Asymptotic relations between total and differential cross sections in crossing channels

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Asymptotic relations for the total and differential cross sections for the reactions $AB \to CD$ and $\bar{C}B \to \bar{A}D$ are derived by the same method by representing the amplitudes $f_{P,A}(s;\,t)$ of these reactions for a fixed $t \leq 0$ in the form of a complex, simple-layer potential. Different modifications of these relations, including the integral forms of the Pomeranchuk theorem, are given. An asymptotic representation for the amplitude phase shifts $\psi_{P,A}(s;\,t)$ at high energies is derived on the basis of analyticity and crossing-symmetry considerations [the formulas (57) and (58)]. It follows directly from these relations that a sufficient, but by no means necessary, condition for the asymptotic equality of the differential cross sections consists in the condition that $\psi_{P,A}(s;\,t) = o(\ln s)$. The most complicated case when the amplitudes $f_{P,A}(s;\,t)$ have an infinite number of real zeros is considered for the first time.

1. INTRODUCTION

Pomeranchuk's assertion of the asymptotic equality of the total cross sections of cross-reactions at high energies—the Pomeranchuk theorem (PT)—did not at one time give rise to doubt. The argument by the author of the theorem[1] showing that the total cross section must eventually assume a constant value, and the diffraction picture in which the real part of the amplitude dies out in comparison with the imaginary part, seemed convincing. The primary attention at the time was given to the improvement of the mathematical proofs. From the purely mathematical point of view, the PT is a theorem on the connection between the behavior of the imaginary and real parts of an analytic function in the vicinity of a boundary point. The theorem has a local character, and the conditions are important only in the neighborhood of the point under consideration, i.e., in the case of the PT, only in the neighborhood of the point at infinity.

The present author used in[2,3] a new—to these problems—mathematical technique: to wit, the generalized Phragmen-Lindelöf principle and some theorems on the behavior of a function at the boundary of a region. In such an approach, the PT appears as if it were a simple physical interpretation of general mathematical theorems, and the role of crossing symmetry is especially clear. In[3] the PT was proved for different regimes of behavior of the difference $\Delta\sigma_{tot}(E)$ between the total cross sections for $E \to \infty$.

Gradually, however, the physics underlying the PT began to be called in question. In particular, such a critical analysis was carried out in Eden's paper[4]. Martin[5] has considered the following limitation on the difference between the amplitudes of cross-reactions: $f_P(E) - f_A(E) = o(E \ln E)$. It is important, in the light of the new experimental data, that the theorem be valid under the less rigid restriction: $\mathrm{Re}[f_P(E) - f_A(E)] = o(E \ln E)$. Simple examples show that this restriction cannot be relaxed.

The second condition contained in the PT, to wit, the existence of a limit for $\Delta\sigma_{tot}(E)$ as $E \to \infty$, has an extremely unpleasant character. No theoretical arguments supporting it exist, and its experimental verification is incomparably more difficult than the verification of the theorem itself. Fortunately, this condition can be replaced by another, significantly simpler and cruder condition: $\Delta\sigma_{tot}(E)$, beginning at some energy, must not change its sign. Weinberg[6] proved the PT under this condition and the restriction that $|f_{P,A}(E)| < \mathrm{const}\cdot E$. In the present paper we demonstrate by a more direct method and in more exact terms that we can restrict ourselves to the above-indicated requirement that $\mathrm{Re}[f_P(E) - f_A(E)] = o(E \ln E)$.

The replacement of the requirement that $\Delta\sigma_{tot}(E)$ should have a limit by the requirement that this difference should preserve its sign is extremely important because in all the experimental data, without exception, $\Delta\sigma_{tot}(E)$ changes its sign only at low energies. Unfortunately, this experimental fact has still not been explained.

In general, the situation with the PT has lost its former definiteness, since the only firmly established limitation on the amplitudes is the Froissart-Martin limit

$$|f_{P,A}(E)| < \mathrm{const}\cdot\ln^2 E. \qquad (1)$$

For differential cross sections, the PT was first proved for certain models by Van-Hove[7] and Logunov et al.[8]. In reality, however, the only difference between the proof of the PT for differential cross sections and the proof for total cross sections consists in the use of the auxiliary function $H_-(E)$ (see the formula (32) below) in place of the function $g_-(E)$ (see the formula (8)). This was performed by the present author in[9]. It was proved at the same time that if the amplitude-phase difference grows more slowly than the logarithm of the energy, then the limit of the ratio of the differential cross sections, if it exists, is equal to unity.

Certain difficulties arise when the amplitudes have an infinite number of zeros. One of such difficulties was recently resolved by Cornille and Martin[10]. In the present paper, a general representation for the phases of the amplitudes of cross-reactions is obtained (see (57)). Finally, amplitudes with an infinite number of real zeros are considered for the first time.

Judging from the experimental data, the logarithm of the ratio of the differential cross sections for a fixed t also changes sign only at comparatively low energies. Therefore, it is quite essential that it be also possible in the PT for differential cross sections to replace the requirement that the ratio of the cross sections should have a limit by the requirement that the logarithm of this ratio should, starting from some value of the energy, preserve its sign.

In the paper we introduce and widely use the concept of a "limit in full measure," without which it is impossible to, for example, investigate amplitudes with an infinite number of real zeros. However, readers who are not interested in subtleties of this sort can assume that we are dealing with an ordinary limit.

