Equilibrium structure of relativistic beams

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We study the structure of a stationary, equilibrium, axially symmetric relativistic beam. We find the conditions under which the collective interaction between electrons and ions guarantees the self-containment of the beam. We show that in the framework of the Maxwell equations and the kinetic equations for classical perfect gases of electrons and ions a stationary equilibrium state of beams with a finite particle density is possible only in the particular case when there is a well-defined relation between the temperatures and the particle concentrations. In the general case, on the other hand, there exist only solutions with a singularity at the origin. An appreciable fraction of the total number of particles can then be condensed on the beam axis. An infinite particle density at the origin indicates that the approximation of a classical perfect gas is inapplicable when we want to determine the beam structure. If the electron temperature is not small, the structure of a compressed relativistic beam is determined by the quantum properties of the electrons. Stationary equilibrium beams are in the framework of quantum statistics realized without there being any relation between the parameters, while the region of their existence increases when the electrons become more degenerate (when their number increases). We study the beam structure analytically in the limiting case of strongly degenerate electrons.

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1. INTRODUCTION

Recently relativistic electron beams have become very popular and have been applied to very varied fields of present-day physics. In the last forty years since the appearance of the basic paper by Bennett [1] a vast number of papers, both theoretical and experimental, has been published which are dealing with relativistic beams (see, e.g., the reviews [2-5]).

However, notwithstanding the large number of papers, the problem of the stationary structure of a relativistic beam can not be considered to be solved. The main achievement of the theory in that direction is the so-called "generalized Bennett distribution" [2] which was obtained for the particular case where the electron and ion concentrations were proportional to one another. However, its interpretation is not adequate for the problem of a relativistic beam when there are no external fields.

The aim of the present paper is a general study of stationary axially symmetric configurations of a non-rotating equilibrium relativistic beam in a vacuum. The starting point for our considerations is the Maxwell equations together with the kinetic equations for the electrons and the ions. We show that in the framework of the classical statistics of perfect gases stationary configurations of a beam with a finite particle density exist only for a well-defined relation between the temperatures and concentrations of the electrons and the ions. In the general case the stationary solutions of the classical equations describe a pinched beam with particle concentrations which increase without bound towards the axis. An appreciable fraction of the total number of particles may then turn out to be condensed on the beam axis. In the light of this, the "generalized Bennett distribution" with a singularity at the origin must be interpreted as a stationary state of a pinched relativistic beam while the current and charge on the axis which are necessary for self-consistency must be thought of as pertaining to the beam itself being produced by the appropriate number of its electrons and ions, compressed into the origin.

The fact that the beam density becomes infinite means, however, that the original kinetic equations lose their meaning as the plasma becomes imperfect. If the electron temperature is not too low, the electron gas becomes degenerate before it becomes imperfect, and this leads to the classical Boltzmann statistics becoming inapplicable for the description of the electrons. A degenerate electron gas still remains perfect and it is thus possible to study in the framework of the quantum statistics of the electrons the beam structure for large densities close to the axis.

2. INITIAL EQUATIONS

We shall consider a relativistic beam in a vacuum free from external fields over times large compared to the times t_e and t_i which are needed to establish equilibrium of the electrons and the ions separately, but small compared to the relaxation time t_{ei} to establish equilibrium between electrons and ions.

In the frame of reference K′ in which the electrons as a whole are at rest $t_e' \approx m_e^{1/2} T_e^{3/2}/e^4 n_e' \Lambda$. [6] Here e, m_e, T_e, and n_e are the charge, mass, temperature, and concentration of the electrons, while Λ is the Coulomb logarithm. In the laboratory frame (l.f.) relative to which the ions as a whole are at rest, while the electrons have an average ordered velocity $V_0 = c\beta_0$, we have

$$t_e = t_e'(1-\beta_0^2)^{-1/2}, \quad t_i \approx m_i^{1/2} T_i^{3/2}/e^4 n_i \Lambda.$$

m_i, T_i, and n_i are the mass, temperature, and concentration of the ions, and c is the light velocity. The characteristic time t_{ei} for the transfer of energy from the electrons to the ions, i.e., practically the relaxation time of the whole beam equals

$$t_{ei} \approx \frac{m_e m_i c^2 V_0}{e^4 n_i \Lambda (1-\beta_0^2)^{1/2}}.$$

The electrons by themselves and the ions by themselves will be in equilibrium if the characteristic time to heat the ions by an amount of the order of their temperature, $\tau_i \approx m_i T_i V_0/e^4 n_e \Lambda$, is large compared to t_i, and the time to decrease the directed electron velocity by an amount of the order of the thermal spread, $\tau_e \approx m_e T_e V_0/e^4 n_i \Lambda$, is large compared to t_e.

