Mutual focusing of high-power light beams in media with quadratic nonlinearity

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Results are reported of numerical experimental investigations of the resonant three-photon interactions of narrow light beams in the case when there is strong energy exchange. A number of new phenomena were observed for the first time in experiments with initially nondivergent Gaussian beams. Parametric diffusion of the beams and anomalous diffraction, in the course of which the wave with the higher frequency acquired a converging wave front, were observed in the linear parametric-amplification regime. The phenomenon of simultaneous mutual focusing was discovered under conditions of strong energy exchange between the beams. The field intensities at the nonlinear focus exceeded the initial field intensity of the fundamental wave by one or two orders of magnitude. The Gaussian beams acquired a ring structure in mutual focusing of quasiperiodic nature. The conditions for self-capture of the beams into coupled waveguides have been established and the structures of some stationary waveguides have been found. Estimates are given of the fundamental-wave intensity threshold for the observation of these effects in laser experiments with crystals.

INTRODUCTION

Three-photon resonance interactions play a major role in wave propagation in nonlinear dispersive media the frequencies and wave vectors of the waves are connected by the relations $\omega_1 + \omega_2 = \omega_3$ and $k_1 + k_2 = k_3$). Such interactions are studied in plasma physics in the analysis of decay instability, in nonlinear optics in the analysis of stimulated scattering and parametric amplification and generation, and in hydrodynamics in the analysis of turbulence development. Three-photon interactions have been sufficiently thoroughly investigated only in the geometrical optics approximation (see, for example,[1-3]). The diffraction phenomena that occur during the resonance interaction between modulated waves—in particular, between wave beams—have practically not been studied before. At the same time, as shown in the present paper, the diffraction of coupled waves acquires qualitatively new features.

The propagation of three light beams in a medium with a quadratic nonlinearity was studied in numerical experiments. At high field intensities, instead of the diffractive spreading of the beams, the following phenomena were observed for the first time: anomalous diffraction, in the course of which the wave with the higher frequency acquires a converging front; the parametric diffusion of the subharmonic beam; the mutual focusing of all the three beams, which participate in a synchronous interaction; the guided propagation of the light beams in the form of coupled optical wave guides. It should be emphasized again that these effects occur in media in the absence of cubic nonlinearity, i.e., without the participation of self-focusing. The physical essence of the phenomena of anomalous diffraction and mutual focusing consists in the following.

During the propagation in a quadratic medium of the three waves

$$\mathbf{E}=\frac{1}{2}\sum_{j=1}^{3}\mathbf{E}_j+\text{c.c.}$$

there arise in the medium at the same frequencies the polarization waves

$$\mathcal{P}_1=\hat{\chi}\mathbf{E}_3\mathbf{E}_2^*,\quad \mathcal{P}_2=\hat{\chi}\mathbf{E}_3\mathbf{E}_1^*,\quad \mathcal{P}_3=\hat{\chi}\mathbf{E}_1\mathbf{E}_2.$$

It is not difficult to see that in the field $\mathbf{E}_3$ of an intense pump wave the diffractively diverging wave $\mathbf{E}_2$ excites a polarization wave $\mathcal{P}_1$ that has a converging front and that, in its turn, tends to focus the wave $\mathbf{E}_1$, and, conversely, a diverging $\mathbf{E}_1$ wave exerts a focusing influence on the $\mathbf{E}_2$ wave. This effect is more important for the wave with the larger wave number (such a wave experiences the usual diffraction to a lesser degree). Thus, in a parametrically active medium, the beam with the higher frequency acquires a converging front and can become focused after emerging from the nonlinear layer.

Under conditions of strong energy exchange when the intensities of all the three waves become of the same order of magnitude, the anomalous diffraction leads to the simultaneous mutual focusing of the three beams. As the numerical experiments showed, the cooperative focusing of the waves is accompanied by nonlinear aberrations that manifest themselves in the formation of a ring structure in the beams and in the existance of several foci.

The waves for which the phase-asynchronism integral is negative are captured and coupled into generally-oscillating waveguides or solitons. We find in the paper the structure of some stationary waveguides with constant cross sections and plane wave fronts.

The above-enumerated phenomena obviously have an intensity threshold, since the characteristic nonlinear-interaction length, which is inversely proportional to the intensity, should be less than the diffractive beam spreading length.

1. QUASI-OPTICAL EQUATIONS AND THE INTEGRALS OF THE MOTION

The interaction between nearly plane waves is described by the system of parabolic equations

$$\partial A_1/\partial z+iD_1\Delta_\perp A_1=-i\gamma_1A_3A_2^*e^{-i\Delta z},\quad (1)$$

$$\partial A_2/\partial z+iD_2\Delta_\perp A_2=-i\gamma_2A_3A_1^*e^{-i\Delta z},\quad (2)$$

$$\partial A_3/\partial z+iD_3\Delta_\perp A_3=-i\gamma_3A_1A_2e^{i\Delta z}.\quad (3)$$

Here $A_j = A_{0j}e^{-i\varphi_j}$ is the slowly-varying complex amplitude of the wave's electric field

$$\mathbf{E}_j=(8\pi/cn_j)^{1/2}\mathbf{e}_jA_j\exp i(\omega_jt-k_jz),$$

z is the coordinate along the axis of the wave beams, $\Delta_\perp = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the Laplace operator in the transverse coordinates, $D_j = 1/2k_j$ is the coefficient of the transverse diffusion of the amplitude,

$$\gamma_j=4\pi\sqrt{2\pi}\chi\omega_j/[c\sqrt{cn_1n_2n_3}],\quad \gamma_3=\gamma_1+\gamma_2,$$

