Singularities in the propagation of sound in moving He II

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From calculations performed in the present paper, it follows that the propagation of waves in liquids having a nonlinear velocity dependence of the friction force is characterized by a number of singularities. An example of such a liquid is superfluid helium, in which the friction is zero at velocities below a certain critical value, and increases sharply at velocities above critical. As a result, the retardation force in moving helium will be different for oscillations of the liquid in the direction of the flow and counter to it. It is shown that this leads, in particular, to a decrease in the time-averaged velocity of the liquid flow. This type of effect has previously been observed experimentally.[1] According to the results of the present paper, the relaxation times of a superfluid liquid can be judged on the basis of the frequency dependence of the effect.

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It was shown by Esel'son, Ivantsov, Mikhaĭlov, and Shcherbachenko[1] that the rate of transport of a superfluid along a film changes sharply when a capacitor is placed on the path of the moving film, with a attenuating voltage applied to the capacitor. The rate of transport $Q = \bar{v}S$ is determined by the time-average velocity $\bar{v}$ of the film and the area of its cross section $S = 2\pi Rh$. Here R is the radius of the ampoule along the surface of which the film moves under the action of a force due to the difference in the levels of the liquid between the ampoule and the chamber, and h is the thickness of the film. As was noted in[1], the rate of transport should decrease due to the change in the film thickness h caused by the propagation of the third sound[2] generated by the alternating electric field. It has been assumed here that the velocity of motion of the film does not change and is equal to the critical value, i.e., $\bar{v} = v_c$. A similar effect has been observed in[3], where a change in Q was brought about by a sharp drop in the value of R on the path of the moving film.

It will be shown below that the propagation of waves in a superfluid that is moving with critical velocity v_c has a number of features. These features should lead, in particular, to a decrease in $\bar{v}$. This latter fact is easily understood if we take into account that there exists an asymmetry in the moving film for oscillations of the liquid along the flow and opposing it. Thus, if the velocity of the oscillatory motion is directed along the flow in one half-period, it should exceed the critical velocity, which leads finally to a very insignificant increase in the total velocity in comparison with v_c. This is connected with the fact that the motion with velocity greater than critical is accompanied by the generation of a significant retarding force, which prevents an increase in the velocity of motion of the film. By virtue of this, $v \approx v_c$ in the considered half-period. Then, in the other half-period, when the velocity of the oscillatory motion is directed counter to the flow, the resultant velocity $v = v_c - |u|$ turns out to be less than critical and v can even be negative in this half-period. Then the average velocity over the period of oscillation will be less than critical, which leads to a decrease of Q relative to the rate of transport in the absence of oscillatory motion, when $\bar{v} = v_c$.

To solve the problem posed, we consider an unbounded film of a superfluid. As the x,y plane, we choose the surface of the substrate on which the film is located. Let all the quantities be independent of the coordinate y. We further assume that the longitudinal velocity of the superfluid component v_x is large in comparison with the transverse v_z. For helium films of thickness $h \sim 10^{-6}$ cm[4], the latter assumption is valid if the distance λ on which the velocity of the liquid undergoes a significant change is much greater than h. So far as the normal component of the superfluid is concerned, it is assumed to be motionless in the film, as usual. Here we can neglect the change in the temperature brought about by motion of the superfluid component relative to the normal component.[5] In accordance with what has been said above, everything that follows applies only to the superfluid component, the density ρ of which is assumed to be of the order of the helium density. This latter observation eliminates from consideration the temperature region near the temperature of the superfluid transition.

We now write out the equations to be satisfied by the variables describing the film: the equation of motion and the equation of mass conservation of the liquid.[6] The second equation can be rewritten without change. So far as the equation of motion is concerned, it is necessary here to take into account a number of forces acting on the film. First, there is the force F_m, which leads to motion of the film as a whole. The differential in the liquid levels provided this force in[1]. Second is the force $-F_r(v)$ which governs the retardation of the film. And, finally, there is the force F_u, which leads to the generation of oscillations in the liquid. In[1], this force was due to the capacitor to which the alternating voltage was applied.

