Three-wave processes in an inhomogeneous plasma in the presence of cutoff points

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Three-wave processes in an inhomogeneous plasma in the presence of cutoff points are investigated. In the general case, a pumping field of frequency ω_0 is connected by decay interaction with three waves with frequencies ω_2 and $\omega_{1,3} = \omega_0 \pm \omega_2$. An integral transformation is proposed, which reduces the set of three coupled second-order differential equations describing these interactions to a single second-order equation. The transformation is used to determine the decay of the pumping field into two waves with close frequencies. The instability is absolute in this case. Relations are derived which describe the dependence of its threshold on the parameters that characterize the wave dissipation and the plasma inhomogeneity throughout their range of variation. Some concrete examples are considered.

Three-wave processes in an inhomogeneous plasma are of considerable interest and have lately been intensively investigated theoretically[1-11]. The results point to the existence of both convective[1-3,5,7] and absolute instabilities[2-4,8-11] and to particular importance of cases when the resonant interaction of the waves takes place at points where the geometrical-optics approximation is violated[8-11]. In this and in following papers, we consider systematically similar situations (with the exception of the case of hybrid resonance of the pump wave). We describe here a method for analyzing systems of coupled differential equations that appear in such problems, and investigate the process of the decay of the pump wave into two waves of equal type with nearly equal frequencies. In a subsequent paper we consider the decay into different modes.

1. DECAY INTERACTION OF WAVES IN THE GEOMETRICAL-OPTICS APPROXIMATION

The characteristic features of three-wave processes in a weakly-inhomogeneous plasma are connected with the fact that the decay conditions for the wave vectors can be satisfied only on individual points of the x axis, in the vicinity of which the resonant interaction is localized[1-3]. A wave passing through the resonance region in the presence of a given pump field undergoes a finite amplification, and accordingly the decay instability becomes convective. This process was considered in most detail by Rosenbluth[3] and by one of us[7], where it was shown that the stationary state is stable (absolute instability is possible[3,8-11] only in a special case that will be discussed later on), and calculated the transition matrix $S_{ik}^{(0)}$ connecting the amplitudes of waves 1 and 2 entering and leaving the resonant layer:

$$S_{11}^{(0)}=S_{22}^{(0)}=e^{\pm\pi Z}, \quad |S_{12}^{(0)}|\sim|S_{21}^{(0)}|\sim|e^{\pm 2\pi Z}-1|^{1/2}, \tag{1}$$

$Z = \gamma_0^2 l^2/|u_1u_2|$, $\gamma_0^2 = |V_{k_1k_0k_2}|^2E_0^2$, $V_{k_1k_0k_2}$ is a suitably normalized matrix element of the wave interaction, calculated without allowance for the spatial inhomogeneity, while the quantities k_0 and E_0 pertain to the pump wave

$$l^2=\left|\frac{\partial}{\partial x}(k_{1x}-k_{2x}-k_{0x})\right|^{-1},$$

and u_i are the x-th components of the group velocity. The plus sign in (1) corresponds to the case of decay ($\omega_0 = |\omega_1| + |\omega_2|$), and the minus sign to the case of coalescence of the waves ($\omega_0 = ||\omega_1| - |\omega_2||$). In the case of decay, γ_0 is the growth rate of the decay instability in a homogeneous plasma without dissipation. At $Z \gg 1$ and in the case of decay, the incident wave becomes amplified and a wave of a second type with an amplitude of the same order of magnitude is excited; in the case of coalescence, the incident wave is practically entirely transformed into the second wave without amplification..

It will be shown later on that the process described by the matrix $S_{ik}^{(0)}$ is the "elementary act" of wave interaction, and is the basis of more complicated situations, particularly those leading to absolute instabilities.

It is seen from (1) that a special analysis is required only for the cases $u_i = 0$ and $l^{-2} = 0$. When one of the group velocities vanishes, say u_1, the parameter Z_1 remains finite. (For example, near the cutoff point x_1 of the wave 1 we have $u_1 \sim k_{1x} \sim (x_1 - x)^{1/2}$ and $l^2 \approx |\partial k_{1x}/\partial x|^{-1} \sim (x_1 - x)^{1/2}$.) It can be assumed that the amplification will be described here, just as before, by the matrix $S_{ik}^{(0)}$. However, the geometrical optics approximation used in the derivation of (1) is violated for the corresponding wave at $u_i = 0$. The interaction near the cutoff of the wave 1 can be described by the system of equations

$$f_1''-\alpha_1(x-x_1)f_1=V_1e^{iKx}a_2, \quad a_2'=V_2e^{-iKx}f_1, \tag{2}$$

where a_2 is the slowly varying amplitude of the wave 2, the function $f_1(x)$ describes the dependence on x of wave 1, in the form $f_1(x)\exp[i(k_yy + k_zz - \omega t)]$, $V_{1,2}$ are constants proportional to E_0, and $K = k_{2x} - k_{0x}$; the slow x-dependence of the wave vectors $\mathbf{k}_2$ and $\mathbf{k}_0$, which have no singularities, is disregarded.