2. THE MATHEMATICAL BASIS OF THEOREMS LIKE THE POMERANCHUK THEOREM

Like $f(z) = u(z) + iv(z)$ be a function that is analytic in the upper half-plane $\operatorname{Im} z > 0$ and that satisfies the once-subtracted dispersion relation (DR):

$$f(z)-f(z_0)=\frac{1}{\pi}\int\left(\frac{1}{z'-z}-\frac{1}{z'-z_0}\right)v(z')\,dz'. \tag{2}$$

An integral sign without limits denotes integration along the entire axis. Let us transform (2) into the form of the complex potential of a simple layer:

$$f(z)-f(z_0)=\frac{1}{\pi}\int\ln\frac{z'-z_0}{z'-z}\frac{dv(z')}{dz'}dz'. \tag{3}$$

The charge density $(2\pi)^{-1}dv(z')/dz'$ is, generally speaking, a generalized function. If at the point z_0' the function $v(z')$ has different limits from the left and from the right, then $dv(z')/dz'$ contains the term $\Delta v(z_0')\delta(z' - z_0')$, where $\Delta v(z_0') = v(z_0' + 0) - v(z_0' - 0)$, and in the complex vicinity of the point z_0',

$$f(z)\sim\frac{1}{\pi}\Delta v(z_0')\ln\frac{1}{z-z_0'},\quad \operatorname{Re}f(z)\sim\frac{1}{\pi}\Delta v(z_0')\ln\frac{1}{|z-z_0'|}. \tag{4}$$

The relations (4) are results of the integral representations (2) and (3); therefore, the requirement that the limits $v(z_0' \mp 0)$ should exist can be relaxed somewhat: to wit, we can introduce the concept of a "limit in full measure" (denoted by "Lim" instead of the conventional "lim"), when the independent variable lets out, as it approaches z_0', the values of a set of zero density at the point z_0'. The exact definition and the necessary properties are given in the Appendix I.

Let us consider the case when at least one of the limits $\operatorname{Lim} v(z_0' \mp 0)$ does not exist. Let us denote by H_- and H_+ the limiting sets of the function $v(z')$ for $z' \to z_0' - 0$ and $z_0' + 0$ respectively. Let us assume that H_- and H_+ are separated by an interval of length 2η. Then the estimate (A.3) yields the result that for any $\epsilon > 0$ in a sufficiently small complex vicinity of the point z_0'

$$|\operatorname{Re}[f(z)-f(z_0)]|>\frac{2\eta(1-\varepsilon)}{\pi}\ln\left|\frac{z_0-z_0'}{z-z_0'}\right|. \tag{5}$$

If the limiting sets H_- and H_+ have a point α in common, but lie on different sides of it, then there are two possibilities: either the $\operatorname{Lim} v(z) = \alpha$ exists at the point z_0', or $\operatorname{Re} f(z)$ increases in the complex vicinity of the point z' like $\ln|z - z_0'|^{-1}$. In fact, if the $\operatorname{Lim} v(z')$ for $z' \to z_0'$ does not exist, then this implies the existence of a set M with a positive density $d(M|z_0')$ and such $\eta > 0$ that the values of $v(z')$ in the set M lie outside the interval $(\alpha - \eta, \alpha + \eta)$, and the estimate (A.3) is valid.

On account of the generalized maximum principle, the order of the growth in the complex vicinity of the point z_0' as $z \to z_0'$ is not higher than the order of the growth in the real vicinity; therefore, in the real vicinity of the point z_0' asymptotic equalities of the type $f(z) \sim \ln|z - z_0'|$ are replaced by assertions that $f(z')$ increases at least like $\ln|z' - z_0'|^{-1}$.

The point z_0' was assumed to be finite and real. The transformation $\zeta = -(z - z_0')^{-1}$ transforms the half-plane $\operatorname{Im} z > 0$ into the half-plane $\operatorname{Im} \zeta > 0$ and the point z_0' into the point $\zeta = \infty$. The quantities $v(z_0' \pm 0)$ then go over into $v(\mp\infty)$, while $\ln|z - z_0'|$ for $z \to z_0'$ goes over into $\ln|\zeta|$ for $\zeta \to \infty$. The relations (4), for example, go over into the relations

$$\operatorname{Re}f(z)\sim\frac{v(-\infty)-v(+\infty)}{\pi}\ln|z|,\quad f(z)\sim\frac{v(-\infty)-v(+\infty)}{\pi}\ln z. \tag{6}$$

The formula (6) should be understood in the wider sense, i.e., in the sense that the growth of $\operatorname{Re} f(z)$ in the vicinity of $z = \infty$ is stronger than the growth of the difference $v(-z') - v(z')$ as $z' \to +\infty$ by the factor $\ln|z|$. For example, for $v(-z') - v(z') \sim \ln^\nu|z'|$ we have

$$\operatorname{Re}f(z)\sim(\nu+1)^{-1}\ln^{\nu+1}|z|.$$

3. TOTAL CROSS SECTIONS FOR CROSS-REACTIONS

The PT and its various refinements for the total cross sections for cross-reactions between spinless particles

$$AB\to AB,\quad \bar{A}B\to\bar{A}B \tag{7}$$

are obtainable at once from the general properties described in Sec. 2. For this purpose, it is sufficient to apply them to the auxiliary function

$$g_-(E)=[f_P(E)-f_A(E)]/k,\quad k=(E^2-\mu^2)^{1/2}, \tag{8}$$

where $f_P(E)$ and $f_A(E)$ are the amplitudes of the direct and cross reactions (7) in the laboratory system of coordinates, while μ and M are the masses of the particles A and B respectively.

Let us recall that in the framework of any formulation of the local theory, as, for example, the conventional formulation that admits of only generalized functions of moderate growth, as well as the more general Jaffe formulation[11] and the most general formulation of the local nature of the theory by the present author in[12], the amplitudes $f(E)$ are, up to single-particle poles, analytic in a plane with the branch cuts $(-\infty, -\mu]$ and $[\mu, +\infty)$, with the exception, perhaps, of some finite region. Furthermore, the amplitudes are bounded from above by the Martin-Froissart limit[1], and satisfy twice-subtracted dispersion relations.

We shall always assume that the amplitudes $f_{P,A}(E)$ are analytic in the upper half-plane, since if the $f_{P,A}(E)$ were not analytic on some half-disk $|E| < a$, then they would, when expressed in terms of the variable $\hat{E} = E + a^2E^{-1}$, be analytic in the half-plane $\operatorname{Im}\hat{E} > 0$.

