Amplification in a γ laser when the Bragg condition is satisfied

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Amplification in a γ laser under Bragg diffraction conditions is considered. It is shown that the largest gain can be attained in crystals consisting of excited isotopes with maximum multipolarity and minimum nuclear-transition energy.

As is well known from the dynamic theory of x-ray scattering, when the Bragg conditions are satisfied for electromagnetic waves propagating in an ideal crystal, an abrupt decrease takes place in their absorption coefficient. The possibility of using this effect in a γ laser was discussed in a recent article by Kagan [1], who analyzed qualitatively the transitions E1, M1, and E2. From quantitative estimates presented in this paper for the characteristic parameters of the problem in question, it follows that the use of anomalous passage of γ rays is more effective for crystals containing nuclei with transitions of multipolarity M2, E3, and higher, i.e., for long-lived nuclear isomers, and that the use of σ polarization is not necessary.

1. THE GAIN

We consider the process of evolution of a γ wave in an ideal crystal, in which some of the nuclei are in the excited state. The equation describing the interaction of a monochromatic γ wave with a crystal is of the form

$$\Delta \mathbf{E}(\mathbf{r}) + \varkappa^2 \mathbf{E}(\mathbf{r}) = -\frac{4\pi i\omega}{c^2}\mathbf{j}(\mathbf{r}), \qquad (1)$$

where $\kappa = \omega/c$.

Owing to Bragg diffraction, a state with definite γ-quantum energy corresponds to a coherent superposition of two plane waves with wave vectors that differ by the reciprocal-lattice vector ($\boldsymbol{\kappa}_1 = \boldsymbol{\kappa}_0 + \mathbf{K}$)

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_0(\mathbf{r}) \exp(i\boldsymbol{\varkappa}_0\mathbf{r}) + \mathbf{E}_1(\mathbf{r}) \exp(i\boldsymbol{\varkappa}_1\mathbf{r}), \qquad (2)$$

where $\mathbf{E}_i(\mathbf{r})$ are slowly varying amplitudes.

Substituting expression (2) in Eq. (1) averaging it over a volume ($\langle\ldots\rangle$) whose linear dimensions are much larger than the distance between the nuclei and much smaller than the characteristic amplification length, we obtain the following system for the field amplitudes:

$$\begin{aligned} \partial E_0(\mathbf{r})/\partial\xi &= -\left\langle \frac{2\pi}{c}(\mathbf{e}_0\mathbf{j})\exp(-i\boldsymbol{\varkappa}_0\mathbf{r}) \right\rangle, \\ \partial E_1(\mathbf{r})/\partial\eta &= -\left\langle \frac{2\pi}{c}(\mathbf{e}_1\mathbf{j})\exp(-i\boldsymbol{\varkappa}_1\mathbf{r}) \right\rangle, \end{aligned} \qquad (3)$$

where the coordinate ξ is directed along $\mathbf{n}_0 = \boldsymbol{\kappa}_0/\kappa$ and the coordinate η is directed along $\mathbf{n}_1 = \boldsymbol{\kappa}_1/\kappa$. To obtain the system (3) we have, as usual, neglected the second derivatives of the slowly-varying amplitudes.

The current density $\mathbf{j}(\mathbf{r})$ in (1) is the current operator averaged over the quantum-mechanical state and over the statistical thermal motion of the nuclei in the lattice. We shall carry out only the first averaging, assuming the nuclei to be immobile. The influence of the nuclear vibrations will be taken into account in the final state.