At large k, the resonance $k_{1x} \equiv (\alpha_1(x_1 - x))^{1/2} = K$ takes place in the region where the geometrical optics approximation is valid for the wave 1, and the system (2) can be easily reduced to the standard equations for the amplitudes a_1 and a_2[7] with parameters $l^2 = \alpha_1/2K$ and $Z = iV_1V_2/\alpha_1$. In accordance with the foregoing, Z turns out to be independent of K. On the other hand, Eq. (2) can be easily solved by using a Laplace transform, and the result for the transition matrix is actually Eq. (1) with $Z = iV_1V_2/\alpha_1$.

2. INTEGRAL TRANSFORMATION FOR SYSTEMS OF COUPLED EQUATIONS

In analogy with (2), the case of cutoff of two waves is described by the system of equations

$$f_1''-\alpha_1(x-x_1)f_1=V_{12}e^{ik_0x}f_2,\quad f_2''-\alpha_2(x-x_2)f_2=V_{21}e^{-ik_0x}f_1, \tag{3}$$

where $k_0 = k_{0x}$; this system is used for a number of three-wave processes in the case of large k, which is of greatest interest. For example, it describes the decay of a transverse wave into oblique hybrid waves and the decay $t \to l + s$ in an isotropic plasma[2,4]. It might be assumed that in other cases the system (3), which imitates the dispersion properties of the noninteracting waves, describes correctly the character of the phenomena. It is assumed in (3) that ω_2 or k_0 are so large that there is no need for simultaneously considering the interaction of wave 2 with the red and violet satellites of the pump wave.

The opposite case ("low-frequency decay") will be considered here at $k_0 = 0$. It is described by the system of equations

$$\begin{gathered} f_1''-\alpha(x-x_1)f_1=V_{12}f_2,\quad f_2''-\alpha_2(x-x_2)f_2=V_{21}f_1+V_{23}f_3,\\ f_3''-\alpha(x-x_3)f_3=V_{32}f_2, \end{gathered} \tag{4}$$

where $\omega_1 = \omega_2 - \omega_0$, $\omega_3 = \omega_2 + \omega_0$ and $\omega_2 \ll \omega_0$. Owing to the proximity of the frequencies $|\omega_1|$ and ω_3, we have put $\alpha_1 = \alpha_3 = \alpha$ and we can assume that $V_{12}V_{21} = V_{32}V_{23}$.

In the literature, the system (4) was considered for $\alpha_2 \to 0$, $x_2 \to \infty$, $\alpha_2 x_2 = \text{const}$, i.e., without allowance for the inhomogeneity for the low-frequency wave, and was solved by discarding terms with derivatives in one or two equations[2,4].

To obtain a solution that is suitable simultaneously for Eqs. (3) and (4), we consider the more general system from which they are obtained as particular cases:

$$\begin{gathered} f_i''-\alpha(x-x_i)f_i=V_{i2}e^{ik_0x}f_2,\quad i=1,3,\\ f_2''-\alpha_2(x-x_2)f_2=(V_{21}f_1+V_{23}f_3)e^{-ik_0x}. \end{gathered} \tag{5}$$

We represent the functions f_i in the form

$$f_i=\frac{V_{i2}}{\alpha s}\int_L \frac{y(\xi)v[\alpha^{1/3}(s\xi+x)]}{\xi+\xi_i}e^{i\varkappa\xi}\,d\xi, \tag{6}$$

$$f_2=e^{-ik_0x}\int_L y(\xi)v[\alpha^{1/3}(s\xi+x)]e^{i\varkappa\xi}\,d\xi, \tag{6$'$}$$

where $y(\xi)$ is an unknown functions, $v(z)$ is an Airy function that decreases as $z \to \infty$, $s = (1/\alpha - 1/\alpha_2)^{1/3}$, $\xi_i = x_i/s$, $\varkappa = k_0/\alpha_2 s^2$, and L is a contour in the complex ξ plane and will be determined later on.