The initial kinetic equations have the form

$$\frac{\partial f_\alpha}{\partial t}+(\mathbf{v}_\alpha\nabla)f_\alpha+e_\alpha\left(\mathbf{E}+\frac{1}{c}[\mathbf{v}_\alpha\times\mathbf{H}]\right)\frac{\partial f_\alpha}{\partial \mathbf{p}}=I_\alpha\{f_\alpha\}, \qquad (2.1)$$

$f_\alpha \equiv f_\alpha(t, \mathbf{r}, \mathbf{p})$ are the distribution functions, the index α takes on the values e, i, respectively, for electrons and ions, $\epsilon_\alpha(\mathbf{p}) = (m_\alpha^2c^4 + p^2c^2)^{1/2}$, $\mathbf{v}_\alpha = \partial\epsilon_\alpha(\mathbf{p})/\partial\mathbf{p}$ are the energy and velocity of the particles, and $I_\alpha\{f_\alpha\}$ is the collision integral for particles of the kind α by themselves.

The electric field strength **E** and magnetic field **H** produced by the particles in the beam satisfy the Maxwell equations

$$\operatorname{rot}\mathbf{H}=\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}+\frac{4\pi}{c}\mathbf{j}, \quad \operatorname{rot}\mathbf{E}=-\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t}, \quad \operatorname{div}\mathbf{E}=4\pi\rho,$$

while the charge density ρ and the current density $\mathbf{j}$ can be expressed in terms of the particle distribution functions

$$\rho(t,\mathbf{r})=\sum_\alpha e_\alpha\int f_\alpha\,d\tau_\mathbf{p}, \quad \mathbf{j}(t,\mathbf{r})=\sum_\alpha e_\alpha\int \mathbf{v}_\alpha f_\alpha\,d\tau_\mathbf{p}, \qquad (2.2)$$

$d\tau_\mathbf{p}$ is a volume element in momentum space.

Buneman[3] wrote these equations in relativistically invariant form. Since, however, to solve the equations we must all the same change to a definite system of reference we thought it expedient to consider everything from the start in the laboratory frame.

The condition that **E** and **H** are fields produced by the particles in the beam is equivalent to boundary conditions for the Maxwell equations. The solution we need can be written in the form of retarded potentials ([7], Sec. 62):

$$\mathbf{E}=-\nabla\varphi-(1/c)\,\partial\mathbf{A}/\partial t, \quad \mathbf{H}=\operatorname{rot}\mathbf{A},$$
$$\varphi=\int\frac{d\mathbf{r}'}{R}\rho\left(t-\frac{R}{c},\mathbf{r}'\right), \quad \mathbf{A}=\frac{1}{c}\int\frac{d\mathbf{r}'}{R}\mathbf{j}\left(t-\frac{R}{c},\mathbf{r}'\right) \qquad (2.3)$$

where $R = |\mathbf{r}-\mathbf{r}'|$. Using Eqs. (2.2) and (2.3) to express the fields **E** and **H** in terms of the electron and ion distribution functions f_α we can write the kinetic Eq. (2.1) in the form

$$\frac{\partial f_\alpha}{\partial t}+(\mathbf{v}_\alpha\nabla)f_\alpha+\mathbf{F}_\alpha\frac{\partial f_\alpha}{\partial \mathbf{p}}=I_\alpha\{f_\alpha\}, \quad \alpha=i,e, \qquad (2.4)$$