With account of the observations made above, we can write down the initial set of equations in the form

$$\frac{dv}{dt} = -F_V\frac{\partial h}{\partial x} + F_m - F_r(v) + F_u, \tag{1}$$

$$\frac{\partial h}{\partial t} + \frac{\partial hv}{\partial x} = 0. \tag{2}$$

Here F_V is the Van der Waals force.[7] The force F_u, which is due to oscillations of the field E in the space between the plates of the capacitor, can be written in the form[8]

$$F_u = \frac{\varepsilon-1}{8\pi\rho}\frac{\partial E^2}{\partial x}, \tag{3}$$

where ϵ is the dielectric constant of the superfluid.

By modeling the experiment of[1], we can write down Eq. (3) in the form

$$F_u = A f_1(t) \frac{\partial f_2(x)}{\partial x}. \qquad (4)$$

Here $f_2(x)$ differs from zero only in the region $|x| \le x_1$, which is determined by the dimensions of the capacitor, and $f_1(t)$ is a function that is periodic in time for $t > 0$ and is equal to zero for $t < 0$. The latter assumes that the source of the oscillations is turned on at the instant $t = 0$. The constant A is simply expressed in terms of the amplitude of the electric field E_0:

$$A = E_0^2 \frac{\varepsilon - 1}{8\pi\rho}. \qquad (5)$$

In accordance with[1], we shall assume for all subsequent calculations that the electric field changes with time in sinusoidal fashion, and that the wavelength is much greater than the distances over which E is inhomogeneous. Then

$$f_1(t) = \Theta(t) \sin^2 \omega t, \quad f_2(x) = \Theta(x + x_1) - \Theta(x - x_1), \qquad (6)$$

where Θ is a function that is equal to zero for negative values of the argument and equal to unity for positive values. If $t < 0$, then the force $F_u = 0$, the flow is stationary, the thickness of the films is equal to the equilibrium h_0, the velocity of motion $v = v_0 > 0$, and $F_m = F_r(v_0)$.

We now consider the case in which the departures of all quantities from their equilibrium values are small and the velocity of motion of the liquid is low. The latter actually assumes that $v = v_0 + u$ is much less than the velocity of propagation of third sound.[2] Limiting ourselves to the linear terms, Eqs. (1) and (2) can be rewritten in the form

$$\frac{\partial u}{\partial t} = -c^2 \frac{\partial \zeta}{\partial x} + F_r(v_0) - F_r(v_0 + u) + F_u, \qquad (7)$$

$$\frac{\partial \zeta}{dt} + \frac{\partial u}{\partial x} = 0, \qquad (8)$$

where $\zeta = h'/h_0$ is the relative departure of the thickness of the film from its equilibrium value and $c = \sqrt{F_V h_0}$ is the velocity of third sound.[2] Here it is appropriate to note that the decrease in the transport velocity associated with the change in the film thickness cannot be taken into account in the linear theory, since it is proportional to ζv.

If the function $F_r(v_0 + u)$ can be expanded in a power series in u, and if we can limit ourselves to terms of first order in u, then the linear system of equations (7), (8) describes the usual oscillations, which are damped in space. Here the time-averaged value of the velocity is $\bar{v} = v_0$, i.e., the oscillations in the linear approximation do not lead to a change in the transport velocity along the film.

As is well known, in the case of motion of a superfluid, the retardation force is virtually equal to zero for velocities less than some critical v_c, and increases sharply for velocities $|v| > v_c$. Such a dependence can be approximated by a function of the type

$$F_r(v) = B(v - v_c \operatorname{sign} v)\Theta(|v| - v_c). \qquad (9)$$

Here $B > 0$ is some constant. Interpretation of the motion of the superfluid in terms of thermal fluctuations (according to[9]) yields

$$F_r = D \operatorname{sign} v \exp(-v_i/|v|), \qquad (10)$$

where D and v_i are quantities that are independent of v.