Notice that the requirement that the amplitudes $f_{P,A}(E)$ be localized is essentially equivalent to the requirement that the generalized Phragmen-Lindelöf maximum principle be applicable in the upper half-plane, since both imply that the $f_{P,A}(E)$ grow at complex infinity more slowly than any linear exponential. The crossing-symmetry relation is, as shown in[9], most conveniently written in the form

$$f_A(E)=f_P^*(-E^*). \tag{9}$$

Besides, it is sufficient to verify that (9) is satisfied along the imaginary axis. The amplitudes are normalized such that the total cross sections

$$\sigma_{P,A}(E)=\mathrm{Im}\, f_{P,A}(E)/k. \quad (10)$$

The auxiliary function $g_-(E)$ is analytic in the upper half-plane, satisfies a once-subtracted dispersion relation, and possesses crossing symmetry: $g_-(-E^*) = g_-^*(E)$. The function $\mathrm{Im}\, g_-(E')$ is odd along the real axis, and for $E' > \mu$ we have $\mathrm{Im}\, g_-(E') = \Delta\sigma_{tot}(E')$.

Applying the results of Sec. 2 to $g_-(E)$, with $\mathrm{Im}\, g_-(E)$ playing a role similar to that of $v(z)$, we obtain:

1) if for $E' \to +\infty$

$$\mathrm{Re}\, g_-(E')=o(\ln E')$$

and the total-cross section difference $\Delta\sigma_{tot}(E')$ has a limit in full measure (the value ∞ is not a priori excluded), then this limit is equal to zero;

2) if $\mathrm{Re}\, g_-(E') = o(\ln E')$ and $\Delta\sigma_{tot}(E')$, starting from some energy value, does not change sign, then the $\mathrm{Lim}\, \Delta\sigma_{tot}(E')$ exists and is equal to zero.

If the condition $\mathrm{Re}\, g_-(E') = o(\ln E')$ in 1) is replaced by the condition $\mathrm{Re}\, g_-(E') \sim C \ln|E'|$, then it follows from (6) that the $\mathrm{Lim}\, \Delta\sigma_{tot}(E') = -C\pi/2$.

The introduction of the symmetric combination of the amplitudes

$$g_+(E)=i[f_P(E)+f_A(E)]/k \quad (11)$$

enables us to derive the following properties:

1a) if for $E' \to +\infty$,

$$\mathrm{Im}\, [f_P(E')+f_A(E')]=o(E' \ln E')$$

and the $\mathrm{Lim}\, \mathrm{Re}[f_P(E') + f_A(E')]/k$ exists, then this limit is equal to zero;

2a) if the first of the conditions in 1a) is fulfilled and, starting from some energy, $\mathrm{Re}[f_P(E') + f_A(E')]$ does not change sign, then the $\mathrm{Lim}\, \mathrm{Re}[f_P(E') + f_A(E')]/k$ exists and is equal to zero.

It is useful to consider the simplest case when the amplitude behaves so regularly in the upper neighborhood of the point at infinity that its asymptotic form is described in the entire half-plane by an analytic formula. It follows from the existence of the Froissart-Martin limit that under this assumption

$$\frac{f_{P,A}(E)}{k} \sim \int_0^2 [a_{P,A}(\nu)+ib_{P,A}(\nu)]\ln^\nu(-iE)\,d\nu. \quad (12)$$

The functions $a(\nu)$ and $b(\nu)$ may contain δ-function terms as well (but not δ'). The crossing-symmetry condition is equivalent to the requirement that the functions $f_{P,A}(E')/k$ assume on the imaginary axis values that are symmetric with respect to the imaginary axis; therefore, it is convenient to introduce the coefficients $a(\nu)$ and $b(\nu)$ according to the formulas

$$a_A(\nu)=-a_P(\nu)=-a(\nu), \quad b_A(\nu)=b_P(\nu)=b(\nu);$$
$$a(\nu)=b(\nu)=0 \quad \text{for} \quad \nu>2. \quad (13)$$

It follows from this that as $E \to \pm\infty$

$$\Delta\sigma_{tot}(E) \sim -\pi \int_1^2 \nu a(\nu) \ln^{\nu-1}|E|\,d\nu, \quad (14)$$

$$\frac{\mathrm{Re}[f_P(E)+f_A(E)]}{k} \sim \pi \int_1^2 \nu b(\nu) \ln^{\nu-1} E\,d\nu. \quad (15)$$

It follows from (14) that the PT is true if $a(\nu) = 0$ for $\nu > 1$ and does not contain the term $\delta(\nu - 1)$. In particular, the PT can also be fulfilled when the cross sections grow at the maximum rate in proportion to $\ln^2 E$. In this case $b(\nu)$ contains the term $\delta(\nu - 2)$ and, according to (15),

$$\mathrm{Re}\, [f_P(E)+f_A(E)]/k \sim \ln E. \quad (16)$$

4. THE INTEGRAL FORM OF THE POMERANCHUK THEOREM

Let us assume that there exists a finite limit $g_-(i\infty)$ for the function $g_-(E)$ as $E \to \infty$ along the imaginary axis. Since $g_-(E)$ satisfies a once-subtracted dispersion relation, we have[1)]

$$g_-(E)-g_-(i\infty)=\frac{1}{\pi}\int \Delta\sigma_{tot}(E')\frac{dE'}{E'-F}. \quad (17)$$

It follows from the odd parity of $\Delta\sigma_{tot}(E')$ and the equality $g_-(0) = 0$ that

$$\frac{1}{\pi}\int_\mu^\infty \frac{\Delta\sigma_{tot}(E')}{E'}\,dE'=-\frac{1}{2}g_-(i\infty). \quad (18)$$

Because of the crossing symmetry, the limit $g_-(i\infty)$ is a real quantity. The existence of the limit $g_-(i\infty)$ and the equality $g_-(i\infty) = \mathrm{Lim}\, \mathrm{Re}\, g_-(E')$ for $E' \to +\infty$ clearly follow, if the latter limit exists, from the representation of $\mathrm{Re}\, g_-(E)$ in the upper half-plane in the form of the Poisson-Lebesgue integral:

$$\mathrm{Re}\, g_-(E)=\frac{1}{\pi}\int \frac{\mathrm{Im}\, E\,dE'}{|E'-E|^2}\mathrm{Re}\, g_-(E') \quad (19)$$