$\chi = \mathbf{e}_1\hat{\chi}\mathbf{e}_2\mathbf{e}_3$, n_j is the refractive index, and $\Delta = k_3 - k_1 - k_2$ is the wave-vector detuning. Obviously, the wave intensity $S_j = A_{0j}^2$, while the power

$$P_j=\iint A_{0j}^2\,dx\,dy.$$

The system (1)—(3) has integrals of the motion that remain constant during the propagation of the waves, i.e., that do not depend on the coordinate z. These integrals are first and foremost the Manley-Rowe relations for the powers:

$$I_{1,1}=\frac{P_2}{\gamma_2}+\frac{P_3}{\gamma_3},\quad I_{1,2}=\frac{P_1}{\gamma_1}+\frac{P_3}{\gamma_3},\quad I_{1,3}=\frac{P_1}{\gamma_1}-\frac{P_2}{\gamma_2},\tag{4}$$

from which follows the law of conservation of the total energy of the waves:

$$I_1=P_1+P_2+P_3.\tag{5}$$

To the law of conservation of the transverse momentum of the wave beams corresponds the motion integral

$$\mathbf{I}_2=\iint\Big(\sum_{j=1}^{3}\frac{D_j}{\gamma_j}A_{0j}^2\nabla_\perp\varphi_j\Big)dx\,dy.\tag{6}$$

For beams with axial symmetry this integral is trivial: $\mathbf{I}_2 \equiv 0$.

A greater amount of information on the nature of the interaction between the waves is provided by the phase-asynchronism integral

$$I_3=\iint\Big[\sum_{j=1}^{3}\frac{D_j}{\gamma_j}|\nabla_\perp A_j|^2-\frac{\Delta}{\gamma_3}A_{03}^2-2A_{01}A_{02}A_{03}\cos\Phi\Big]dx\,dy,\tag{7}$$

where $\Phi = \varphi_1 + \varphi_2 - \varphi_3 - \Delta z$ is the phase difference, which determines, first, the direction and rate of energy transfer between the waves. In fact, from (1) we can derive the expression

$$dP_1/dz=2\gamma_1\iint A_{01}A_{02}A_{03}\sin\Phi\,dx\,dy.\tag{8}$$

Clearly, to the coherent process in which the energy-transfer rate is maximal corresponds $\Phi = \pm\pi/2$. According to (7), the coherence of the three-photon interaction is destroyed because of the detuning Δ of the mean magnitudes of the wave vectors and, which is particularly important, because of the diffraction of the waves. It should be emphasized here that the diffractive detuning of the phase velocities is nonuniform over the cross section and varies with distance, so that it is not possible to cancel it out by a judicious choice of the value of Δ.

Secondly, with the diffractive incoherence is connected a nonlinear distortion of the wave fronts of the interacting beams. For example, for the first wave we have from (1) the equation

$$\frac{\partial\varphi_1}{\partial z}+D_1(\nabla_\perp\varphi)^2=\frac{\gamma_1A_{02}A_{03}}{A_{01}}\cos\Phi+\frac{D_1\Delta_\perp A_{01}}{A_{01}}.\tag{9}$$

It can be seen from this that we can expect the effects of focusing or defocusing of spatially modulated waves to be observable in strong interactions.

2. PARAMETRICALLY ACTIVE MEDIA ($\gamma_3 = 0$)

Let us first consider the diffraction of the beams A_1 and A_2 in the uniform field of the pump wave $A_3 = E_3$. The angular components

$$s_j(k_x,k_y,z)=\iint A_j\exp\{-i(k_xx+k_yy)\}\,dx\,dy$$

of the fields being amplified vary in a parametrically active medium according to the law (see[4]):

$$\begin{gathered}s_1=[s_1(k_x,k_y,0)\;G_{11}(k_\perp,z)+s_2^*(-k_x,-k_y,0)G_{12}(k_\perp,z)]\\ \times\exp\{-i[\Delta+(D_2-D_1)k_\perp^2]z\},\\ G_{11}=\operatorname{ch}(\Gamma z)+\frac{i\Delta_k}{2\Gamma}\operatorname{sh}(\Gamma z),\quad G_{12}=-i\Big(\frac{\omega_1}{\omega_2}\Big)^{1/2}e^{-i\varphi_3}\operatorname{sh}(\Gamma z),\\ \Gamma=(\Gamma_0^2-\Delta_k^2/4)^{1/2},\quad \Gamma_0=(\gamma_1\gamma_2)^{1/2}E_3,\quad \Delta_k=\Delta+(D_1+D_2)k_\perp^2,\quad k_\perp^2=k_x^2+k_y^2.\end{gathered}\tag{10}$$

It can be seen from (10) that the contour of the angular amplification $\Gamma(k_\perp)$ is inhomogeneous, as a result of which the angular spectrum of the waves narrows down and the beams accordingly broaden with distance. For example, the coordinated, axially-symmetric Gaussian beams $A_1(r, 0) = E_1\exp(-r^2/a^2)$ and $A_2(r, 0) = E_2\exp(-r^2/a^2)$ get diffracted, as follows from (10), in the following manner:

$$\begin{gathered}A_1(r,z)\\ =(a^2/2)\int_0^\infty(E_1G_{11}+E_2^*G_{12})J_0(k_rr)\exp\{-k_r^2[a^2+iz(k_2^{-1}-k_1^{-1})]/4\}k_r\,dk_r,\end{gathered}\tag{11}$$

where $k_r = (k_x^2 + k_y^2)^{1/2}$ and J_0 is a Bessel function. Interchanging the indices 1 and 2, we obtain the expression for the amplitude A_2.