A number of experimental studies[10,11] resulted in sufficiently good agreement with the dependence $F_r(v)$ given by Eq. (10). It is true that the constants D and v_i differ somewhat from those calculated theoretically. However, both calculation and experiment give values for v_i that are much greater than the experimentally measured values of v. Substitution of numerical values of v_i and D from the paper of Liebenberg[10] into (10) shows that even a physically infinitesimally small differential of the levels ΔH leads to motion with velocity $v \sim 10^2$ cm-sec^{-1}. Subsequent changes of ΔH give a very weak increase in the velocity.

It is clear that in the sense considered here, the dependence given by (9) is similar to the dependence (10). In what follows, we shall assume that $F_r(v)$ is determined by relation (9).

It is evident that the expansion in u in (9) is possible only if $|u| < v_0 - v_c$, i.e., for a sufficiently large force F_m and relatively small amplitude A. In this case, as has already been noted, the velocity of motion of the liquid, averaged over time, remains unchanged and equal to v_0. The situation changes significantly if the inequality given above is not satisfied. Then the retardation force, which prevents a change in the velocity u of the oscillatory motion, will be different for oscillations in the direction of the flow and counter to it. Thus, as $F_m \to 0$ (i.e., at $v_0 = v_c$) the retardation force is equal to $-Bu$ during the half-period when the velocity of the oscillatory motion is directed along the flow ($u > 0$). In the other half-period, when $u < 0$, the retardation force is correspondingly equal to zero. Such an asymmetry in the oscillations produced against a background of stationary flow v_0, should lead to the result that the time-averaged velocity $\bar{v} = v_0 + \bar{u}$ turns out to be less than v_0.

We note that this effect is characteristic of any continuous medium with a nonlinear retardation force. For example, it should be observed in flows of He II in narrow channels. In this case, c is the velocity of fourth sound in (7), (8),[2] and ζ is the relative deviation of the pressure from its equilibrium value. The same effect should appear in principle in the case of flow of an electron liquid if the resistance turns out to be a sufficiently nonlinear function of the current.

Something similar was observed when alternating and direct currents were passed simultaneously through a superconducting wire.[12] As is well known, the dependence of the resistance of the superconducting wire on the current is similar to the dependence given by Eq. (9) where v is now the rate of flow of the electron liquid and F_r is the resistance. The situation that develops here is in no way different from that considered above. It is true that the cited paper[12] dealt with the simplest limiting case of a wavelength much greater than the dimensions of the system. However, it is clear from what has been said above that the physics of all the phenomena considered is the same. We note that the nonlinear retardation force, in the case of the corresponding $F_r(v)$ dependence, can lead not only to a decrease but also to an increase in $\bar{v}$.

To simplify all further calculations, we limit ourselves to the case $F_m \to 0$ (i.e., $v_0 = v_c$) and $|u| \le 2v_c$. Then the system (7), (8) can be rewritten in the form

$$\frac{\partial u}{\partial t} = -c^2 \frac{\partial \zeta}{\partial x} - Bu\Theta(u) \mp F_u, \qquad (11)$$

$$\frac{\partial\zeta}{\partial t}+\frac{\partial u}{\partial x}=0 \tag{12}$$

with the initial conditions

$$\zeta|_{t=0}=0, \quad u|_{t=0}=0. \tag{13}$$

Differentiating (11) with respect to time and (12) with respect to the coordinate and eliminating ζ, we obtain

$$\frac{\partial^2 u}{\partial t^2}-c^2\frac{\partial^2 u}{\partial x^2}+B\Theta(u)\frac{\partial u}{\partial t}=\frac{\partial F_u}{\partial t} \tag{14}$$

with the initial conditions

$$u\Big|_{t=0}=0, \quad \frac{\partial u}{\partial t}\Big|_{t=0}=0. \tag{15}$$

We now seek a solution of (14) in the form

$$u=u_0+u_1, \tag{16}$$

where u_0 is the solution of (14) for $B = 0$ with the initial conditions (15). As is known, this solution represents undamped plane waves which propagate from a source located at the point $x = 0$ (here it is assumed for simplicity of notation in the final result that $x_1 \to 0$).