To find the nuclear part of the current density, we first obtained the current of one nucleus $\mathbf{j} = \mathrm{Tr}(\rho\hat{\mathbf{j}})$. The Hamiltonian of the interaction of the nucleus with the electromagnetic field can be written in the form

$$\hat{\mathscr{H}}_{\text{int}} = \frac{i}{\omega}\int \hat{\mathbf{j}}^{-}\mathbf{E}(\mathbf{r},t)\,dV + \text{h.c.}$$

where $\hat{\mathbf{j}}^{-}$ is the negative-frequency part of the current density operator. The equation of motion for the density matrix in the interaction representation then takes the form [2]

$$i\hbar\frac{\partial\langle J_2m_2|\rho|J_1m_1\rangle}{\partial t} = (\rho_1-\rho_2)\frac{i}{\omega}\int\langle J_2m_2|\hat{\mathbf{j}}^{-}|J_1m_1\rangle\mathbf{E}(\mathbf{r}) \times\exp\left[-i\omega t+\frac{i}{\hbar}(E_2-E_1)t\right]dV - \frac{i\hbar\langle J_2m_2|\rho|J_1m_1\rangle}{T}$$

where $\rho_\alpha = \langle J_\alpha m_\alpha|\rho|J_\alpha m_\alpha\rangle$, $\alpha = 1, 2$, and $1/T$ is the width of the Mössbauer-emission line.

Using the stationary solution of this equation in place of exact resonance ($\hbar\omega = E_2 - E_1$), we obtain the following expression for the current density

$$\mathbf{j}_n(\mathbf{r}) = \mathrm{Sp}\left\{-\frac{T(\rho_2-\rho_1)}{\hbar\omega}\sum_{i=0}^{1}\mathbf{E}_i(\mathbf{r}) \times\int\exp(i\boldsymbol{\varkappa}_i\mathbf{r}')\langle J_2m_2|\hat{\mathbf{j}}^{-}(\mathbf{r}')|J_1m_1\rangle\,dV'\langle J_1m_1|\hat{\mathbf{j}}^{+}(\mathbf{r})|J_2m_2\rangle\right\}.$$

Substituting the last expression in the right-hand side of the system (3), we obtain

$$-\left\langle\frac{2\pi}{c}(\mathbf{e}_i\mathbf{j}_n)\exp(-i\boldsymbol{\varkappa}_i\mathbf{r})\right\rangle = \sum_j \alpha_{ij}E_j \qquad (4)$$

where

$$\alpha_{ij} = \frac{2\pi}{c}\sum_k\frac{T(\rho_2-\rho_1)}{\Omega\hbar\omega}\mathrm{Sp}\,\{\langle J_2m_2|\mathbf{e}_j\hat{\mathbf{j}}_k^{-}(\boldsymbol{\varkappa}_j)|J_1m_1\rangle\langle J_1m_1|\mathbf{e}_i\hat{\mathbf{j}}_k^{+}(\boldsymbol{\varkappa}_i)|J_2m_2\rangle\},$$

$$i, j = 0, 1.$$

The summation over k is carried out over the resonant nuclei in the unit cell (its volume is Ω). To separate the explicit dependence on the multipolarity of the nuclear transition, we expand the operators $\hat{\mathbf{j}}^{+}(\boldsymbol{\kappa})$ and $\hat{\mathbf{j}}^{-}(\boldsymbol{\kappa})$ in spherical vectors:

$$\hat{\mathbf{j}}^{+}(\boldsymbol{\varkappa}) = \sum_m \hat{c}_{Jm}^{+(\lambda)}\mathbf{Y}_{Jm}^{(\lambda)}(\boldsymbol{\varkappa}). \qquad (5)$$

After substituting this expansion into the coefficients α_{aj}, the latter take the form

$$\alpha_{ij} = \frac{1}{4c}\sum_k\frac{T(\rho_2-\rho_1)}{\hbar\omega}|\langle J_2|\hat{c}_{Jk}^{-(\lambda)}|J_1\rangle|^2 p_{ij}(\lambda J), \qquad (6)$$

where

$$p_{ij}(\lambda J) = 8\pi\sum_{m_1m_2}(\mathbf{e}_j\mathbf{Y}_{J\,m_2-m_1}^{(\lambda)*}(\boldsymbol{\varkappa}_j))(\mathbf{e}_i\mathbf{Y}_{J\,m_2-m_1}^{(\lambda)}(\boldsymbol{\varkappa}_i))\begin{pmatrix}J_2 & J & J_1\\ -m_2 & m_2-m_1 & m_1\end{pmatrix}^2$$