where

$$\mathbf{F}_\alpha=e_\alpha\sum_{\alpha'}e_{\alpha'}\int d\mathbf{r}'\int d\mathbf{p}'\left\{-\left[1-\frac{\mathbf{v}_{\alpha'}'\mathbf{v}_\alpha}{c^2}\right]\frac{\partial}{\partial\mathbf{r}}-\frac{\mathbf{v}_{\alpha'}'}{c^2}\left[\frac{\partial}{\partial t}+\left(\mathbf{v}_\alpha\frac{\partial}{\partial\mathbf{r}}\right)\right]\right\}\frac{f_{\alpha'}'}{R}, \quad \alpha=i,e \qquad (2.5)$$

are the forces exerted on a particle of the α-th kind by all other particles through the electromagnetic field produced by them, while $f'_{\alpha'} \equiv f_{\alpha'}(t-R/c, \mathbf{r}', \mathbf{p}')$.

Equation (2.4) enables us, in principle, to study not only equilibrium stationary beam configurations, but also to study their stability, to find the radiation by the beam, and to follow the evolution of an arbitrarily given initial state of the beam.

The aim of the present paper is to study the stationary equilibrium beam configurations and in this connection we assume that the distribution functions are time-independent. We restrict ourselves to beams which have axial symmetry and which are uniform along the z- and φ-coordinates, and we shall assume that the ions are non-relativistic in the laboratory frame: $T_i \ll m_ic^2$. Let the electrons in the frame of reference comoving with them also be non-relativistic: $T_e \ll m_ec^2$.

Under those assumptions the forces (2.5) are functions of a single spatial coordinate, r. They are in the radial direction (i.e., they are potential forces) and equal to

$$F_e=-\frac{dU_e}{dr}=-\frac{4\pi e^2}{r}\int_0^r r'\,dr'[n_i(r')-(1-\beta_0^2)n_e(r')], \qquad (2.6)$$

$$F_i=-\frac{dU_i}{dr}=-\frac{4\pi e^2}{r}\int_0^r r'\,dr'[n_e(r')-n_i(r')], \qquad (2.7)$$

where

$$n_\alpha(r)=\int f_\alpha\,d\tau_\mathbf{p}, \quad \alpha=e,i \qquad (2.8)$$

are the electron and ion concentrations.

Notwithstanding the fact that both electrons and ions are in the same collective interaction electromagnetic field, the forces F_e and F_i are different as they depend on the particle velocities. The force due to the collective interaction of an electron with the other electrons contains a factor $(1-\beta_0^2)$ (the second term in the square brackets in (2.6)) and tends to zero as $V_0 \to c$. This is connected with the fact that as $V_0 \to c$ the electrostatic repulsion between electrons is compensated by their magnetic attraction as parallel currents. If $n_e > n_i$ and β_0^2 sufficiently close to unity (and it is just this situation which is of interest from the point of view of the self-containment of a beam), both the forces acting on an electron, $F_e < 0$, and the forces acting on an ion, $F_i < 0$, will be directed towards the axis. In that case the ions attract the electrons and the surplus electron charge attracts the ions, and as a whole under the action of the collective interaction forces the beam tends to be compressed.

Under our assumptions the kinetic Eq. (2.4) is satisfied by an equilibrium distribution function which is a function of the additive constants of motion ([8], Sec. 4). In the frame of reference in which the particles as a whole are at rest their distribution function depends solely on the total energy. The ion equilibrium state is described by two parameters—the number of particles N_i per unit length of the beam and the spread in energy, i.e., the temperature T_i. On the other hand, the electrons as a whole move in the l.f. with a velocity V_0. Their equilibrium distribution function depends thus on three parameters: N_e, T_e, and V_0.

3. RELATIVISTIC BEAM IN THE CLASSICAL PERFECT GAS APPROXIMATION

In the classical perfect gas approximation the ion equilibrium distribution function has the form

$$f_i=A_i\exp\{-[p^2/2m_i+U_i(r)]/T_i\},$$

A_i is a normalization constant, and $U_i(r)$ the potential of the force (2.7) which acts upon an ion. To determine the form of the electron distribution function in the l.f. we change to the frame of reference K′ in which they are at rest. In the frame K′ the equilibrium distribution function for the electrons treated as a classical perfect gas is determined by the total energy E′:

$$f_e^{(K')}(E')=A_e\exp(-E'/T_e).$$

Belyaev and Budker[9] have shown that the distribution function is invariant under a Lorentz transformation. Hence in the l.f.