In the case of a linear medium, when $\Gamma_0 = 0$, the expression (11) goes over into the well-known formula

$$A_j(r,z)=\frac{E_j}{1-iz/R_{d,j}}\exp\Big\{-\frac{r^2}{a^2(1-iz/R_{d,j})}\Big\},\tag{12}$$

in which the diffractive beam spreading length is equal to

$$R_{d,j}=k_ja^2/2.\tag{13}$$

For $z > R_{d,j}$ the beam acquires a diverging spherical front and the amplitude decreases with the distance $\propto R_{d,j}/z$, the wave with the lower frequency spreading more rapidly.

In a parametrically active medium, the diffraction of the waves being amplified begins to proceed completely differently if, as can be seen from (11), the wave detuning is relatively small within the limits of the angular divergence of the beam, i.e., if $\Delta_k \leq 2\Gamma_0$ when $k_ra \leq 2$. In other words, there exists a critical pump-wave intensity

$$S_{cr1}=\frac{S_0n_3^4\lambda_3^4}{(1-\mu^2)^2a^4},\qquad S_0=\frac{c}{8\pi^7\chi^2n_1n_2n_3},\tag{14}$$

where $\mu = (2\lambda_3/\lambda_1) - 1$ is a parameter characterizing the frequency degeneracy: $-1 < \mu < 1$. When the threshold is exceeded, i.e., when $S_3 > S_{cr\,1}$, new diffraction phenomena arise.

Let us first consider the case when the wave detuning can be neglected in the evaluation of the integral (11). It is not difficult to show that this can be done if $S_3 > \mu^{-2}S_{cr\,1}$. Setting $\Delta_k = 0$, and assuming, for definiteness, that $\omega_1 > \omega_2$, we find the wave amplitudes

$$A_1=\frac{E_1G_{11}(0,z)+E_2^*G_{12}(0,z)}{1+iz/R_{an}}\exp\Big\{-\frac{r^2}{a^2(1+iz/R_{an})}\Big\},\tag{15}$$

$$A_2=\frac{E_2G_{22}(0,z)+E_1^*G_{21}(0,z)}{1-iz/R_{an}}\exp\Big\{-\frac{r^2}{a^2(1-iz/R_{an})}\Big\},\tag{16}$$

where the common—for both beams—spreading length is equal to

$$R_{an}=k_1k_2a^2/(k_1-k_2). \tag{17}$$

A comparison of the formulas (15)—(17) with (12) and (13) shows that the diffraction of the waves acquires an anomalous character in the case of parametric amplification: first, although the wavelengths are different, the rates of spreading of the beams are equal; second, during the diffraction the beam with the shorter wavelength acquires a converging front, and not a diverging one, as in the case of a linear passive medium. Such a beam, after emerging from a nonlinear layer of thickness $z = L$ gets focused, and in the constriction formed over the distance $z^{(c)} = L(k_1 - k_2)/2k_2$ the beam radius attains the initial value a.

Let us now turn to the analysis of the behavior of the waves when the second threshold pump intensity is not exceeded: $S_{cr\,1} < S_3 < \mu^{-2} S_{cr\,1}$. For this purpose, let us follow the variation of the field on the axis of the beam; setting $\Delta = 0$, $\Delta_k \ll \Gamma_0$, and $\Gamma_0 z \gg 1$ in (11), we find

$$A_1(0,z)=[E_1G_{11}(0,z)+E_2^*G_{12}(0,z)](\pi R_{an}/4z)^{1/2}(1+\operatorname{erf}\xi)e^{\xi^2},\quad \xi=(1+iz/R_{an})[R_p/(4z)]^{1/2}. \tag{18}$$

Here erf ξ is the probability integral, while the length

$$R_p=2\Gamma_0a^4k_1^2k_2^2/k_3^2 \tag{19}$$

characterizes the effect of the wave-detuning-induced nonuniformity of the parametric-amplification contour.

For pump intensities $S_3 \gg \mu^{-2} S_{cr\,1}$, when $|\xi| \gg 1$ for any z and we can use the asymptotic expansion $\operatorname{erf}\xi = 1 - \pi^{-1/2}\xi^{-1}e^{-\xi^2}$, we arrive at the previous result (15), which confirms the existence of the anomalous diffraction.

If $S_3 < \mu^{-2} S_{cr\,1}$, then two diffraction regions are observed. In the first region, $0 < z < R_{an}$, the amplitude decreases with distance relatively slowly as $(R_p/z)^{1/2}$, while the phase front remains practically plane. This zone can be called a region of parametric diffusion of the amplitudes of the waves being amplified. Further, there develops at $z > R_{an}$ an anomalous diffraction in which the amplitude decreases more rapidly: $\propto R_{an}/z$. Obviously, only parametric diffusion can occur in a frequency-degenerate amplifier ($\omega_1 = \omega_2 = \omega_3/2$), since $R_{an} \to \infty$ as $\mu \to 0$.

In real experiments the pump wave is bounded in space. Therefore, it is of interest to consider the diffraction phenomena in, for example, the field of a Gaussian beam made up of the fundamental radiation. Owing to the nonuniformity of the amplification factor over the cross section, the excited beams undergo the constriction $a_{1,2} \approx a_3(\Gamma_0 z)^{-1/2}$, which is opposed by the diffraction. As a result, there gets established the equilibrium beam radius:

$$a_p=a_3(S_{cr\,2}/S_3)^{1/4} \tag{20}$$

in the parametric diffusion ($S_{cr\,2} < S_3 < \mu^{-6} S_{cr\,2}$) and

$$a_p=a_3(S_{cr\,2}\mu^2/S_3)^{1/4} \tag{21}$$

in the anomalous diffraction ($S_3 > \mu^{-6} S_{cr\,2}$), the second critical intensity of the pump wave being equal to (cf. (14))

$$S_{cr\,2}=S_0\lambda_3^4/(1-\mu^2)^2a_3^4. \tag{22}$$

The condition $S_3 \gg S_{cr\,2}$ automatically ensures large amplification factors over the diffractive-spreading length of the fundamental beam: $\Gamma_0 R_{d3} \gg 1$, $R_{d3} = k_3a_3^2/2$.