We substitute (6) and (6′) in (5), eliminate the terms containing x with the aid of the Airy equation, and integrate (where necessary) by parts. We can then easily show that the function $y(\xi)$ should satisfy the equation

$$\begin{gathered} y''+\left(\beta-\xi+\frac{\lambda_1}{\xi+\xi_1}+\frac{\lambda_3}{\xi+\xi_3}\right)y=0;\\ \beta=k_0^2/s(\alpha_2-\alpha)-\xi_2,\quad \lambda_i=V_{i2}V_{2i}/\alpha\alpha_2 s^2. \end{gathered} \tag{7}$$

Equations of this type describe wave propagation in a cold inhomogeneous plasma in the presence of the hybrid-resonance points ξ_1 and ξ_2[12]. We are interested in a solution that decreases in the "opacity region," i.e., as $\xi \to \infty$. It has an asymptotic form

$$y\approx(\beta-\xi)^{-1/4}\left[\exp\left(-i\frac{2}{3}(\beta-\xi)^{3/2}\right)+R\exp\left(i\frac{2}{3}(\beta-\xi)^{3/2}\right)\right],\quad \xi\to-\infty,$$

$$y\approx C(\xi-\beta)^{-1/4}\exp\left(-\frac{2}{3}(\xi-\beta)^{3/2}\right),\quad \xi\to\infty. \tag{8}$$

The points ξ_i are branch points of the function y:

$$y\approx C_i[1-\lambda_i(\xi+\xi_i)\ln(\xi+\xi_i)],\quad \xi\to\xi_i,$$

therefore the "reflection coefficient" R in (8) depends on the path along which the solution that decreases as $\xi \to \infty$ is continued, i.e., on the positions of the cuts on the ξ plane. In the wave equation (7), wave absorption or generation takes place at the singular points, so that $|R| \neq 1$ even for real parameters[12].

Three solutions of the system (5), which do not increase in the opacity region for the waves f_i, can be obtained by using $y(\xi)$ in the asymptotic form of (8) and different integration contours in (6). These contours should satisfy the usual conditions that ensure convergence of the integrals and validity of integration by parts. These conditions are in fact satisfied by contours that go off to $\pm\infty$ along the real axis. Indeed, if $\alpha^{1/3}s < 0$, then the integrands decrease as $\xi \to \infty$ because of $y(\xi)$ and as $\xi \to -\infty$ because of $v(z)$. If $\alpha^{1/3}s > 1$, then we can assume that the Airy function is represented in the form $v = w_1 + w_2$, where $w_{1,2}$ decrease in the sectors $\pi/3 < \arg z < \pi$ and $\pi/3 > \arg z > -\pi$ and, accordingly, all the integrals in (6) break up into sums of two terms. In each of these terms the integration is carried out from ∞ to a certain point ξ', $(-\xi') \gg 1$, along the same path, and then the contour goes off to a sector where the Airy function $w_{1,2}$ decreases. At $0 < \alpha^{1/3}s < 1$ we can represent the function $y(\xi)$ analogously.

It is now easy to determine the asymptotic behavior of the integrals (6) and (6′) as $|x| \to \infty$. In (6′) at $\alpha_2^{1/3}x \to -\infty$, i.e., in the transparency region, the main contribution for the wave 2 is made by the stationary-phase point, in which the functions v and y can be replaced by their asymptotic expressions. As a result we obtain

$$\begin{gathered} f_2=\sqrt{\pi}\left|\frac{\alpha_2}{\alpha}\right|^{1/3}X_2^{-1/4}\exp\left[\frac{ik_0^3(\alpha_2+\alpha)}{3(\alpha_2-\alpha)^2}-\frac{ik_0x_2}{\alpha s^3}-i\frac{\pi}{4}(1\pm1)\right]\\ \times\left[\exp\left(\pm i\frac{2}{3}X_2^{3/2}\right)\mp R\exp\left(\mp i\frac{2}{3}X_2^{3/2}\right)\right], \end{gathered} \tag{9}$$

where $X_2 = \alpha_2^{1/3}(x_2 - x)$ and the upper and lower signs are taken at $\alpha_2^{1/3}s > 0$ and $\alpha_2^{1/3} < 0$, respectively.