If, as was assumed earlier, the real part of the amplitude dies out as $E \to +\infty$, i.e., if the $\mathrm{Lim}\, \mathrm{Re}\, g_-(E') = 0$, then

$$\frac{1}{\pi}\int_\mu^\infty \frac{\Delta\sigma_{tot}(E')}{E'}\,dE'=0. \quad (20)$$

Applying the Cauchy formula to $g_-(E)/k$, we obtain

$$\frac{1}{\pi}\int_\mu^\infty \Delta\sigma_{tot}(E')\frac{dE'}{(E'^2-\mu^2)^{1/2}}=-\frac{1}{2}g_-(i\infty)+\frac{f_P(\mu)-f_A(\mu)}{2\mu}-\frac{c+d}{\mu^2-E_0^2}, \quad (21)$$

where c and d are the residues of the amplitude $f_P(E)$ at the single-particle poles $\pm E_0 = \mu^2/2M$.

The function

$$g_{-,1}(E)=g_-(E)/\ln(-ik) \quad (22)$$

tends to zero as $E \to i\infty$ if $g_-(iE)$ increases more slowly than $\ln|E|$ as $E \to +\infty$. This, for example, is necessarily true when

$$\mathrm{Lim}\, \mathrm{Re}\, g_{-,1}(E)=0 \quad \text{for} \quad E\to\pm\infty. \quad (23)$$

Calculations similar to those carried out above show that from the vanishing of the limit $g_{-,1}(i\infty)$ follows the conditional convergence of the integral

$$\int_\mu^\infty \mathrm{Im}\, g_{-,1}(E')\frac{dE'}{E'}. \quad (24)$$

For $E \to +\infty$,

$$\mathrm{Im}\, g_{-,1}(E) \sim \frac{\Delta\sigma_{tot}(E)}{\ln E}+\frac{\pi}{2}\frac{\mathrm{Re}\, g_{-,1}(E)}{\ln^2 E} \quad (25)$$

and, therefore, if we strengthen somewhat the condition (23), to wit, if we assume the conditional convergence of the integral

$$\int_\mu^\infty \frac{\mathrm{Re}\, g_{-,1}(E')}{(E'^2-\mu^2)^{1/2}\ln E'}\,dE', \quad (26)$$

then from (24) will follow the conditional convergence of the integral

$$\int_\mu^\infty \frac{\Delta\sigma_{tot}(E')}{\ln E'}\frac{dE'}{E'} \quad (27)$$

5. THE DIFFERENTIAL CROSS SECTIONS

The asymptotic relations for the differential cross sections of the two cross-reactions

$$AB\to CD,\quad \bar{C}B\to \bar{A}D \tag{28}$$

of spinless particles can, as in the case of the total cross sections, be obtained from the crossing symmetry and the general properties considered in Sec. 2. Let s, u, and t be the Mandelstam variables and let the square of the momentum transfer $t \le 0$. We denote the symmetrized variable $(s - u)/\Sigma M_i$ by E and the reaction amplitudes by $f_{P,A}(E; t)$. We shall drop the indices P, A, and the fixed value t if this will not lead to any confusion. The variable E has been introduced because the crossing-symmetry relation can be written in terms of this variable in the form

$$f_A(E;\,t)=f_P^*(-E^*;\,t). \tag{29}$$

This form retains its meaning also at those t when the gap between the cuts vanishes.

Let us, for brevity, introduce the special designation $Q^2(E; t)$ for the ratio of the differential cross sections:

$$Q^2(E;t)=\frac{d\sigma_P(E;t)}{dt}\Big/\frac{d\sigma_A(E;t)}{dt}. \tag{30}$$

Let us recall that

$$Q^2(E;\,t)=|f_P(E;\,t)/f_A(E;\,t)|^2. \tag{31}$$

Let us introduce the auxiliary function

$$H_-(E;\,t)=i\ln\,[f_P(E;\,t)/f_A(E;\,t)]. \tag{32}$$

If the ratio of the amplitudes does not have in the upper half-plane Im E > 0 zeros and poles at the given t, then $H_-(E; t)$ is an analytic function in this half-plane. On the imaginary axis $H_-(E)$ is real and, therefore, crossingwise symmetric:

$$H_-(-E^*)=H_-^*(E). \tag{33}$$

On the positive semiaxis,

$$H_-(E)=[\psi_A(E;\,t)-\psi_P(E;\,t)]+i\ln Q(E;\,t), \tag{34}$$

where $\psi_{P,A}(E; t)$ are the phases of the amplitudes $f_{P,A}(E; t)$. Since $f_P(E; t)$ in the local theory is polynomially bounded, $H_-(E)$ increases not faster than $\ln|E|$.

It can be seen from (33) and (34) that $H_-(E; t)$ plays in the derivation of the asymptotic value of the ratio $Q^2(E; t)$ the role that $g_-(E)$ played for $\Delta\sigma_{tot}(E)$. So, the following properties obtain:

I. If the ratio $Q^2(E; t)$ of the total cross sections possesses a limit in full measure for $E\to+\infty$ (the values 0 and ∞ are not excluded) and the amplitude-phase difference $\psi_P(E) - \psi_A(E) = o(\ln E)$, then

$$\mathrm{Lim}\, Q^2(E;\,t)=1. \tag{35}$$

In the elastic case, on account of the unitarity condition $\mathrm{Im}\, f_{P,A} \ge 0$, the phase shift $\psi_{P,A}(E; t = 0)$ does not exceed π, and the condition on the growth of the phases is clearly fulfilled.

Ia. If the ratio of the differential cross sections for elastic cross-reactions possesses a limit in full measure for $E\to+\infty$, then this limit is equal to unity.

II. If, starting from some energy value, the differential cross section for one of the reactions (28) is not less than the differential cross section of the cross-reaction and $\psi_P(E) - \psi_A(E) = o(\ln E)$, then the ratio of the differential cross sections has a limit in full measure and this limit is equal to unity.