Thus, if the radiation at the fundamental frequency ω_3 is a Gaussian beam, then for the observation of the new effects arising in the diffraction of the wave being amplified, it is necessary that the pump intensity exceed $S_{cr\,2}$, (22). In this case if the initial radius of the beams being amplified is larger than the radius of the pump beam, i.e., if $a_{1,2} > a_3$, then $S_{cr\,1} > S_{cr\,2}$, and the weak beams contract to the equilibrium radius; in the opposite case, when $a_{1,2} < a_3$, $S_{cr\,1} < S_{cr\,2}$ and the beams spread, as a result of anomalous diffraction or parametric diffusion, to the equilibrium radius.

The diffraction phenomena in a parametric amplifier were also studied by us with the aid of the numerical solution of the system (1)—(3) reduced to a dimensionless form (see the Appendix). In the numerical experiments we investigated the propagation of cylindrical beams having at the entrance to the nonlinear medium plane wave fronts and Gaussian amplitude profiles

$$B_j(r,0)=B_j(0,0)\exp(-r^2/a_j^2). \tag{23}$$

In the series of experiments under discussion in the present paper, we studied the interaction of two narrow weak beams with a wide fundamental-radiation beam ($a_1 = a_2 \approx 1$, $a_3^2 = 20$, and $B_{1,2}(0, 0) \ll B_3(0, 0)$) in the degenerate ($\mu = 0$, $\lambda_1 = \lambda_2$), as well as in the nondegenerate ($\mu = 0.6$ and $\lambda_3 < \lambda_1 < \lambda_2$) cases for $S_3 \cong S_{cr\,1}$. On account of the last condition, the beams at first undergo the ordinary diffraction described by (12). But since the primary requirement $S_3 > S_{cr\,2}$ is clearly satisfied, the diffraction undergoes significant changes in the subsequent propagation of the waves.

In Fig. 1 we show the results of the numerical experiment on the amplification of the subharmonic beams ($S_3 = S_{cr\,1} = 400 S_{cr\,2}$, $R_{d3} = 2.5$, and $R_{d1} = R_{d,2} = 1/16$). In the course of the amplification the beams first expand by roughly a factor of two and then become stabilized (notice that the beam radii would, in a linear medium, for which $\gamma_1 = \gamma_2 = 0$, increase by a factor of 16 by the time the beams exit from the medium). According to the estimate (20), the equilibrium radius is equal to $\rho_e = 1.4$, which is in good agreement with experiment. It can also be seen from Fig. 1 that in parametric amplification the distortion of the wave front within the boundaries of the beam is considerably less than in a

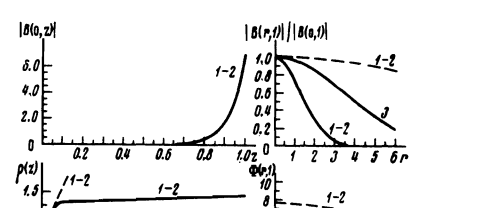

FIG. 1. Parametric diffusion of the subharmonic beams ($\mu = 0$, $B_{1,2}(r, 0) = 10^{-5}e^{-r^2}$, and $D_1^{-1} = D_2^{-1} = 0.25$) in a parametrically active medium ($\gamma_1 = \gamma_2 = 16$ and $\gamma_3 = 0$) pumped by a high-power wave with $B_3(r, 0) = e^{-r^2/20}$, $D_3^{-1} = 0.5$, and $S_3 = S_{cr1}$. Shown are the plots of the variations with distance of the amplitude $|B_j(0, z)|$ on the axis and the beam radius $\rho(z)$, as well as the plots of the profile of the amplitude $|B_j(r, 1)|/|B_j(0, 1)|$ and of the phase $\Phi(r, 1)$ at the exit from the medium at z = 1. The dashed lines show the behavior of the waves during ordinary diffraction in a linear medium ($\gamma_1 = \gamma_2 = \gamma_3 = 0$); the numerals 1, 2, and 3 are the numbers of the waves.

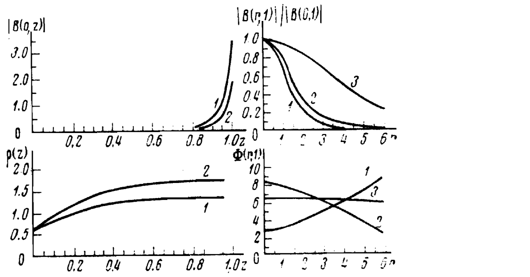

FIG. 2. Anomalous diffraction of the beams ($\mu \neq 0$, $D_1^{-1} = 0.8$, $D_2^{-1} = 0.2$, and $B_{1,2}(r, 0) = 10^{-9}e^{-2r^2}$) in a parametric amplifier ($\gamma_1 = 24$, $\gamma_2 = 6$, and $\gamma_3 = 30$) excited by a pump wave with $B_3(r, 0) = 2e^{-r^2/20}$, $D_3^{-1} = 1$, and $S_3 = 0.92S_{cr1}$.

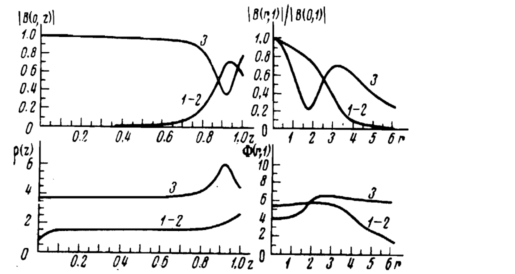

FIG. 3. Strong interaction among the subharmonic- and fundamental-radiation beams ($D^{-1} = D_2^{-1} = 0.25$, $D_3^{-1} = 0.5$; $\gamma_1 = \gamma_2 = 13$, $\gamma_3 = 26$; $B_{1,2}(r, 0) = 10^{-4}e^{-r^2}$, $B_3(r, 0) = e^{-r^2}$; $S_3 = 0.66S_{cr1}$).

linear medium, as a result of which the angular width of the diagram of the radiation of the waves under amplification decreases.