In the integrals (6), the principal terms are determined asymptotically by the vicinity of the singular points:

$$f_i=-\frac{\pi V_{i2}C_i}{2|\alpha s|X_i^{1/4}}\exp\left[\pm i\frac{2}{3}X_i^{3/2}\pm i\frac{\pi}{4}\right],\quad X_i=\alpha^{1/3}(x_i-x). \tag{10}$$

At $\alpha^{1/3}s > 0$, the upper and lower sign are taken as the corresponding singularity is bypassed from above and below; the situation is reversed when $\alpha^{1/3}s < 0$.

By choosing three integration contours that bypass the singular points ξ_i differently, we can construct a system of fundamental solutions of Eqs. (5), containing a decreasing wave of definite type. The amplitudes of the outgoing waves determine in this case the transition matrix S_{ik}. It is physically clear, and it can be proved formally, that the presence of the instability can be assessed from any one of its elements. To save space, we therefore confine ourselves to the quantity S_{22}, which is determined by a solution, continuous on the real axis, of Eq. (7). Indeed, in the presence of damping, the quantities x_i in (5) are complex:

$$x_i=x_i'+ix_i'',\quad x_i''=\frac{\partial x_i}{\partial\omega}\nu_i, \tag{11}$$

where ν_i is the linear damping decrement. Considering the wave of type i near the cutoff but in the region where the geometrical-optics approximation is valid, we obtain from the identity $u_i^{-1} \equiv \partial k_{1x}/\partial\omega$, $k_{1x} = (\alpha_i(x_i - x))^{1/2}$:

$$\frac{\partial x_i}{\partial \omega} = \frac{2k_{ix}}{u_i \alpha_i}. \tag{12}$$

With allowance for this equality, which determines the sign of the imaginary part of ξ_i, we can easily establish that the contour passing along the real axis always leads to the appearance of outgoing waves 1 and 3 in (10) (assuming that the energy of these waves is positive).

Under the assumption $\omega_2 > \omega_2 k_{2x} u_2 > 0$, the incident wave in (9) corresponds to a term with a minus sign in the argument of the exponential. Thus,

$$S_{22} = -R^{-1} \quad \text{at} \quad \alpha_2^{1/3} s > 0, \qquad S_{22} = R \quad \text{at} \quad \alpha_2^{1/3} s < 0. \tag{13}$$

The quantity S_{22} can be regarded as a function of ω_2. Then the presence in it of poles in the upper half-plane indicates in the usual manner, absolute instability. The inequality $|S_{22}| > e^A$, $A = 5-10$, means the presence of convective instability. Thus, the problem of parametric instability in the system of coupled waves described by the systems (3) and (4) reduces to an investigation of the "reflection coefficient" R in the solution of the second-order equation (7) which is continuous on the real axis.

3. DECAY INTO WAVES WITH COMPARABLE FREQUENCIES

We now consider in greater detail the case of "high-frequency decay" (3), which is obtained from (5) at $V_{23} = V_{32} = 0$. In (7) at $\lambda_3 = 0$, it is convenient to make the change of variables $\xi + \xi_1 \to \xi$, which yields for $y(\xi)$ the equation

$$y'' + (\beta - \xi + \lambda/\xi) y = 0, \quad \beta = \xi_1 - \xi_2 + k_0^2/s(\alpha_2 - \alpha), \quad \lambda \equiv \lambda_1. \tag{14}$$

In this paper we consider only the case $\lambda > 0$. For a homogeneous pump field ($k_0 = 0$), the parameter β vanishes when the cutoff point of the two waves coincide. It is easy to verify that at $k_0 \neq 0$ the equality $\beta = 0$ means tangency of the dispersion curves $k_{1x} = \sqrt{\alpha(x_1 - x)}$ and $k_{2x} = k_0 = (\alpha_2(x_2 - x))^{1/2} - k_0$. Thus, both singular cases at which $Z \to \infty$ (see (1): $u_1 \to 0$, $u_2 \to 0$ and $l^2 \to \infty$) turn out to be perfectly equivalent mathematically.