λJ	$p_{ij}^{(\sigma)}(\lambda J)$	λJ	$p_{ij}^{(\sigma)}(\lambda J)$
E1	1	M3	$\frac{1}{16}(15\cos 3\varphi_{ij}+\cos\varphi_{ij})$
M1	$\cos\varphi_{ij}$	E4	$\frac{1}{16}(7\cos 3\varphi_{ij}+9\cos\varphi_{ij})$
E2	$\cos\varphi_{ij}$	M4	$\frac{1}{8}(7\cos 4\varphi_{ij}+\cos 2\varphi_{ij})$
M2	$\cos 2\varphi_{ij}$	M5	$\frac{1}{128}(105\cos 5\varphi_{ij}+21\cos 3\varphi_{ij}+\cos\varphi_{ij})$
E3	$\frac{1}{8}(5\cos 2\varphi_{ij}+3)$		

Note: $\varphi_{ij}=(\widehat{\varkappa_i,\varkappa_j})$.

The coefficients $p_{ij}(\lambda J)$ for transitions with different multipolarity are listed in the table for the case of σ polarization.

For π polarization, as expected, the following change takes place:

$$p_{ij}^{(\pi)}(MJ)=p_{ij}^{(\sigma)}(EJ),\quad p_{ij}^{(\pi)}(EJ)=p_{ij}^{(\sigma)}(MJ). \tag{7}$$

We write down the electronic part of the current in the form

$$\mathbf{j}_e=-i\omega\chi(\mathbf{r})\mathbf{E}, \tag{8}$$

where $\chi(\mathbf{r})$ is the complex polarizability averaged in the sense indicated above. Substituting (8) in the right-hand side of the system (3), we obtain

$$-\left\langle\frac{2\pi}{c}(\mathbf{e}_i\mathbf{j}_e)\exp(-i\varkappa_i\mathbf{r})\right\rangle=\sum_j\beta_{ij}E_j \tag{9}$$

where

$$\beta_{ij}=\frac{i2\pi\varkappa C}{\Omega}\int\chi(\mathbf{r})\exp[i(\varkappa_i-\varkappa_j)\mathbf{r}]dV,$$

$$C=\begin{cases}1 & \text{for } \sigma \text{ polarization}\\ \cos\varphi_{ij} & \text{for } \pi \text{ polarization}\end{cases}$$

We assume that the amplitudes $E_i(\mathbf{r})$ are given by

$$E_i(\mathbf{r})=E_i\exp(k\mathbf{n}\mathbf{r}), \tag{10}$$

where

$$\mathbf{n}=(\mathbf{n}_0+\mathbf{n}_1)/|\mathbf{n}_0+\mathbf{n}_1|.$$

Then, taking (4), (9), and (10) into account, the system (3) takes the form

$$\begin{aligned}k\gamma E_0&=g_{00}E_0+g_{01}E_1,\\ k\gamma E_1&=g_{10}E_0+g_{11}E_1,\end{aligned} \tag{11}$$

where $g_{ij}=\alpha_{ij}+\beta_{ij}$, and $\gamma=\cos(\mathbf{n}_i,\mathbf{n})$.