In the next experiment (Fig. 2), we investigated the amplification of waves with different frequencies ($\lambda_1 = 1.25\lambda_3$, $\lambda_2 = 5\lambda_3$; $S_3 = 0.92S_{cr1} = 1475S_{cr2}$; $R_{d1} = 0.1$, $R_{d2} = 1/40$, and $R_{d3} = 5$). Since the frequencies of the waves under amplification differed greatly from each other, there distinctly appeared in the experiment an anomalous diffraction effect: the first beam with the shorter wavelength had at the exit from the medium a converging front, while the second beam had a diverging front. According to the estimate (20), the equilibrium radius of the beams is equal to $\rho_e = 1.55$, which is in good agreement with the mean radius found in the experiment.

3. MUTUAL FOCUSING OF WAVES IN A QUADRATIC MEDIUM

In the regime of high amplification, the amplitudes of the initially weak waves can be comparable to the amplitude of the fundamental wave, and it is necessary to take the exhaustion of the pump wave into account, i.e., it is necessary to solve the complete system of equations (1)–(3) with the right-hand sides ($\gamma_j \neq 0$). The exact solutions of this system in the geometrical-optics approximation ($D_j = 0$) are known[2]. For example, under the conditions of our experiments, when $\Phi(r, 0) = 0$ and $A_1(r, 0) = A_2(r, 0) \ll A_3(r, 0)$, the intensity of the pump wave varies with distance in the following fashion:

$$S_3(r,z) = S_{3,min} + [S_3(r,0) - S_{3,min}]\,\mathrm{sn}^2\left\{(\gamma_1\gamma_2)^{1/2}A_3(r,0)z;\ 1-\frac{S_{3,min}}{S_1(r,0)}\right\}, \quad (24)$$

where sn is the elliptic sine, the square of which has the half-period

$$L_h = \frac{1}{2(\gamma_1\gamma_2)^{1/2}A_3(r,0)}\ln\frac{16\gamma_1\gamma_2S_3(r,0)}{\gamma_3^2S_1(r,0)}. \quad (25)$$

It can be seen that the amplitude oscillates periodically between the initial value $A_3(r, 0)$ and the minimum value

$$A_{3,min} = S_{3,min}^{1/2} \doteq \frac{2A_1^2(r,0)}{(1-\mu^2)^{1/2}A_3(r,0)}, \quad (26)$$

while the amplitudes of the other two waves vary within the limits $A_1(r, 0)$ and $A_3(r, 0)$.

Thus, in the absence of diffraction, the maximum values of all the amplitudes do not exceed the initial amplitude of the pump wave, i.e., $A_j(r, z) \leq A_3(r, 0)$. The nonuniform distribution of the intensity over the cross section of, for example, a Gaussian beam leads to a situation in which the beat period for the peripheral

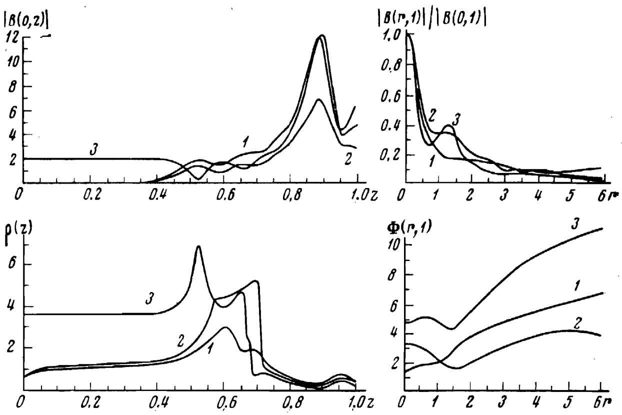

FIG. 4. Mutual focusing of Gaussian beams without initial divergence ($\mu = 0.6$, $B_{1,2}(r, 0) = 10^{-4}e^{-2r^2}$, $B_3(r, 0) = 2e^{-r^2/20}$, $D_1^{-1} = 0.8$, $D_2^{-1} = 0.2$, and $D_3^{-1} = 1$) in a medium with quadratic nonlinearity ($\gamma_1 = 24$, $\gamma_2 = 6$, $\gamma_3 = 30$; $S_3 = 0.92S_{cr1}$).

rays is much longer than for the axial rays. As a result of this the beams acquire in the course of the interaction a ring structure, the number of rings in the primary beam being equal to the number of intensity minima on its axis.

To study the influence of diffraction on the resonant interaction of the beams, we carried out numerical experiments, the results of which are shown in Figs. 3 and 4. The initial parameters of the waves were chosen to be roughly the same as in the first two experiments, only the amplitudes were chosen such that an intense energy exchange developed inside the nonlinear medium.

In the degenerate case (Fig. 3) the subharmonic beams stabilize at the first stage of the parametric amplification in a manner similar to how the corresponding beams stabilized in the first experiment (cf. Fig. 1). Further, there begins to form at the center of the primary beam an intensity dip that, as a result of diffraction, fills up, as it were, owing to the influx of energy from the peripheral part of the beam. As a result of this, the minimum amplitude turns out to be considerably larger than in the case when diffraction is neglected, and the location of the minimum recedes a little (according to the formulas (25) and (26), $L_h = 0.76$ and $A_{3,min}(0) = 2 \times 10^{-8}$; cf. Fig. 3). By considering the variation of the effective radii with distance, we can follow the formation of the ring structure of the beams: the radius increases sharply at the place where a new ring appears.