Strictly speaking, the coefficients V_{12} and V_{21} in (3) are smooth functions of x. But since it is clear from physical considerations that the interaction of the waves is determined mainly by the "resonant" region near the point of tangency of the dispersion curves, we can substitute in the system (3) their values at this point. Taking into account the symmetry of Eqs. (3) and (4) relative to the waves 1 and 2, we can assume, without loss of generality, that $\alpha < \alpha_2$; then the singular point $\xi = 0$ of Eq. (14) is bypassed from below. This bypass direction, at $\lambda > 0$, corresponds to absorption of the incident wave, if we regard (14) as a wave equation. According to (13), absolute instability sets in at total absorption, and convective instability at absorption close to total. At a given λ the total absorption is possible only at definite, generally speaking complex, values of β. These eigenvalues form an infinite discrete set $\beta_n(\lambda) = \beta_n' + i\beta_n''$. Consequently, the frequencies and the growth rates of the unstable oscillates are determined by the dispersion equation

$$\beta(\omega_2) = \beta_n(\lambda). \tag{15}$$

The right-hand side of this equation is a universal function of $\lambda \sim E_0^2$. Unfortunately, analytic expressions for β_n can be obtained only in limiting cases. It is clear from physical considerations that $|\beta_n''|$ should increase monotonically with increasing λ, going through zero at a certain λ_n, whereas β_n' can have at $\lambda \sim \lambda_n$ any value, depending on n. It follows from (14) that strong absorption is possible only at $\beta' > 0$ and at sufficiently high transparency of the barrier lying to the left of the point $\xi = 0$, i.e., at $\lambda/\beta^{1/2} \lesssim 1$. We consider therefore (14) in the region of the parameters $\beta' > 0$, $(\beta')^2 \gg 1 + \lambda$, $\beta' \gg |\beta''|$. In this case, Eq. (14) can be solved in the geometrical-optics approximation. This approximation is violated at small $|\xi|$; in this region we can neglect ξ in comparison with $\beta + \lambda/\xi$, after which the solution is obtained in terms of Whittaker functions. The regions of applicability of both approximations overlap. As a result we obtain for R the expression

$$R = e^{-2\pi Z}[-i + \exp(-i(4/3\,\beta^{3/2} + \psi))] - \exp(-i(4/3\,\beta^{3/2} + \psi)), \tag{16}$$

$$Z = \lambda/2\beta^{1/2}, \qquad \psi \lesssim 1. \tag{17}$$

The considered case $\mathrm{Re}\,\beta \gg 1$ corresponds to the intersection of the dispersion curves k_{1x} and $k_{2x} - k_0$ in the region of applicability of the geometrical-optics approximation. Here, as can be readily verified, the parameter Z in (1) turns out to be the same at both resonant points and coincides with (17). Equating (16) to zero, we obtain (at $\beta_n' \gg |\beta_n''|$):

$$\beta_n' = \left(\frac{3\pi n}{2}\right)^{2/3}, \quad \beta_n'' = -\frac{1}{2}\left(\frac{2}{3\pi n}\right)^{1/3} \ln\left[\exp\left(\pi\lambda\left(\frac{2}{3\pi n}\right)^{1/3}\right) - 1\right]. \tag{18}$$

These expressions are valid for $(3\pi n/2)^{2/3} \gg 1$ and $\lambda \ll (3\pi n/2)^{4/3}$. Another limiting case that can be considered corresponds to $\lambda \to \infty$. Here, too, $|\beta_n| \to \infty$ and geometrical optics can still be applied to (14). The condition for the existence of a solution corresponding to total absorption takes in this case the form

$$\int_{\xi^{(1)}}^{\xi^{(2)}} \left(\beta - \xi + \frac{\lambda}{\xi}\right)^{1/2} d\xi = \pi\left(n + \frac{1}{2}\right), \tag{19}$$

where the integration is carried out in the complex plane between the turning points $\xi^{(1,2)}$. As $\lambda \to \infty$, Eq. (19) can be satisfied only as $4\lambda/\beta^2 \to -1$ (i.e., in the case when the points $\xi^{(1)}$ and $\xi^{(2)}$ approach each other). Taking this into account, we easily obtain for $\lambda^{3/4} \gg n$

$$\beta_n' = (n + 1/2)/(4\lambda)^{1/4},$$
$$\beta_n'' = -(4\lambda)^{1/2} + (n + 1/2)(4\lambda)^{-1/4}. \tag{20}$$

Expressions (18) and (20) enable us to trace the behavior of $\beta_n(\lambda)$ at $n \gg 1$. Figure 1 shows numerically calculated plots of β_n' and β_n'' at n = 0 and 1. We note that, with accuracy not worse than several percent, $\beta_0''(\lambda)$ is described by the formula $\beta_0'' = -(4\lambda)^{1/2} + (4\lambda)^{-1/4}$ for all λ for which $\beta_0'' < 0$.