III. If the limit in full measure of the ratio $Q^2(E; t)$ exists when $t\to+\infty$ and this limit is equal to γ^2, then

$$\ln\gamma^2=\frac{1}{\pi}\,\mathrm{Lim}\,\frac{\psi_P(E)-\psi_A(E)}{\ln E} \tag{36}$$

(this follows from the general relation (6)). This also proves the existence of the $\mathrm{Lim}[\psi_P(E) - \psi_A(E)]/\ln E$. The equality $\gamma = \infty$ or $\gamma = 0$ implies that the limit from the right is equal to ∞ or $-\infty$.

The assumption that the amplitudes $f_{P,A}(E; t)$ have no zeros or poles in the region Im E > 0 is not realistic, but the presence of a finite number of zeros and poles does not affect the obtained results, since we can, with the aid of the method indicated at the beginning of Sec. 3, exclude from consideration the region $|E| \le a$ in which all the zeros and poles are located.

The auxiliary function

$$H_+(E;\,t)=\ln\,[f_P(E;\,t)f_A(E;\,t)]/\ln\,(-iE) \tag{37}$$

plays a role similar to that played by $g_+(E)$. The function $H_+(E)$ possesses crossing symmetry and for $E\to+\infty$

$$\begin{aligned} H_+(E;t)\sim{}&\frac{\ln|f_P(E;t)f_A(E;t)|}{\ln|E|}-\pi\frac{\psi_P+\psi_A}{2\ln^2|E|}\\ &+i\left(\frac{\psi_P+\psi_A}{\ln|E|}+\pi\frac{\ln|f_P(E;t)f_A(E;t)|}{2\ln^2|E|}\right). \end{aligned} \tag{38}$$

Since the ratio $\ln|f_P(E;t)f_A(E;t)|/\ln|E|$ is bounded, from (38) follows the result:

IV. If for $E\to+\infty$ the $\mathrm{Lim}[\psi_P(E; t) + \psi_A(E; t)]/\ln|E|$ exists, then this limit is equal to zero.

From I and IV follows the property:

V. A necessary and sufficient condition for the equality $\mathrm{Lim}\, Q^2(E; t) = 1$ to hold consists in the limitation on the growth of the phases: $\psi_{P,A}(E; t) = o(\ln E)$.

Let us recall that the assertions I–IV have been obtained under the assumption that the amplitudes $f_{P,A}(E; t)$ have a finite number of zeros.

We omit the proof of the conditional convergence of the integrals

$$\int^\infty\frac{dE'}{E'}\ln Q^2(E';t),\quad \int^\infty\frac{dE'}{E'\ln E'}\ln Q^2(E';t). \tag{39}$$

A necessary condition for the convergence of the first integral is the existence of the $\mathrm{Lim}[\psi_P(E'; t) - \psi_A(E'; t)]$; for the convergence of the second, the conditional convergence of the integral

$$\int^\infty\frac{\psi_P(E)-\psi_A(E)}{E\ln^2 E}dE. \tag{40}$$

In the case of elastic scattering the integral theorem can be refined; to wit:

$$\begin{aligned} \frac{1}{\pi}\int_\mu\frac{dE'}{(E'^2-\mu^2)^{1/2}}\ln Q^2(E';t{=}0)={}&\mathrm{Lim}\,[\psi_P(E')-\psi_A(E')]\\ &-\pi\sum_a\,[w(a)-w(-a^*)], \end{aligned}$$

where w(E) is the potential of a plane electromagnetic field in a plane with cuts and boundary values $\pm1/2$ on the right and left cuts. The existence of the limit on the right-hand side is assumed and the summation is over all the zeros of the amplitude $f_P(E)$ in the plane with the cuts.

Let N and ν be the numbers of zeros the amplitude has in the plane with cuts and on the cuts. From the inequality $\Delta\sigma_{tot}(E') \geq 0$ and the Froissart-Martin limit we can obtain the estimates

$$N+\nu \leqslant \begin{cases} 4 \text{ when } \operatorname{sign} f_P(\mu) = \operatorname{sign} f_A(\mu) = 1 \\ 3 \text{ when } \operatorname{sign} f_P(\mu) = -\operatorname{sign} f_A(\mu) \\ 2 \text{ when } \operatorname{sign} f_P(\mu) = \operatorname{sign} f_A(\mu) = -1 \end{cases}$$

6. AMPLITUDES WITH AN INFINITE NUMBER OF COMPLEX ZEROS

Let $E_k = E_k(t)$ be the complex zeros of the function $f(E; t)$ in the upper half-plane. From the polynomial boundedness of $f(E)$ clearly follows the convergence of Blaschke product over these zeros (see, for example,[14,15])

$$\pi(E;t) = \prod_k \left(1-\frac{E}{E_k}\right)\left(1-\frac{E}{E_k^*}\right)^{-1}. \tag{41}$$

Let us represent the amplitudes $f(E; t)$ in the form

$$f(E;t) = \pi(E;t)\hat{f}(E;t), \tag{42}$$

where $\hat{f}(E; t)$ does not have zeros in the upper half-plane $\operatorname{Im} E > 0$. Accordingly,

$$\psi(E;t) = \arg\pi(E;t) + \arg\hat{f}(E;t). \tag{43}$$

It is well known[14,15] that the condition for the convergence of the Blaschke product $\pi(E)$ is the convergence of the series

$$\sum \left|\operatorname{Im}\frac{1}{E_k}\right| < +\infty, \tag{44}$$

Notice that $2|\pi(E)|$ is none other than the potential of the field induced in the upper half-plane with a grounded boundary by unit charges located at the points E_k.

The sequence of complex zeros $\{E_k\}$ does not have real limiting values, since otherwise $\arg f(E)$ would have at such a point an infinite jump. Therefore, $\{E_k\}$ either contains a finite number of points or it tends to infinity. In both cases $\pi(E)$ is continuous on the real axis; $|\pi(E)| = 1$; $\arg\pi(E)$ is uniquely determined by the normalization $\pi(0) = 0$ and is strictly an increasing function provided $\{E_k\}$ is not empty. In the last case $\pi(E) \equiv 1$ and it is not considered.