Beyond the point $z = 0.9$, there is observed in the near-axial region a reverse transfer of energy from the subharmonic to the pump wave: This is essentially a process of generation of the second harmonic of the subharmonic radiation. In such an interaction the wave fronts of the beams become, as can be seen from Fig. 3, converging near the axis. The beams can be expected to focus inside a more extended medium.

A new phenomenon of mutual focusing of waves in a quadratic medium was observed by us in the subsequent experiments under certain conditions, including those under which harmonic generation occurred with high efficiency (the characteristics of mutual focusing involving frequency doubling will be communicated in the next paper). It should be especially emphasized that mutual focusing occurs in the absence of cubic nonlinearity, i.e., without the participation of the self-focusing mechanisms.

The mutual focusing of the three beams during a nondegenerate interaction is shown in Fig. 4. After the anomalous diffraction and the stabilization of the beams being amplified (cf. Fig. 2), there develops a strong interaction among all the three beams (the formulas (25) and (26) give $L_h = 0.43$ and $A_{3,\min}(0) = 1.25 \times 10^{-8}$), the spatial beats in the amplitudes being nonperiodic, as in the geometrical-optics case, (24). In the course of the formation of the annular zones the wave amplitudes on the axis begin to, on the average, increase, and attain their maximum values at the nonlinear focus $z_f = 0.9$ simultaneously, the beam radii sharply decreasing in the process. During the mutual focusing the wave fronts of the beams with the higher frequencies ω_1 and ω_3 become converging, while the wave fronts of the beams with the lower frequency ω_2 become diverging in the near-axial region (cf. Fig. 2).

It is extremely difficult to analytically compute the field at the nonlinear focus, but it can be shown with the aid of the integrals (5) and (7) of the motion that it is bounded. Let us introduce the following two quantities:

$$M(z)=\sum_{j=1}^{3}\max_r A_{0j}(r,z), \tag{27}$$

$$m=\min_j (D_j/\gamma_j). \tag{28}$$

By using a number of inequalities to estimate the integrals entering into (7), we can derive (see the Appendix) fundamental inequalities determining the upper bound of the amplitudes at the focus:

$$3^{q-2}mI_1\left(\frac{M^2}{6I_1}\right)^{2/q}\leqslant I_3+\frac{MI_1}{3}+\frac{\Delta}{\gamma_3}P_3(z), \tag{29}$$

where $q = 1$ for one-dimensional beams ($\Delta_\perp = \partial^2/\partial x^2$), $q = 2$ for cylindrical beams ($\Delta_\perp = \partial^2/\partial r^2 + r^{-1}\partial/\partial r$), and

$$I_1=\int\int\sum_{j=1}^{3}A_{0j}^2 r^{q-1}dr.$$

In a slightly nonlinear medium ($\chi \to 0$) the amplitudes are restricted to the level

$$M\leqslant\sqrt{6I_1}\,(3^{2-q}I_3/mI_1)^{q/4}. \tag{30}$$

If, however, the conditions $I_3 < MI_1/3$ or

$$3^{1+(q-1)^2}m^q I_3^{4-q}\leqslant 4I_1^{6-q}, \tag{31}$$

are satisfied, then mutual focusing of the waves develops in the medium, and the upper bound is raised to

$$M\leqslant\sqrt{6I_1}\,(2I_1/3m^2)^{q/2(4-q)}. \tag{32}$$

When a high-power pump beam with a Gaussian profile enters the medium, to the threshold condition (31) corresponds the critical intensity (cf. (14) and (22)):

$$S_{cr\,3}\cong\frac{S_0 n_1 n_2 n_3^2\lambda_3^4}{8^{4-q}a_3^4}\left(1+\frac{\alpha_{03}^2}{\alpha_{d3}^2}\right)^{(4-q)/2}, \tag{33}$$

where α_{03} is the initial divergence and $\alpha_{d,3} = 2/k_3a_3$. For $S_3 > S_{cr\,3}$ we have, according to (32), the upper bound

$$M\leqslant E_3(S_3/S_{cr\,3})^{q/2(4-q)},\quad \alpha_{03}=0. \tag{34}$$

Under the conditions of our experiments the formulas (33) and (34) yield $S_{cr\,3} = 5.6 \times 10^4$ and $M < 240$ (the nonlinear focus is not visible in Fig. 3, since it is located beyond the point $z = 1$) in the degenerate case and $S_{cr\,1} = 1.2 \times 10^5$ and $M < 700$ in the nondegenerate case (in Fig. 4, $M = 30$).

To decrease the critical intensity or power, it is necessary to reduce the divergence of the pump beam and increase its radius. In this case, however, the mutual-focusing length $z_f > L_h(0)$, (25), increases if the power of the beam remains constant. Obviously, there exists an optimum radius $a_3 \approx (k_3L)^{1/2}$ for which the critical power is a minimum.

4. COUPLED WAVE GUIDES AND SOLITONS IN A QUADRATIC MEDIUM

The behavior of the waves at large distances $z \to \infty$ depends on the sign of I_3. If $I_3 > 0$, then after mutual focusing (in the process of which several foci can appear) the field decays into three beams that no longer interact and each of which carries a positive integral $I_{3,j}^{(\Lambda)}$ where

$$I_3=\sum_{j=1}^{3}I_{3,j}^{(\Lambda)}.$$

If, on the other hand, $I_3 < 0$, then the beams propagate in the form of three coupled, generally-oscillating wave guides. Indeed, in the last case it follows from (7) that the maxima of the amplitudes are bounded from below:

$$\max_r A_{0j}(r,z)>|I_3|/I_1. \tag{35}$$

This implies that self-capture of the beam occurs when $I_3 < 0$. The propagation of the Gaussian beams