4. PARAMETRIC FREQUENCY DIVISION

Let us apply the results to a case of practical importance, that of decay into identical modes (parametric frequency division). For the qualitative analysis presented here, we can replace V_{12} and V_{21} in (3) by their values near the cutoff points of waves 1 and 2. Considering the system (3) in the geometrical-optics approxi-

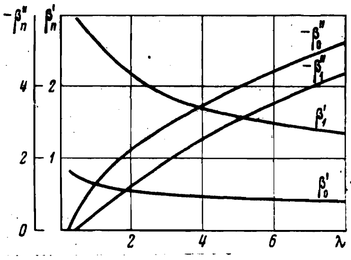

FIG. 1. The quantities β_n'' and β_n' for the fundamental and first modes as functions of the parameter λ.

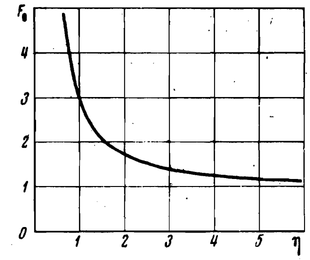

FIG. 2. The function $F_0(\eta)$ for the fundamental mode.

mation and using (12), we can express the parameter λ in terms of the growth rate γ_0 of the instability in a homogeneous plasma,

$$\lambda=\gamma_0^2\left|\frac{\partial x_1}{\partial\omega_1}\right|\left|\frac{\partial x_2}{\partial\omega_2}\right|s^{-2}.$$

We assume that $k_0 \ll |\alpha|^{1/3}$. At small k_0, resonant interaction is possible only at $\delta = ||\omega_1| - |\omega_2/\omega_0| \ll 1$. In this case we can regard the differences $\alpha_2 - \alpha$ and $\xi_1 - \xi_2$ in (14) as proportional to δ.[1)] It is also convenient to separate in explicit form the dependence of the parameters on the layer inhomogeneity scale $a = (\partial \ln n/\partial x)^{-1}$, putting

$$\alpha_i=\frac{\bar{k}_i^2}{a},\quad\left|\frac{\partial x_i}{\partial\omega}\right|=\frac{a}{\bar{\omega}_0},$$

where $\bar{k}_i$ has the meaning of the maximum value of $k = (k_y^2 + k_z^2)^{1/2}$ allowed by the dispersion equation, and $\bar{\omega}_0 \sim \omega_0$. Then, taking (11) into account, we obtain

$$\beta'=\left(\frac{k_0a}{\mu}\right)^{2/3}t^{-1/3}\left(\frac{1}{t}-t\right),\quad\beta''=-2\left(\frac{\bar{k}^3a^2}{k_0\mu^{1/2}}\right)^{1/3}t^{-1/3}\bar{\nu},$$

$$\lambda=\frac{\gamma_0^2}{\bar{\omega}_0^2}\left(\frac{\bar{k}^3a^2}{k_0\mu^{1/2}}\right)^{2/3}t^{-2/3}=\lambda_0t^{-2/3};$$

$$t=\frac{\delta\bar{k}\mu^{1/2}}{k_0},\quad\mu=\left|\frac{\bar{\omega}_0}{\bar{k}^2}\frac{\partial(\bar{k}^2)}{\partial\omega}\right|\sim1,\quad\bar{k}=\bar{k}_i|_{\omega_i=\omega_0/2},$$

$$\bar{\nu}=\frac{\nu_1+\nu_2}{2\bar{\omega}_0},\quad\delta=\left|\frac{|\omega_1|-|\omega_2|}{\bar{\omega}_0}\right|.$$

The instability threshold of the n-th mode is determined obviously by that value of $\lambda = \lambda_n$ at which (15) has real solutions, i.e., in our case, by the system of equations

$$\left(\frac{k_0a}{\mu}\right)^{2/3}t^{-1/3}\left(\frac{1}{t}-t\right)=\beta_n'(\lambda_0t^{-2/3}),$$

$$2\left(\frac{\bar{k}^3a^2}{k_0\mu^{1/2}}\right)^{1/3}t^{-1/3}\bar{\nu}=-\beta_n''(\lambda_0t^{-2/3}),$$

where the unknowns are t and λ_0.