$$f_e=f_e^{(K')}(E'),$$

where E′ can be expressed through the formulae of the

Lorentz transformation in terms of E and **p**. The total electron energy equals $E = \epsilon_e(\mathbf{p}) + U_e(r)$ and the electron distribution function has the form

$$f_e=A_e \exp\{-[\varepsilon_e(\mathbf{p})-V_0p_z+U_e(r)]/T_e(1-\beta_0^2)^{1/2}\}. \quad (3.1)$$

In the case of non-relativistic temperatures T_e and relativistic velocities V_0 the distribution function is different from zero only in a small vicinity near the momentum $\mathbf{p} = \mathbf{p}_0$ for which the index of the exponential is a minimum, i.e.,

$$\mathbf{v}_e(\mathbf{p}_0)=\partial\varepsilon_e(\mathbf{p})/\partial\mathbf{p}|_{\mathbf{p}=\mathbf{p}_0}=\mathbf{V}_0$$

($\mathbf{p}_0$ is the momentum of an electron moving with the average beam velocity). In the vicinity of $\mathbf{p} = \mathbf{p}_0$

$$\varepsilon_e(\mathbf{p})-V_0p_z=m_ec^2(1-\beta_0^2)^{1/2}+\frac{p_\perp^2}{2m_e}(1-\beta_0^2)^{1/2}+\frac{(p_z-p_0)^2}{2m_e}(1-\beta_0^2)^{3/2}. \quad (3.2)$$

If we denote the transverse and longitudinal electron temperatures in the l.f. by $T_\perp = T_e(1 - \beta_0^2)^{1/2}$ and $T_\parallel = T_e(1 - \beta_0^2)^{-1/2}$ and the effective electron mass by $m^* = m_e(1 - \beta_0^2)^{-1/2}$, we can write the distribution function (3.1) in the form

$$f_e=A_e \exp\left\{-\frac{m_ec^2}{T_e}-\frac{(p_z-p_0)^2}{2m^*T_\parallel}-\frac{p_\perp^2/2m^*+U_e(r)}{T_\perp}\right\}.$$

The normalization constants A_i and A_e are connected with the number of particles per unit length of the beam through the relations ($d\tau_{\mathbf{p}} = d^3p$)

$$N_e=2\pi\nu_e \exp\{-m_ec^2/T_e\}(2\pi m^*)^{3/2}T_\parallel^{1/2}T_\perp A_e, \qquad \nu_e=\int_0^\infty r\exp\{-U_e(r)/T_\perp\}dr,$$

$$N_i=2\pi\nu_i(2\pi m_iT_i)^{3/2}A_i, \qquad \nu_i=\int_0^\infty r\exp\{-U_i(r)/T_i\}dr.$$

After performing the integration in (2.8) we find

$$n_e(r)=(N_e/2\pi\nu_e)\exp\{-U_e(r)/T_\perp\}, \quad (3.3)$$

$$n_i(r)=(N_i/2\pi\nu_i)\exp\{-U_i(r)/T_i\}. \quad (3.4)$$

As should be the case, the electron and ion equilibrium densities are connected with the potentials of the forces acting upon them through the Boltzmann factor. Substituting the values of $n_e(r)$ and $n_i(r)$ into the integrals in (2.6) and (2.7) we are led to the equations determining the potentials U_e and U_i:

$$r\frac{dU_e(r)}{dr}=\frac{2e^2N_i}{\nu_i}\int_0^r r'\exp\left\{-\frac{U_i(r')}{T_i}\right\}dr' - \frac{2e^2N_e}{\nu_e}(1-\beta_0^2)\int_0^r r'\exp\left\{-\frac{U_e(r')}{T_\perp}\right\}dr', \quad (3.5)$$

$$r\frac{dU_i(r)}{dr}=\frac{2e^2N_e}{\nu_e}\int_0^r r'\exp\left\{-\frac{U_e(r')}{T_\perp}\right\}dr'-\frac{2e^2N_i}{\nu_i}\int_0^r r'\exp\left\{-\frac{U_i(r')}{T_i}\right\}dr'. \quad (3.6)$$