$$A_j(r,0)=E_j\exp\left\{-\frac{r^2}{a_j^2}\left(1+\frac{i\alpha_{0j}}{\alpha_{dj}}\right)-i\varphi_{0j}\right\}$$

in the form of waveguides begins when

$$\sum_{j=1}^{3}\delta_j E_j^2\leqslant E_1E_2E_3\cos(\Phi_0-\Phi_\alpha),$$

$$\delta_j=\frac{2^{q-2}D_j}{\gamma_j a_j^q}\left(1+\frac{\alpha_{0j}^2}{\alpha_{dj}^2}\right)\left[\left(\sum_{j=1}^{3}a_j^{-2}\right)^2+\left(\sum_{j=1}^{3}\frac{\alpha_{0j}}{a_j^2\alpha_{dj}}\right)^2\right], \tag{36}$$

$$\Phi_\alpha=\frac{q}{2}\operatorname{arctg}\left\{\sum_{j=1}^{3}\frac{\alpha_{0j}}{\alpha_{dj}a_j^2}\Big/\sum_{j=1}^{3}a_j^{-2}\right\},$$

from which it is easy to derive the necessary condition for the self-capture of the beams:

$$(\delta_2\delta_3)^{1/2}/E_1+(\delta_1\delta_3)^{1/2}/E_2+(\delta_1\delta_2)^{1/2}/E_3<\cos(\Phi_0-\Phi_\alpha). \tag{37}$$

It follows from the last expression that in the case when the non-optimal—for energy exchange—phase relation $\Phi_0 = 0$ obtains collimated (i.e., $\alpha_{0j} = 0$) beams with intensities of the same order of magnitude are more easily captured and coupled to form waveguides, the diffraction effects being determined by the smallest radius. When the interacting beams are identical (i.e., when $a_j = a$ and $\alpha_{0j} = 0$) their intensities should exceed some critical value

$$S_{cr\,4}=S_0\lambda_3^4/(1-\mu^2)a^4. \tag{38}$$

If we change the phase relation such that $\cos\Phi_0 < 0$, or if we make the beams highly convergent (divergent), then the coupled waveguides may not go into the regime of self-capture.

The structure of nonoscillating waveguides with amplitudes

$$A_j = A_{w,j}(x,y)e^{-i\beta_j z},\quad A_{w,j} = \left(\frac{D_1 D_2 D_3 \gamma_j}{\gamma_1\gamma_2\gamma_3 D_j a_w^4}\right)^{1/2} B_{w,j}\left(\frac{x}{a_w},\frac{y}{a_w}\right) \quad (39)$$

can be found by solving the boundary-eigenvalue problem for the system of equations ($\beta_3 = \beta_1 + \beta_2 - \Delta$)

$$\Delta_\perp B_{w,j} = (\beta_j a_w^2/D_j)B_{w,j} - B_{w,1}B_{w,2}B_{w,3}/B_{w,j}, \quad (40)$$

which follows from (1)—(3) when allowance is made for (39) and which corresponds to the variational problem of the absolute extremum of the functional

$$J = I_3 + \beta_1 I_{1,2} + \beta_2 I_{1,1}. \quad (41)$$

The coupled waveguides propagate more slowly than waves in a linear medium ($\beta_j > 0$, $v_w < c/n_j$), and carry with them the power

$$P_{w,j} = \frac{S_0 n_j \lambda_3^4 \tilde{P}_{w,j}}{a_w^4}\begin{cases}(1-\mu^2)^{-2}, & j=1,2\\ 1, & j=3\end{cases}, \quad (42)$$

where $\tilde{P}_{w,j} = \int\int A_{w,j}^2 dx_w dy_w$. The asynchronism integral then assumes a negative value:

$$I_s = -\int\int A_{w,1}A_{w,2}A_{w,3}\,dx\,dy = -\frac{1}{2}\sum_{j=1}^{3}\frac{\beta_j}{\gamma_j}P_{w,j}. \quad (43)$$

In the general case the formulated problem of finding the amplitude profiles of the waveguides is fairly complicated and requires a separate treatment. Here we shall discuss only some waveguide solutions to Eqs. (40).

If $\beta_j = D_j/a_w^2 = \Delta D_j/(D_3 - D_1 - D_2)$, then there exists a degenerate wave guide with $B_{w,1} = B_{w,2} = B_{w,3}$. Such a one-dimensional waveguide ($q = 1$) has the profile $B_{w,j} = \frac{3}{2}\cosh^{-2}(x/2a_w)$, and its power is $\tilde{P}_w = 3$.

For one-dimensional beams the problem is generally somewhat simpler, since the Hamiltonian for the functional (41)

$$H = r^{q-1}\left\{\sum_{j=1}^{3}\left[\frac{D_j}{\gamma_j}\left(\frac{\partial A_j}{\partial r}\right)^2 - \frac{\beta_j}{\gamma_j}A_j^2\right] + 2A_1A_2A_3\right\} \quad (44)$$

is an integral of motion of the waveguide equations (40) (H = const for $q = 1$). If we allow for the fact that $H = 0$ on account of the conditions $A_{w,j} = 0$ for $|x| \to \infty$, and that $dA_{w,x}/dx = 0$ for $x = 0$, then we can relate the amplitudes on the axis of the beams:

$$\sum_{j=1}^{3}\frac{\beta_j}{\gamma_j}A_{w,j}^2(0) = 2A_{w,1}(0)A_{w,2}(0)A_{w,3}(0). \quad (45)$$

The numerical solution of the Eqs. (40) with allowance for (45) in the degenerate case $\mu = 0$, $\Delta = 0$, $\beta_1 = \beta_2$, $B_{w,1} = B_{w,2}$ yields $B_{1,2}(0) = 2.9$, $B_3(0) = 1.7$, $\tilde{P}_{1,2} = 9.6$, and $\tilde{P} = 2.8$.

