t)}\right) \tag{2.21}$$

is comparable to the square of the initial thermal velocity:

$$\Delta_\parallel(t_a)\approx v_{T0}^2.$$

At this moment of time the ratio of the transverse to the longitudinal (relative to E_0) electron energy attains, according to (2.19) and (2.20), the following maximum value:

$$\frac{\Delta_\perp(t_a)}{\Delta_\parallel(t_a)}\approx\frac{v_E}{v_{T0}}\frac{1}{\pi}\left(\frac{e_i}{e}\frac{m_e}{m_i}\right)^{1/4}\left[\ln\left(\frac{v_E}{v_{T0}}\frac{1}{\pi}\left[\frac{e_i}{e}\frac{m_e}{m_i}\right]^{1/4}\right)\right]^{-1/2}, \tag{2.22}$$

which is determined by the ratio of the velocity of oscillation of the electrons in the pump-wave field to their initial thermal velocity.

Further, at $t > t_a$, the distribution function anisotropy begins to decrease rapidly, since the longitudinal diffusion coefficient (2.21) increases roughly exponentially in time with an exponent two times larger than the corresponding value for the transverse coefficient (2.19), and the electron distribution becomes nearly isotropic by the end of the hydrodynamic phase (i.e., at $t\approx t^*$), when the average—over the period of the pump field—value of either the transverse or the longitudinal electron energy reaches

$$m_e\Delta_\perp(t^*)\approx m_e\Delta_\parallel(t^*)\approx\frac{m_e v_E^2}{2}\frac{1}{\pi}\left[\frac{e_i}{e}\frac{m_e}{m_i}\right]^{1/2}. \tag{2.23}$$

Thus, in the case of an isotropic initial electron distribution and an isotropic initial-perturbation spectrum, the transverse energy of the electrons exceeds their longitudinal (relative to E_0) energy only at the definite stage of the quasilinear relaxation when $t\approx t_a$, (2.20).

The growth of the mean energy when $t > t^*$, where t^* is defined by (2.14) and is given by

$$t^*=\frac{1}{2\gamma_{max}}\ln\left(\frac{v_E^2 x_0}{v_{T0}^2}n_e r_E^3\right),$$

leads to a situation in which the maximum increment value (which is less than the value given by the hydrodynamic expression (2.11)) is attained for wave vectors parallel to E_0. This, in its turn, leads to a further relatively more rapid growth of the longitudinal energy $m_e\Delta_\parallel(t)$ of the electrons in comparison with the value of their transverse energy $m_e\Delta_\perp(t)$, giving rise to an anisotropy in the distribution function, but now with $\Delta_\parallel(t) > \Delta_\perp(t)$.

In the case when the initial spectrum of the perturbations is anisotropic (so that, for example, $|\varphi_0(\mathbf{k})|^2$ is different from zero only for $k_\perp < k_\parallel$), the longitudinal electron energy exceeds the transverse energy from the very beginning of the relaxation process, i.e., at $t > t_T$. The evolution of the electron distribution function is then of quasi-one-dimensional nature. The following section of the paper is devoted to the investigation of one-dimensional relaxation.

3. ONE-DIMENSIONAL RELAXATION

In the case when the initial perturbation spectrum is anisotropic (which can be caused by, for example, charged-particle beams), as well as at large $t > t^*$ and in the case of an isotropic initial perturbation spectrum, when the oscillations excited in the plasma propagate chiefly along the direction of the external pump field E_0, there occurs a one-dimensional relaxation of the electron distribution in the direction $0z \parallel E_0$ (since $\Delta_\parallel(t) \gg \Delta_\perp(t)$). In this case, as the expressions (2.3) for the moments show, the one-dimensional electron distribution function is determined in the following manner by its initial value $F_e(v_z, t = 0)$:

$$F_e(v_z,t)={}^1/_2[F_e(v_z-a(t),t=0)+F_e(v_z+a(t),t=0)], \tag{3.1}$$

where

$$a(t)=\sqrt{2\Delta_\parallel(t)[1-\sin\tau]},\quad \tau=2\omega_0 t. \tag{3.2}$$