It follows from the crossing-symmetry relation $f_A(E) = f_P^*(-E^*)$ that the zeros of the amplitudes $f_P(E)$ and $f_A(E)$ are connected by the relation $E_{A,k} = -E_{P,k}^*$, and $\pi_A(E) = \pi_P^*(-E^*)$, i.e., the corresponding Blaschke products and, consequently, the functions $f_{P,A}(E)$ also satisfy the crossing-symmetry condition. On the real axis

$$|\hat{f}_{P,A}(E)| = |f_{P,A}(E)|. \tag{45}$$

The functions $\ln\hat{f}_{P,A}(E)$ are analytic in the upper half-plane and satisfy, in their turn, the crossing-symmetry condition. Following from the fact that $f(E)$ satisfies a dispersion relation is the result than $\ln\hat{f}(E)$ satisfies a dispersion relation is the following form:

$$\ln\frac{\hat{f}(E)}{\hat{f}(0)} = ilE + \frac{1}{i\pi}\int\left(\frac{1}{E'-E}-\frac{1}{E'}\right)\ln|f(E')|\,dE', \tag{46}$$

where

$$l = \lim_{R\to\infty}\frac{2}{\pi R}\int_0^\pi \ln^-|f(Re^{i\varphi})|\sin\varphi\,d\varphi, \quad 0 \leqslant l < \infty, \tag{47}$$

$$\ln^+|a| = \max\{\ln|a|, 0\}, \quad \ln^-|a| = \max\{\ln|a|^{-1}, 0\}.$$

It follows from the crossing symmetry that $l_P = l_A$. If $l > 0$, then this implies that the amplitude $f(E)$ tends in the complex direction to zero like e^{ilE}, and instead of $f(E)$ and $\hat{f}(E)$ it is more convenient to consider $e^{-ilE}f(E)$ and $e^{-ilE}f(E)$.

If the amplitudes $f_{P,A}(E; t)$ do not have real zeros and poles, then (see (46))

$$\arg[\hat{f}_P(E)\hat{f}_A(E)] = \operatorname{Im}\frac{1}{i\pi}\int\left(\frac{1}{E'-E}-\frac{1}{E'}\right)\ln|\hat{f}_P(E')\hat{f}_A(E')|\,dE' + 2lE. \tag{48}$$

Since on account of crossing symmetry $|f_P(E)f_A(E)| = |f_P(-E^*)f_A(-E^*)|$, the integral in (48) is equal to $o(\ln E)$ and

$$\arg[\hat{f}_P(E)\hat{f}_A(E)] = 2lE + o(\ln E). \tag{49}$$

Turning to the phases of the amplitudes, we obtain

$$\psi_P(E;t) + \psi_A(E;t) = \arg\pi_P(E;t) + \arg\pi_A(E;t) + 2lE + o(\ln E). \tag{50}$$

All the terms on the right-hand side strictly increase and cannot cancel each other out. Hence:

a) If $\psi_P(E) + \psi_A(E) = o(\varphi(E))$, where $\varphi(E)$ is a positive nondecreasing function ($\varphi(E) = o(E)$), then $l = 0$ and $\arg\pi_{P,A}(E) = o(\varphi(E))$.

b) In particular, if $\psi_P(E) + \psi_A(E) = o(\ln E)$, then $l = 0$ and $\arg\pi_{P,A}(E) = o(\ln E)$.

The property b) was obtained by Cornille and Martin[10] by a method that required some additional assumptions.

If for $E \to +\infty$ the $\operatorname{Lim} Q^2(E; t) = \gamma^2$ exists, then from (46) and the crossing symmetry follow the following representations for the phases $\psi_{P,A}(E; t)$:

$$\psi_{P,A}(E;t) = \arg\pi_{P,A}(E;t) \pm \frac{\ln\gamma}{\pi}\ln E + lE + o(\ln E). \tag{51}$$

The upper sign is for ψ_P and the lower sign is for ψ_A.

The condition for the convergence of $\pi_{P,A}(E)$ is the convergence of the series (44). Let $\varphi_{P,A}(E)$ be two arbitrary increasing functions. We can always choose the zeros E_k such that for $E \to \pm\infty$

$$\arg\pi_P(E) = \varphi_P(E) + o(\ln E), \quad \arg\pi_A(E) = \varphi_A(E) + o(\ln E). \tag{52}$$

These relations hold if the numbers of zeros with real parts in the intervals $(0, E)$ and $(-E, 0)$ are respectively equal to the integral part of $\varphi_{P,A}(E)/\pi$ and the imaginary parts decrease sufficiently rapidly.

Let $\gamma > 0$ be arbitrary, and let us choose an arbitrary real function $p(E')$ that is sufficiently smooth at the point $E = 0$ and that satisfies the conditions

$$\int\frac{dE'}{1+E'^2}|p(E')| < +\infty, \quad \operatorname{Lim}[p(E') - p(-E')] = \ln\gamma, \quad E' \to +\infty. \tag{53}$$

Let us now determine

$$\hat{f}_P(E) = \exp\left[\frac{1}{i\pi}\int\left(\frac{1}{E'-E}-\frac{1}{E'}\right)p(E')\,dE'\right], \quad \hat{f}_A(E) = \hat{f}_P^*(-E^*), \tag{54}$$

$$f_P(E) = \pi_P(E)\hat{f}_P(E), \quad f_A(E) = \pi_P^*(-E^*)\hat{f}_A(E). \tag{55}$$

It follows from (52) and (53) that the phases of the two functions $f_{P,A}(E)$, which are analytic in the upper half-plane, are related by the crossing symmetry, and satisfy the condition

$$\operatorname{Lim}|f_P(E)/f_A(E)| = \gamma, \tag{56}$$

can be represented in the form

$$\psi_{P,A}(E) = \varphi_{P,A}(E) \pm \pi^{-1}\ln\gamma\ln E + o(\ln E). \tag{57}$$

Thus, there exist for any positive increasing functions $\varphi_{P,A}(E)$ and any $\gamma > 0$ an odd number of pairs of analytic functions $f_{P,A}(E)$ which are related by crossing symmetry and which satisfy (56) and (57). Furthermore, the only real limitation on the asymptotic forms of the phases $\psi_{P,A}(E)$ for a given γ that follows from the analyticity and crossing-symmetry conditions is the condition that $\varphi_{P,A}(E)$ be positive and increasing functions. Precisely because of this, the condition $\psi_{P,A}(E) = o(\ln E)$ entails the equality $\gamma = 1$.