Notice, finally, that we can, with the aid of (43), estimate the effective radius of the waveguides.

5. CONCLUSION

Thus, the diffractive incoherence of the three-photon interactions of bounded beams leads to mutual focusing or the propagation of the beams in the form of waveguides. In the optical band these phenomena can be observed in non-centrosymmetric crystals. For typical crystals with $\chi = 10^{-8}$ cgs esu and $L = 5$ cm, the critical intensity of beams with $\lambda_3 = 0.53 \times 10^{-4}$ cm, $\mu = 0$, and $a_3 = 0.1$ mm (in this case $R_{d,3} \cong L$) is equal to $S_{cr} \cong 1$ MW/cm^2, which is entirely attainable in laser experiments. Upon going over to the ultraviolet region of the spectrum, the critical intensity decreases sharply ($S_{cr} \sim \lambda_3^4$). The mutual-focusing-induced increase of the wave intensity may lead to the breakdown of the crystals.

It is necessary to take similar phenomena into account in the investigation of stimulated scattering, which can be treated as a parametric interaction of primary and scattered waves with the motions of the medium, as well as in the investigation of four-photon interactions.

The diffractive incoherence imposes a limitation on the efficiency of optical frequency converters. For this reason, it is, in particular, impossible to obtain 100% efficiency in second-harmonic generators.

On account of the space-time analogy[5] in the resonant interaction of narrow pulses, there can arise the effects of anomalous dispersive spreading, mutual compression of the pulses, and, under certain conditions when the phase-asynchronism integral is negative, the formation of coupled solitons is possible. The wave-packet diffusion coefficients in the basic equations (1)—(3) are then equal to $D_j = \frac{1}{2}\partial^2 k_j/\partial\omega_j^2$ and the Laplacian $\Delta_\perp = \partial^2/\partial\eta^2$. In this case the critical intensity $S_{cr} \sim S_0 k_{\omega\omega}^2 \lambda_3^2 \tau^{-4}$, where τ is the pulse length. In contrast to the interaction of beams, for which $D_j > 0$, the diffusion coefficients for pulses can have different signs, which can significantly change the nature of the compression of the pulses.

The resonant interaction of pulses in a nonlinear dispersive medium is of interest for acoustics, hydrodynamics, and plasma physics; the detailed investigation of such interactions merits a separate analysis.

APPENDIX

1. Formulation of the numerical experiments. To carry out the numerical experiments, the system of differential equations (1)—(3) was reduced to a dimensionless form. The longitudinal coordinate z was normalized to the thickness L of the nonlinear medium, i.e., we set $\tilde{z} = z/L$; the transverse radius vector—to the characteristic beam dimension: $(\tilde{x}, \tilde{y}) = (x/a, y/a)$; the field amplitudes—to the characteristic value of the pump-wave amplitude: $B_j = A_j/E_0$. The coefficients entering into the equations then become equal to the following quantities: $\tilde{D}_j = L/2k_j a_0^2$ and $\tilde{\gamma}_j = \gamma_j E_0 L$.

In the numerical solution the system (1)—(3) was approximated by implicit and symmetric conservative difference schemes, i.e., schemes for which the difference analogues of the relations (4) and (5) are realizable on the net (see[6]). To find the solution in a new z layer, we used the method of successive approximations and difference trial runs.

Equations (1)—(3) were solved inside a cylinder, $r < R$. The radius of the cylinder was chosen to be sufficiently large, so that the influence of the lateral boundary was negligible. The wave amplitudes at the lateral surface $r = R$ were set equal to zero, i.e., $B_j(0, z) = 0$.

In the numerical experiments we followed the modification of the wave fronts and the intensity profiles of

Gaussian beams interacting in a nonlinear medium (the results are presented in the form of graphs). The beam radius $\rho(z)$ was determined from the relation $|B(\rho, z)| = \frac{1}{2}|B(0, z)|$.

2. Derivation of the fundamental inequality. From the phase-asynchronism integral (7) follows the obvious inequality

$$m\iint\sum_{j=1}^{3}(\nabla_\perp A_{0j})^2 dx\,dy \leqslant I_2+\frac{\Delta}{\gamma_3}P_3(z) +2\left|\iint A_{01}A_{02}A_{03}\cos\Phi\,dx\,dy\right|, \tag{A.1}$$

where m is given by the formula (28). Notice further that

$$\left|\iint A_{01}A_{02}A_{03}\cos\Phi\,dx\,dy\right| \leqslant MI_1/6, \tag{A.2}$$

where M is given by the formula (27). Let us now transform the left-hand side of (A.1).

In the case of the interaction of two-dimensional waves (one-dimensional beams) we can use the Cauchy-Bunyakovskiĭ inequalities

$$\int\sum_{j=1}^{3}\left(\frac{\partial A_{0j}}{\partial x}\right)^2 dx \geqslant \sum\frac{\max_x A_j^4}{4P_j} \leqslant \frac{M^4}{108 I_1}. \tag{A.3}$$

For axially symmetric beams we have

$$\int\sum_{j=1}^{3}\left(\frac{\partial A_{0j}}{\partial r}\right)^2 r\,dr \geqslant \frac{1}{2}\sum_{j=1}^{3}A_{0j}^2(0,z) \leqslant \frac{M^2}{6}. \tag{A.4}$$

Here we have assumed that

$$\int A_{0j}\frac{\partial^2 A_{0j}}{\partial r^2}r\,dr<0,$$

which is valid for beams with bell-shaped amplitude profiles. It is precisely such a profile that is observed for beams in the region of their mutual focusing.

Substituting (A.2)–(A.4) into (A.1), we arrive at the fundamental inequality (29).

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