At not very large n and at $(k_0a) \gg 1$ (i.e., in the case when the geometrical-optics approximation is valid for the pump wave), the approximate solution of the first equation is $t = 1$, i.e.,

$$\delta=k_0\mu^{-1/2}/\bar{k}.$$

The second equation determines then the threshold value of $\gamma_0/\bar{\omega}_0^2$ (and consequently also of E_0, since $\gamma_0^2 \sim E_0^2$) in the form of a universal function of one parameter $\eta = 2(\bar{k}^3a^2/k_0\mu^{1/2})^{1/3}\bar{\nu}$

$$\gamma_0^2/\bar{\omega}_0^2=\bar{\nu}^2F_n(\eta).\tag{21}$$

At large n, according to (18) and (20), we have

$$F_n(\eta)=\frac{4}{\pi}\left(\frac{3\pi n}{2}\right)^{1/3}\eta^{-2}\ln\left\{1+\exp\left[2\left(\frac{3\pi n}{2}\right)^{1/3}\eta\right]\right\},\quad\eta\ll1,\tag{22}$$

$$F_n(\eta)=1+(2n+1)/\eta^{3/2},\qquad\eta\gg1.$$

A plot of the function $F_0(\eta)$ at $n = 0$ is shown in Fig. 2. Thus, in the case of weak damping or relatively strong inhomogeneity, the threshold is independent of $\bar{\nu}$ and is determined by the relation $(n = 0)$

$$\frac{\gamma_0^2}{\bar{\omega}_0^2}\approx\frac{1}{4}\left(\frac{\bar{k}^3a^2}{k_0\mu^{1/2}}\right)^{-2/3}\sim a^{-4/3}.$$

With increasing inhomogeneity scale, the threshold ceases to depend on a and tends to a limiting value that coincides in order of magnitude with the threshold of the corresponding instability in a homogeneous plasma. The influence of the inhomogeneity becomes therefore significant at values of a for which $\eta \lesssim 1$, i.e., at $\bar{k}a \sim \bar{\nu}^{3/2}(k_0/\bar{k})^{1/2}$.

By way of illustration, we consider a concrete example of the decay of an ordinary wave with frequency $\omega_0 \ll \omega_H$ (ω_H is the electron cyclotron frequency) into "oblique" hybrid waves with nearly equal frequencies. For simplicity we assume that the pump wave propagates along the x axis. In this case the coefficients β and λ which enter in (14) can be easily obtained by comparing the system (3) with the corresponding equations for a homogeneous plasma, obtained in the hydrodynamic approximations:

$$\beta=(k_0a)^{2/3}t^{2/3}\left(t^{-2}-1-\frac{i\bar{\nu}}{2\delta}\right),$$

$$\lambda=(k_0a)^{4/3}t^{4/3}\frac{k^2v_E^2}{\omega_0^2}\cos^2\theta,$$

$$t=\frac{4\delta k}{k_0},\quad\delta=\left|\frac{|\omega_1|-|\omega_2|}{\omega_0}\right|.$$

$$\cos^2\theta=\frac{k_z^2}{k^2},\quad k^2=k_z^2+k_y^2,\tag{23}$$

$$v_E=\frac{eE_{0z}}{m\omega_0},\quad k_0=\frac{\omega_0}{c}\left(1-\frac{1}{2\cos^2\theta}\right)^{1/2}$$

$$\bar{\nu}=\frac{2\nu_e}{\omega_0}+\frac{\pi^{1/2}\omega_0^3}{4k_z^3v_T^3}\exp\left[-\frac{\omega_0^2}{4k_z^2v_T^2}\right],$$

where a is the scale of the plasma inhomogeneity ($a = |d \ln n/dx|^{-1}$ at $(4\omega_p^2(x)/\omega_0^2)\cos^2\theta = 1$).

To assess the accuracy of the presented formulas, we calculated the matrix elements V_{ik} for plasma parameters corresponding to the tangency of the quasiclassical curves, which occurs at $\delta \approx k_0/4k$. Using formulas (15), (18), and (20), we can obtain relations analogous to (21) and (22) for threshold values of the pump field

$$\bar{\gamma}^2/\omega_0^2=\bar{\nu}^2F_n(\eta);$$

$$\bar{\gamma}^2/\omega_0^2=k_0^2v_E^2\cos^2\theta/\omega_0^2,\quad\eta=2\bar{\nu}ka^{2/3}k_0^{-1/3},\tag{24}$$

$$4\bar{\nu}k/k_0\ll1,\qquad k_0a\gg3\pi n/2.$$

The spectrum of the instability frequencies is determined by the relation

$$\delta=\frac{k_0}{4k}\left(1-\frac{2n+1}{4(2\bar{\nu}k_ck)^{1/2}a}\right),\quad\frac{2n+1}{\eta^{3/2}}\ll1,$$

$$\delta=\frac{k_0}{4k}\left[1-\frac{1}{2}\left(\frac{3\pi n}{2k_0a}\right)^{2/3}\right],\quad\eta\ll1.$$