In addition, with the help of the Green's function in regions where $u_0 \gg u_1$, we obtain

$$u_1(x,t)=-\frac{B}{2c}\int_0^t dt'\int_{x-c(t-t')}^{x+c(t-t')}\Theta(u_0)\frac{\partial u_0(x',t')}{\partial t'}dx'. \tag{17}$$

Omitting unimportant terms after integration, we can write the final result in the form

$$u(x,t)=u_0(x,t)\left\{1-\frac{B}{2c}|x|\Theta(u_0)-\frac{B}{4}\left(t-\frac{|x|}{c}\right)\frac{\overline{|u_0|}}{u_0}\right\}. \tag{18}$$

Here and everywhere, the bar denotes time averaging. It is seen from (18) that, in the regions of the x, t plane where the velocity is positive, u is smaller than u_0, and at negative values of the velocity its modulus turns out to be larger than $|u_0|$. Then, as was to be expected, $\overline{v} = v_0 + \overline{u} < v_0$.

However, it must be kept in mind that the solution (18) gives values of the velocity only in a region that is sufficiently close to the source, and for relatively small times. In the scheme of the problem at hand, we are interested in the opposite limiting case: the behavior of the solution after a sufficiently large interval of time in which the system "forgets" the initial conditions, i.e., in the regions $x, t > 0$ sufficiently far from the straight lines $ct \pm x = -x_1$.

For solution of this problem it is convenient to write down the set (11), (12) in Riemann invariants with derivatives along the characteristics:

$$\frac{\partial r_1}{\partial\gamma}=-\frac{B}{4c}(r_1+r_2)\Theta(r_1+r_2)+\frac{1}{2c}F_u, \tag{19}$$

$$\frac{\partial r_2}{\partial\eta}=-\frac{B}{4c}(r_1+r_2)\Theta(r_1+r_2)+\frac{1}{2c}F_u, \tag{20}$$

where

$$r_1=u-c\zeta, \quad r_2=u+c\zeta, \tag{21}$$

$$\eta=ct+x, \quad \gamma=ct-x. \tag{22}$$

The functions r_1 and r_2 should be different from zero on the characteristics $\eta = -x_1$ and $\gamma = -x_1$.

Since the effect of the initial data weakens with passage of time, the desired solution should, after a sufficiently long time, have the same symmetry as the initial set of equations. Thus, it follows from (6), (11), and (12) that the solution in the steady regime should be periodic in time with period $2\tau_c = \pi/\omega$.

In the region of $x, t > 0$ sufficiently far from the initial lines $\eta = -x_1$ and $\gamma = -x_1$, in accord with (14) and (6), the equality

$$u(-x, t+\tau_c)=u(x, t). \tag{23}$$

should be satisfied. It follows from Eqs. (19), (20) that

$$r_2(x<-x_1)<0, \quad r_1(x>x_1)<0. \tag{24}$$

Here, according to (19), (20), and (23), $r_2(x < -x_1)$ can be obtained by specular reflection of $r_1(x > x_1)$ in the plane $x = 0$ with a subsequent time shift τ_c. Further, if we neglect the damping the region of the source, it follows from (19) and (20) that

$$r_1(-x_1,t)=r_1\left(x_1, t-\frac{2x_1}{c}\right)+F_\Sigma\left(t-\frac{x_1}{c}\right), \tag{25}$$

$$r_2(x_1,t)=r_2\left(-x_1, t-\frac{2x_1}{c}\right)-F_\Sigma\left(t-\frac{x_1}{c}\right), \tag{26}$$

where

$$F_\Sigma\left(t-\frac{x_1}{c}\right)=\frac{A}{c}\sin 2\omega\frac{x_1}{c}\sin 2\omega\left(t-\frac{x_1}{c}\right). \tag{27}$$

According to the equations obtained above, we can write in the region of the x, t plane of interest to us, outside the source,

$$r_1(-x, t+\tau_c)=r_2(x, t). \tag{28}$$

We recall that in the scheme of the problem that has been advanced, it is actually necessary for us to find

$$\overline{u}=(\overline{r}_1+\overline{r}_2)/2 \tag{29}$$

in a region of x, t sufficiently far away from the lines $ct \pm c = -x_1$. We note that the desired value of (29), in accord with the conservation law (12), does not depend on the x coordinate. Therefore, equating the values of (29) at the points $x = -x_1$ and $x \to -\infty$, we can express $\overline{u}$ in terms of the driving force F_Σ.