From the condition that the system (11) have a solution, we obtain the following quadratic equation for the coefficients k:

$$k^2\gamma^2-2k\gamma g_{00}+g_{00}^2-g_{10}g_{01}=0.$$

The real part of k determines the character (amplification or damping) of the development of the γ wave. We call it, for the sake of argument, the gain:

$$\begin{aligned}\mu=\operatorname{Re}k=&\gamma^{-1}(\alpha_{00}-2\pi\varkappa|\chi_{0i}|)\pm\gamma^{-1}\operatorname{Re}[\alpha_{01}^2-4\pi^2\varkappa^2|\chi_{1r}|^2C^2\\&+4\pi^2\varkappa^2|\chi_{1i}|^2C^2+4\pi\varkappa\alpha_{01}|\chi_{1i}|C\cos\beta\\&-4i\pi\varkappa|\chi_{1r}|C(\alpha_{01}\cos\alpha+2\pi\varkappa|\chi_{1i}|C\cos(\alpha-\beta))]^{1/2},\end{aligned} \tag{12}$$

where we have introduced the notation [3]:

$$\chi_{0i}=\operatorname{Im}\frac{1}{\Omega}\int_\Omega\chi(\mathbf{r})dV,$$

$$\chi_1=|\chi_{1r}|e^{i\alpha}+i|\chi_{1i}|e^{i\beta}=\frac{1}{\Omega}\int_\Omega\chi(\mathbf{r})e^{i\mathbf{K}\mathbf{r}}dV,$$

$$\chi_{-1}=|\chi_{1r}|e^{-i\alpha}+i|\chi_{1i}|e^{-i\beta}=\frac{1}{\Omega}\int_\Omega\chi(\mathbf{r})e^{-i\mathbf{K}\mathbf{r}}dV.$$

2. INVESTIGATION OF THE CHARACTER OF DEVELOPMENT OF THE γ WAVE

As follows from the form of the system (11), the first two terms in the right-hand side of (12) determine respectively the gain ($\mu_{0n}=\alpha_{00}/\gamma$) and the damping ($\mu_{0e}=2\pi\kappa|\chi_{0i}|/\gamma$) of the γ wave far from the region where the Bragg conditions are satisfied. To simplify (12), we assume that these two parameters are of the same order of magnitude, i.e., $\alpha_{00}\sim 2\pi\kappa|\chi_{0i}|$.

On the other hand, for frequencies ω that are not too close to the absorption boundary on the K or L shell, the following inequality holds:

$$|\chi_{1i}|\ll|\chi_{1r}|. \tag{13}$$

Since $|\chi_{0i}|\sim|\chi_{1i}|$, it follows that in addition to the inequality (13) there are satisfied the conditions

$$\alpha_{00}\ll 2\pi\varkappa|\chi_{1r}|,\quad \alpha_{01}\ll 2\pi\varkappa|\chi_{1r}|, \tag{14}$$

inasmuch as $\alpha_{01}\sim\alpha_{00}$.

Taking the conditions (13) and (14) into account, we can obtain the following approximate expression for the gain:

$$\mu=\mu_{0n}\left(1-\frac{\alpha_{01}}{\alpha_{00}}\right)-\mu_{0e}\left(1-\frac{|\chi_{1i}|}{|\chi_{0i}|}|C|\right).$$

Taking the nuclear oscillations into account, this formula becomes

$$\mu=\mu_{0n}\left(1-\frac{\alpha_{01}}{\alpha_{00}}\right)-\mu_{0e}\left(1-\frac{|\chi_{1i}|}{|\chi_{0i}|}|C|e^{-Z(\mathbf{K})}\right), \tag{15}$$

where Z(K) is the Debye-Waller factor.

Thus, the absorption due to inelastic interactions of γ quanta with electrons (mainly the photoeffect—for γ quanta of energy on the order of hundreds of keV) is described by the same term as in the dynamic theory, i.e., when the Bragg conditions are satisfied, the absorption coefficient decreases by approximately two orders of magnitude in comparison with its value far from these conditions. Let us estimate the change of the gain for interactions of different multipolarities. We assume the linear dimensions of the unit cell to be a = 3.5 Å, and then for γ quanta of wavelength λ = 0.3 Å (E = 40 keV), the Bragg-diffraction angle is $\varphi=\lambda/a=0.086$. Substituting this value into the formulas for the coefficients $p_{ij}(\lambda J)$, we obtain the following dependences of $\mu_{0n}/\mu_g^{(\sigma)}$ on the multipolarity of the transition, where $\mu_g^{(\sigma)}=\mu_{0n}(1-\alpha_{01}^{(\sigma)}/\alpha_{00})$ is the true gain at the nuclei when the Bragg conditions are satisfied:

λJ:	M1, E2	M2	E3	M3	E4	M4	M5
$\mu_{0n}/\mu_g^{(\sigma)}$:	300	70	100	30	60	20	10

We see therefore that the use of the anomalous passage in the case of σ polarization is the most effective for transitions of multipolarity M3, M4, and M5. On the other hand, for the electric multipole transitions it is more convenient to use π polarization, inasmuch as in view of the smallness of the diffraction angles the coefficient $C^{(\pi)}=\cos\varphi$ is close to unity, i.e., the increase of the γ-ray absorption due to the fact that the electric field intensity at the corners of the crystal lattice is now different from zero is completely overlapped by the increase of the gain.

Let us consider the paired Bragg states of the γ quanta in the case when **K** is not the principal reciprocal-lattice vector $\mathbf{K}_0$, but is its multiple, $\mathbf{K} = n_1\mathbf{K}_{01} + n_2\mathbf{K}_{02} + n_3\mathbf{K}_{03}$, and let us attempt to find the optimal ratio K/K_0 at which the gain μ assumes a maximum value. For simplicity we confine ourselves to transitions of multipolarity M3.

When the condition $Z(\mathbf{K}) \ll 1$ is satisfied, formula (15) for the gain takes the form

$$\mu=\mu_0\left\{\frac{15}{16}\left[\frac{9}{2}\left(\frac{K}{\varkappa}\right)^2-3\left(\frac{K}{\varkappa}\right)^4+\frac{1}{2}\left(\frac{K}{\varkappa}\right)^6\right]-K^2\left(\frac{3\hbar^2}{4Mk_0\Theta_D}+\delta\right)\right\}, \quad (16)$$

where $\mu_0 \sim \mu_{0n} \sim \mu_{0e}$, M is the mass of the nucleus, k_0 is Boltzmann's constant, Θ_D is the Debye temperature and

$$\delta=\frac{1}{K^2}\left(1-\frac{|\chi_{1i}|}{|\chi_{0i}|}\right)\approx\frac{1}{2}\chi_{0i}^{-1}\int\chi_i(\mathbf{r})r^2\cos^2(\widehat{\mathbf{Kr}})\,dV.$$

In the derivation of (16) we used the approximate formula for the Debye-Waller factor at $T \ll \Theta_D$. A simple calculation shows that the maximum value of (16) is reached at

$$\frac{K}{K_0}\approx\frac{\varkappa}{K_0}\sqrt[6]{f}=\frac{a}{\chi}\sqrt[6]{f}, \quad (17)$$

where f is the Mössbauer-effect probability. We note that the last formula is valid when f does not differ strongly from unity. Substituting in (17) the values of the characteristic parameters employed by us, we obtain $K/K_0 = 5$. Taking into account the approximate character of our calculations, we can state that the use of paired Bragg states of γ quanta with multiple reciprocal-lattice vectors is effective at $K = (2-5)K_0$.

CONCLUSION

The considered singularities of the gain in a γ laser under Bragg-diffraction conditions enable us to estimate in greater detail the effectiveness of utilization of the anomalous passage of γ rays. The results indicate that the largest gain is reached in crystals consisting of excited isotopes, with largest multipolarity and with smallest nuclear-transition energy. In view of the smallness of the diffraction angles, the use of the σ polarization is not necessary. For electric multipole transitions, the use of π polarization is more effective. The possibility of using paired Bragg states of γ quanta with multiple reciprocal-lattice vectors is determined in a wide interval of values of the γ-quantum energy by the condition of smallness of the Debye-Waller factor $Z(\kappa)$.

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