In the case of an elastic reaction, to the crossing-symmetry condition must be added such consequences of unitarity as the boundedness of the phases $\psi_{P,A}(E; t = 0)$, which is possible only when $\gamma = 1$. For $\gamma = 0$ or $\gamma = \infty$, the difference $p(E) - p(-E) \to \pm\infty$ and (57) gets replaced by the relation

$$\psi_{P,A}(E)=\varphi_{P,A}(E)\pm\pi^{-1}[p(E)-p(-E)]\ln E+o\{[p(E)-p(-E)]\ln E\}. \quad (58)$$

A defect of the above-formulated results is the additional assumption that the amplitudes have a finite number of real zeros. Furthermore, it is not possible in the axiomatic approach to even regard the real zeros as discrete points.

Let us denote the total number of zeros of the functions $f_{P,A}(E)$ in the interval $(0, E)$ by $\nu_{P,A}(E)$, considering the poles to be zeros of negative multiplicity. Let us consider the electrostatic field with the complex potential $e^{-il E}f_{P,A}(E)$ and with lines of force defined by the relation

$$\arg f_{P,A}(E)-l\operatorname{Re}E=\text{const.} \quad (59)$$

This field is induced by charges distributed along the real axis. The magnitude of the charge in the interval $(0, E)$ is equal to the flux of the field-lines of force passing through the interval divided by 2π, and has the form

$$-\frac{1}{2}\nu_{P,A}(E)+\frac{1}{2\pi}\int_0^E d\ln|\tilde{f}_{P,A}(E')|,\quad \nu_A(E)=\nu_P(-E). \quad (60)$$

This partition is determined by the fact that $\nu(E)$ is its singular part and $d\nu(E') = 0$ almost everywhere. The function $\ln|\tilde{f}(E')|$ is absolutely continuous, and the analytic function $\tilde{f}(E)$ has neither zeros nor poles in the closed upper half-plane.

In contrast to the phases $\psi_{P,A}(E')$, the functions $\psi_{P,A}(E') + \pi\nu_{P,A}(E')$ are continuous functions that are defined for all real E'. It can be shown that the results obtained under the assumption that the number of real zeros is finite remain valid provided that a symmetry relation of the form

$$\nu_P(E)-\nu_A(E)=o(\ln E) \quad (61)$$

is satisfied and $\psi_{P,A}(E)$ is replaced by $\psi_{P,A}(E) + \pi\nu_{P,A}(E)$.

7. ANOTHER APPROACH TO THEOREMS OF THE TYPE OF THE POMERANCHUK THEOREM

The most adequate method for obtaining the asymptotic relations between the cross sections is a method based on the generalized Phragmen-Lindelöf-Nevanlinna maximum principle and certain theorems given in the Appendix to the present author's paper[9] 2).

Let the function $g_-(E)$ (see (8)) be bounded for $E \to +\infty$. If the difference $\Delta\sigma_{tot}(E)$ between the total cross sections preserved its sign as $E \to +\infty$, then it would follow from the general theorems given in[9] that the amplitudes $f_{P,A}(E)$ increase in the complex direction like a linear exponential function, which, in the local theory, is forbidden.

We can get rid of the superfluous limitation on $g_-(E)$ for $E \to +\infty$ by replacing it by a necessary condition for the validity of the PT: to wit, by the condition $\operatorname{Re} g_-(E) = o(E \ln E)$. For this purpose, we had to prove the following special property:

Let $F(E)$ be analytic in the region $\operatorname{Im} E > 0$, $|E| > E_0$, continuous right up to the boundary, crossingwise symmetric, and bounded, and let $\operatorname{Im} F(E) \to 0$ for $E \to +\infty$. If for $E \geq E_0$ the imaginary part $\operatorname{Im} F(E)$ does not change sign, then in any interval (E_0, E_1)

$$\min|\operatorname{Im}F(E)|<{}^1/_2\pi^2\max\operatorname{Re}|F(E)|/\ln(E/E_0). \quad (62)$$

The meaning of this proposition is that although the function $F(E)$ itself need not even tend to zero, its imaginary part tends, on account of the crossing symmetry, to zero not more slowly than $(\ln E)^{-1}$ at least along some sequence. We omit the proof of the theorem: it can very easily be derived from harmonic-measure theory, which is expounded in, for example,[14].

To prove the PT under the condition that $\operatorname{Re} g_{-1}(E) = o(E \ln E)$, it is sufficient to take as $F(E)$ the function[3)] $g_-(E)/\ln(-iE)$. A shortcoming of such a proof is that we have to a priori assume the existence of the limit of $\Delta\sigma_{tot}(E')$ for $E' \to +\infty$. We can eliminate this shortcoming by proving that (62) is valid in full measure. In order to prove the asymptotic relations for the differential cross sections, it is necessary to take as $F(E)$ the auxiliary function (see (42))

$$i\ln[\hat{f}_P(E)/\hat{f}_A(E)]/\ln(-iE), \quad (63)$$

If the number of real zeros of the amplitudes is finite, and the function (see (60))

$$i\ln[\tilde{f}_P(E)/\tilde{f}_A(E)]/\ln(-iE), \quad (64)$$

if the relation (61) holds.