From (25) we have at the point $x = -x_1$, with account of (27) and (28),

$$\overline{u}=\overline{r_2(-x_1)}. \tag{30}$$

It follows further from (19) and (24) that if $r_1(-x, t) < 0$, its value does not change upon removal from the source of oscillations along the line η = const, i.e., $r_1(\gamma, \eta = ct - x_1) = r_1(-x, t)$.

So far as $r_1(-x_1, t) > 0$ and $r_2(x < -x_1)$ are concerned, according to (19) and (20), both these quantities damp out with increasing distance from the source. As a result, we have as $x \to -\infty$,

$$\overline{u}={}^1/_2\overline{r_1(-\infty)}. \tag{30a}$$

Inasmuch as only the positive values of r_1 are damped out with increase in γ(η = const), in accordance with what is given in (25), while the negative values are propagated without damping, then we have for $\overline{r_1(-\infty)}$, with account of (28),

$$\overline{r_1(-\infty)}=\overline{\{r_2(-x_1, t-x_1/c-\tau_c)+F_\Sigma(t)\}\Theta(-r_2-F_\Sigma)}. \tag{31}$$

Equating the $\overline{u}$ at the points $x = -x_1$ (30) and $x \to -\infty$ (30a), we finally obtain

$$2\overline{r_2(-x_1)}=\overline{r_1(-\infty)}. \tag{32}$$

For $B\tau_c \ll 1$, $r_2(x < -x_1)$ does not depend on the time at all in zero-order approximation. Moreover, if we take into account that the basic contribution in averaging in (31) is made only by those intervals of time during which $F_\Sigma(t) < 0$, it then follows from (32) that

$$4r_2(-x_1)=r_2(-x_1)-\overline{|F_\Sigma|}, \quad (33)$$

whence we have

$$\bar{u}=-\frac{1}{3}\overline{|F_\Sigma|}. \quad (34)$$

Taking the time average of the modulus of (27), we finally obtain

$$\bar{u}=-\frac{2}{3\pi}\frac{A}{c}\left|\sin 2\omega\frac{x_1}{c}\right|. \quad (35)$$

But if the parameter $B\tau_c$ is not small, then the result does not coincide with (34). However, as is not difficult to prove, it follows from (31) and (33) that the quantity $\bar{u}$ changes little in this case, remaining of the order of $-\overline{|F_\Sigma|}$.

At first glance, it comes as somewhat of a surprise that $\bar{u}$ should remain finite at $B\tau_c \ll 1$ and not depend on the small parameter at all in zero-order approximation. The same result can be obtained by iteration of the system (19), (20) in the region $x < -x_1$. Here it becomes quite clear that $r_2(-x_1)$ turns out to be finite due to integration over a sufficiently wide interval of the variable η. This means that the effect will always be finite for sufficiently long times in a system whose dimensions are greater than the damping distance $L_3 = c/B$. We recall that the film was assumed to be unbounded in our problem. Then the result that is obtained is just as natural as the fact that for damping as small as we please over distances that are large compared with the damping distance, the result will always be finite—the wave is damped.

We note further that a wave propagating in a system with nonlinear retardation has an unusual surface form. As was pointed out earlier, it follows from Eqs. (19) and (20) that, for sufficiently large $|x|$ and t, $r_2 \approx 0$ in the left half-plane, while positive values in r_1 are practically completely damped, and the negative values are propagated along the lines η = const without damping. In accord with (21) and (28), this leads to the result that "valleys" $\zeta(x \gg x_1) \le 0$ are propagated to the right of the source of the oscillations and "hills" $\zeta(x \ll -x_1) \ge 0$ to the left. The velocity turns out to be negative both at $x \gg x_1$ and at $x \ll -x_1$.