It is seen from the foregoing formulas that in a weakly inhomogeneous plasma $\omega_0a/c \gg 1$, the most stable are the waves with $k_y \ll k_z$ and $k \sim k_z$ large enough to satisfy the relation $\eta \sim 1$. For the minimum threshold we then obtain (at a = const)

$$\frac{v_E^2}{c^2}\approx\left(\frac{\omega_0}{2kv_T}\right)^2\left(\frac{2v_T^3}{\omega_0^2a^2c}\right)^{2/3},\quad\left(\frac{\omega_0}{2kv_T}\right)^2\approx\ln[q_1(\ln q_1)^2],$$

$$q_1 = \frac{8\sqrt{\pi}}{3}\left(\frac{\omega_0^2 a^2 c}{v_T^3}\right)^{1/3} \tag{25}$$

at

$$v_T(\omega_0^2 a^2 c)^{-1/3} \gg 2\nu_e/\omega_0,$$

$$\frac{v_E^2}{c^2} = 8\frac{\nu_e^2}{\omega_0^2} \quad \text{at} \quad \frac{\omega_0^2 a^2 c}{v_T^3} \gg \frac{\omega_0^3}{8\nu_e^3}\left(\frac{2kv_T}{\omega_0}\right)^3,$$

$$\left(\frac{\omega_0}{2kv_T}\right)^2 = \ln[q_2(\ln q_2)^{1/4}], \quad q_2 \approx \frac{8\sqrt{\pi}}{3}\left(\frac{2\omega_0\nu_e c a^2}{v_T^3}\right)^{1/2}.$$

Thus, the minimum threshold is determined by collisions only in a very weakly inhomogeneous plasma for $\omega_0^2 a^2 c/v_T^3 \gg (\omega_0/2\nu_e)^3$. At smaller values of the parameter $\omega_0^2 a^2 c v_T^{-3}$, the decisive role is played by the Landau damping, although the thresholds are determined as before by formula (24) at $\eta \sim 1$, corresponding to the limit of almost homogeneous plasma. When k decreases from the values determined by formulas (25), the plasma inhomogeneity causes the thresholds to increase.

$$\frac{v_E^2}{c^2} = \left(\frac{2v_T^3}{\omega_0^2 a^2 c}\right)^{2/3}\left(\frac{\omega_0}{2kv_T}\right)^2, \qquad \left(\frac{c^2\omega_0}{4k^3 v_T^3 a^2}\right)^{1/3} > \bar{\nu}.$$

Similar estimates can be obtained also for the decay of an extraordinary wave propagating along the x axis. In this case we have at $4\omega_H^2\cos^2\theta/\omega_0^2 \gg 1$

$$\lambda \approx t^{4/3}(k_0 a)^{4/3}\frac{k^2 v_E^2 \omega_0^2 \sin^2\theta}{4\omega_H^4\cos^4\theta}(2\cos^2\theta - 1)^2,$$

$$k_0 = \frac{\omega_0}{c}\left(1 - \frac{\omega_0^2}{4\omega_H^2\cos^4\theta}\right)^{1/2}, \quad v_E = \frac{eE_{0y}}{m\omega_0},$$

and β is determined, as before by formula (23). It is easy to find that for the threshold values of the pump field one can use formula (24) in which we put

$$\frac{\bar{\gamma}^2}{\omega_0^2} = \frac{\omega_0^2 k_0^2 v_E^2 \sin^2\theta}{4\omega_H^4\cos^4\theta}(2\cos^2\theta - 1)^2.$$

The minimal threshold $v_E^2/c^2 = 16\nu_e^2\omega_H^2/\omega_0^4$ is reached in this case at $\cos^2\theta \approx \omega_0/\omega_H\sqrt{2}$ and under the condition

$$\left(\frac{\omega_0^2 a^2 c}{v_T^3}\right)^{1/3} \gg \frac{\omega_0}{\nu_e}\left(\frac{\omega_0}{\omega_H}\right)^{1/2}.$$

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[1)]For simplicity, we consider the case $k_{0y} = k_{0z} = 0$. At $k_{0yz} \neq 0$, these differences are of the same order, but naturally, depend also on k_{0y} and k_{0z}.

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