That the distribution (3.1) is the solution to the system of equations (1.12), is most easily verified by neglecting the initial electron dispersion, when

$$F_e(v_z,\ t=0)=n_e\delta(v_z) \tag{3.3}$$

and the distribution function (3.1) can be written in the form:

$$F_e(w,t)=\frac{n_e}{|w|}\delta\left(\frac{w^2}{1-\sin\tau}-1\right), \tag{3.4}$$

where $w = v_z/\sqrt{2\Delta_\parallel(t)}$ is the dimensionless velocity, which (in the case (3.3) under consideration) is the sole self-similar variable for the slowly-varying—in time—harmonics of the distribution function (3.4). Computing directly the harmonics $\varphi^{(n)}(w)$ defined by

$$F_e(w,t)=\sum_{n=-\infty}^{\infty}\varphi^{(n)}(w)e^{-n\tau},$$

and substituting them into the equations

$$\left\{i2n+x_0\left(1+w\frac{d}{dw}\right)+\frac{d^2}{dw^2}\left[i\frac{2n}{4n^2-1}-x_0\frac{4n^2+1}{(4n^2-1)^2}w\frac{d}{dw}\right]\right\}\varphi^{(n)}(w)$$
$$=\frac{1}{2}\frac{d^2}{dw^2}\left\{\left[\frac{1}{2n-1}+\frac{ix_0}{(2n-1)^2}w\frac{d}{dw}\right]\varphi^{(n-1)}(w)\right.$$
$$\left.-\left[\frac{1}{2n+1}+\frac{ix_0}{(2n+1)^2}w\frac{d}{dw}\right]\varphi^{(n+1)}(w)\right\}, \tag{3.5}$$

which, in the one-dimensional case under consideration here, follow from Eqs. (1.12) when the coefficient $\Delta_\parallel(t)$ varies exponentially:

$$d\Delta_\parallel/d\tau=x_0\equiv\gamma_{max}/\omega_0,$$

we can easily verify that the distribution function (3.4) and, consequently, (3.1) indeed describe a one-dimensional quasilinear electron relaxation. Let us only note that a more exact (than in the formulas (2.3)) allowance for the terms of order x_0 in the Eqs. (3.5) leads to a small shift in the time τ defined in (3.2):

$$\tau=2\omega_0 t+2x_0.$$

It is clear that the distribution (3.1), like (3.4), to which the arbitrary distribution (3.1) asymptotically tends when $\Delta_\parallel(t) \gg T_{e0}$, possesses all the above-indicated general properties following from the expressions (2.3) for the moments. In particular, the formula (3.4) allows us to speak of an oscillating electron energy whose mean value increases in time in the same way as the energy of the internal perturbation potential field parametrically excited in the plasma during the instability.

[1]V. P. Silin, Zh. Eksp. Teor. Fiz. 48, 1679 (1965) [Sov. Phys.-JETP 21, 1127 (1965)].
[2]V. P. Silin, Zh. Eksp. Teor. Fiz. 57, 183 (1969) [Sov. Phys.-JETP 30, 105 (1970)].

[3]V. P. Silin, Parametricheskoe vozdeĭstvie izlucheniya bol'shoĭ moshchnosti na plazmy (The Parametric Action of High-Power Radiation on a Plasma), Nauka, 1973.
[4]N. E. Andreev, A. Yu. Kiriĭ, and V. P. Silin, Izv. Vyssh. Uchebn. Zaved. Radiofiz. 13, 1321 (1970) [Sov. Radiophys. 13, 1018 (1970)].
[5]V. P. Silin, Zh. Eksp. Teor. Fiz. 63, 1686 (1972) [Sov. Phys.-JETP 36, 891 (1973)].

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