The frequency dependence of $\bar{u}$ is given by Eq. (35) and is quite clear physically. Actually, the solution of Eqs. (19) and (20) in the region $|x| \le x_1$, which is occupied by the source, can be written in the following form if we do not take damping into account:

$$u=-\frac{A}{2c}\sin 2\omega\left(t-\frac{x_1}{c}\right)\sin 2\omega\frac{x}{c},$$
$$\zeta=\frac{A}{2c^2}\left\{1-\cos 2\omega\left(t-\frac{x_1}{c}\right)\cos 2\omega\frac{x}{c}\right\}. \quad (36)$$

It is then clear that at frequencies $2\omega x_1/c = \pi n$ (where n = 0, 1, 2, 3, ...) the standing wave that is formed in the space $|x| \le x_1$ has velocity nodes at $x = \pm x_1$. Here the surface at the ends of the source vibrates in such a way that the force due to the oscillations of E is canceled. As a result, the wave exists only in the space $|x| \le x_1$.

It should be noted here that the above valid accurate to the small parameter $2x_1/L_3$, i.e., as long as we can neglect the damping in the region occupied by the source.

The frequency dependence obtained is similar to that which was observed in the experiment of[1]. The only difference is that the rate of transport not only oscillated with rising frequency, but also increased. At frequencies of the order of 1000 kHz, the effect was completely absent. This discrepancy becomes understandable if we take two facts into account. First, the finiteness of the film (the length of the film $\mathscr{L}$ in[1] was ~10 cm) is important. Second, the parameter B, as follows from[13-16], should depend significantly on the frequency. This is because the retardation mechanism is not fully operative at sufficiently high frequencies. As a result, B should decrease with increasing frequency. Then, beginning with certain frequencies, the damping length $L_3 = c/B$ turns out to be larger than the film length and the effect will be virtually nonexistent. In this , the experimental frequency dependence of the transport rate can be used for influences as to the $B(\omega)$ dependence.

In the scheme considered above, similar experiments with fourth sound are also of interest. Here equalization of the levels should take place due to the flow of helium along a channel, plugged with a fine powder. In this case, the dependence of the mass transport on the frequency of the fourth sound would also make possible inferences as to relaxation times.

We now make a number of numerical estimates and compare them with the experimental data. If, in accord with[1], we assume that the dimensions of the region occupied by the source are $2x_1 \approx 1$ cm, then the first maximum, according to (35), should be expected at frequencies ~50 Hz. In[1] the first maximum was observed at a frequency of ~20 Hz. Some shift of the maximum in the direction of lower frequencies can be due to the fact that at $\lambda \gtrsim L_3$, the modulus of the mean velocity according to (31) and (32) turns out to be somewhat larger than follows from Eq. (34).

Starting out from the fact that $v_0 \approx 30$ cm-sec^{-1}, we estimate the amplitude of the field at which the effect will be significant. According to (5) and (35), we have, at the maximum in the frequency,

$$E_0^2=\frac{12\pi^2 c\rho}{\varepsilon-1}|\bar{u}|. \quad (37)$$

If we assume that $|\bar{u}| \approx 10$ cm-sec^{-1}, the field E_0 for He II should be of the order 10^5 V-cm^{-1}, which agrees with the experimental data.

All the calculations given above are valid as $F_m \to 0$. With increasing F_m, as was pointed out at the beginning of the paper, $|\bar{u}|$ will decrease, so that for a sufficiently large force, the mean velocity of the oscillatory motion vanishes, inasmuch as ordinary damped sound will be propagated in the liquid in this case. Actually, as was discovered in[1], the rate of transport at a given frequency increases with increasing level difference, so that above some critical level difference the alternating field has, for practical purposes, no farther influence on the flow of the film.

We note that in a number of papers[16-18] the critical velocity was determined in reduction of the experimental data from formulas given by the Doppler effect. It is clear from what has been said that this can be done either for sufficiently high frequencies or when the force F_m is relatively large. Estimates show that the first condition was evidently always satisfied in the papers cited above.

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