APPENDIX 1

Let M be some set on the real axis and $u(z; M)$ a harmonic function defined in the upper half-plane by the boundary values $u(z') = 1$ for z' w M and $u(z') = 0$ for the remaining points. Then

$$u(z;M)=\frac{1}{\pi}\int_M\frac{y\,dx'}{(x'-x)^2+y^2},\quad z=x+iy. \quad (A.1)$$

This formula is equivalent to $\delta(z' - z_0')$ being regarded as the boundary value of the function $\pi^{-1}\operatorname{Im}(z - z_0')^{-1}$. Therefore, it is natural to define the density, denoted by $d(M|z_0')$, of the set M at the point z_0' by the relation

$$d(M|z_0')=\lim\frac{1}{\pi}\int_M\frac{y\,dz'}{(z'-z_0')^2+y^2},\quad y\to+0. \quad (A.2)$$

We shall call the set M a set of full or zero measure at the point z_0' according as $d(M|z_0') = 1$ or 0. We shall call M a set of full measure to the right or to the left at z_0' if M is located to the right or to the left of the point z_0' and $d(M|z_0') = 1/2$. Let us call the limit, if it exists, to which the function $v(z')$ tends as $z' \to z_0'$ along some set of full measure at z_0' the limit in full measure of the function $v(z')$ at the point z_0' and denote it by $\operatorname{Lim} v(z')$, $z' \to z_0'$.

It follows from the definition that if the function $v(z')$ has different signs at the points of M lying to the left and right of z_0' and $|v(z')| > \eta > 0$, then we find that in a sufficiently small upper complex vicinity of the point z_0'

$$\left|\operatorname{Re}\frac{1}{\pi}\int_A\left(\frac{1}{z'-z}-\frac{1}{z'-z_0}\right)v(z')dz'\right|>\frac{2\eta}{\pi}d(M|z_0')\ln\frac{|z_0-z_0'|}{|z-z_0'|} \quad (A.3)$$

The addition of a constant to $v(z')$ does not significantly change the left-hand side of (A.3), and therefore the estimate is also correct in the case when there eixsts a constant α such that the quantity $v(z')-\alpha$ has different signs at the points of M lying to the left and right of z_0' and $v(z')$ lies outside the interval $(\alpha-\eta, \alpha+\eta)$.

Let us recall that M is called a harmonic null-set if for some z_0 the quantity $u(z_0; M) = 0$. In this case $u(z; M) \equiv 0$. It is easy to verify that a null set has zero density at all points and, conversely, that a set of zero density at all points is a null set. It is claimed that a property obtains almost everywhere on the real axis if it obtains everywhere except, perhaps, on some null set. For example, it can be shown on the basis of Fatou's theorem (see, for example,[14]) that functions satisfying dispersion relations have almost everywhere on the axis limiting values.

APPENDIX 2

The following conditions are sufficient for the validity of the representation (2), i.e., for the once-subtracted dispersion relation:

1a. The function $v(z)$ has limiting angular values almost everywhere on the real axis, i.e., for any $\pi/2 > \delta > 0$, $v(z') = \lim v(z)$ for $z\to z'$ and

$$\delta<\arg(z-z')<\pi-\delta. \quad (A.4)$$

2a. The equality

$$\lim\int\frac{dz'}{1+z'^2}|v(z'+i\varepsilon)|=\int\frac{dz'}{1+z'^2}|v(z')|<+\infty,\quad \varepsilon\to+0, \quad (A.5)$$

is valid.

3a. The equality

$$\lim\frac{1}{R}\int_0^\pi|v(\mathrm{Re}^{i\varphi})|\sin\varphi\,d\varphi=0 \text{ for } R\to\infty. \quad (A.6)$$

is satisfied.

Only $v(z) = \operatorname{Im} f(z)$ figures in these conditions. If we know the behavior of not only $|v(z)|$, but also of $|f(z)|$ as we approach the boundary, then the sufficient conditions can be modified in the following manner:

1b. The function $f(z)$ has limiting angular values almost everywhere on the real axis.

2b. The equality

$$\lim\int\frac{dz'}{1+z'^2}|f(z'+i\varepsilon)|=\int\frac{dz'}{1+z'^2}|f(z')| \quad (A.7)$$

$$\text{for } \epsilon\to+\infty,$$

is valid.

3b. The equality

$$\lim\frac{1}{R}\int_0^\pi\ln^+|f(Re^{i\varphi})|\sin\varphi\,d\varphi=0,\quad R\to+\infty, \quad (A.8)$$

is satisfied.

Comparing the conditions a) with the conditions b), we can see that the condition 3a is incomparably more rigid than the condition 3b, but it can be shown that this is completely balanced by the fact that the condition 2b is more rigid than 2a. The conditions 1a and 3a (and, correspondingly, 1b and 3b) are necessary conditions at the same time. The conditions a) are altogether necessary if we consider a representation of the form (2) in which the integral converges absolutely.

The impression may be created that the condition 3a (consequently 3b) separates out the point at infinity. In fact, it follows from the condition 2a (and, consequently, 2b) that the equality

$$\lim\rho\int_0^\pi|v(z'+\rho e^{i\varphi})|\sin\varphi\,d\varphi=0,\quad \rho\to0; \quad (A.9)$$

is valid for any real point; consequently,

$$\lim\rho\int_0^\pi\ln^+|f(z'+\rho e^{i\varphi})|\sin\varphi\,d\varphi=0,\quad \rho\to0. \quad (A.10)$$

It follows from the conditions a) and b) that although the limiting values $v(z')$ and $f(z')$ are, generally speaking, generalized functions, they can almost everywhere be regarded as ordinary local functions; and that in integrations over the harmonic measure $dz'/\pi(1+z'^2)$ their behavior on an exclusive null set is of no importance.

1) The convergence of (17) under slightly more rigid conditions, i.e., under the conditions that the limits of $f_{P,A}(E)/E$ for $E\to+\infty$ along the real axis should exist, was asserted in [13], but the proof given there is incorrect.

2) The proof of the fundamental theorem in Sugawara and Kanazawa's paper [16] is not correct.

3) This proof was expounded by us at the summer school in Uzhgorod in 1967.

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