Solely for the sake of the greatest simplicity we shall in what follows assume the electrons to be ultrarelativistic ($\beta_0 \to 1$) and neglect the last term in (3.5) which takes into account the incomplete compensation of the electrostatic repulsion by the magnetic attraction of the electrons. We change to dimensionless functions $\psi_e = U_e/T_\perp$, $\psi_i = U_i/T_i$ and introduce the notation

$$K_1=e^2N_i/T_\perp, \quad K_2=e^2N_e/T_i, \quad K_3=e^2N_i/T_i. \quad (3.7)$$

After differentiation the equations for ψ_α, $\alpha = e, i$, take the form

$$\frac{1}{r}\frac{d}{dr}r\frac{d\psi_e}{dr}=\frac{2K_1}{\nu_i}e^{-\psi_i}, \quad (3.8)$$

$$\frac{1}{r}\frac{d}{dr}r\frac{d\psi_i}{dr}=\frac{2K_2}{\nu_e}e^{-\psi_e}-\frac{2K_3}{\nu_i}e^{-\psi_i}, \quad (3.9)$$

where

$$\nu_\alpha=\int_0^\infty re^{-\psi_\alpha(r)}dr, \qquad \alpha=i, e$$

are normalization integrals.

The parameters (3.7) have an intuitive physical meaning. For instance, K_1 is the ratio of the energy of the collective interaction of an electron with all ions to its temperature.

Equations (3.8) and (3.9) are scale and gauge invariant and at the same time the total number of electrons and ions per unit length of the beam are fixed. We are interested in their solution which satisfies the condition that the total number of particles N_i and N_e is finite. Far from the beam axis, as $r \to \infty$ the electron and ion densities, i.e., the right-hand sides of Eqs. (3.8) and (3.9) must decrease sufficiently fast. We can write Eqs. (3.5) and (3.6) after integration in integral form:

$$\psi_e(r)=\frac{2K_1}{\nu_i}\int_0^r \ln\frac{r}{r'}e^{-\psi_i(r')}r'\,dr',$$

$$\psi_i(r)=\frac{2K_2}{\nu_e}\int_0^r \ln\frac{r}{r'}e^{-\psi_e(r')}r'\,dr'-\frac{2K_3}{\nu_i}\int_0^r \ln\frac{r}{r'}e^{-\psi_i(r')}r'\,dr'.$$

If the normalization integrals ν_α exist, they cancel as $r \to \infty$, and the potentials increase logarithmically:

$$\psi_e(r)\approx 2K_1\ln r, \quad \psi_i(r)=2(K_2-K_3)\ln r, \quad r\to\infty. \quad (3.10)$$

The particle concentrations then decrease as powers:

$$n_e(r)\sim e^{-\psi_e(r)}\sim r^{-2K_1}, \qquad n_i(r)\sim e^{-\psi_i(r)}\sim r^{-2(K_2-K_3)}.$$

In order that the normalization integrals converge as $r \to \infty$, it is therefore necessary that the following inequalities hold:

$$K_1>1, \quad K_2-K_3>1, \quad \text{i.e.,} \quad e^2N_i>T_\perp, \quad e^2(N_e-N_i)>T_i \quad (3.11)$$

The equilibrium configurations of relativistic beams with a given finite number of particles can thus exist only when the energy of the collective containment of the electrons and the ions is larger than their temperatures.

Equations (3.8) and (3.9) enable us to find a first integral

$$\frac{K_3}{2K_1}\left(r\frac{d\psi_e}{dr}-2\right)^2+\left(r\frac{d\psi_e}{dr}-2\right)\left(r\frac{d\psi_i}{dr}-2\right)-r^2\left(\frac{2K_2}{\nu_e}e^{-\psi_e}+\frac{2K_1}{\nu_i}e^{-\psi_i}\right)=C. \quad (3.12)$$

The integration constant C is determined by the behavior of the potentials ψ_e and ψ_i as $r \to 0$. In the class of potentials which are finite as $r \to 0$, $r\,d\psi_\alpha/dr \to 0$ as $r \to 0$, and $C = 2K_3/K_1 + 4$. Substituting the value of C into (3.12), letting r tend to infinity, and noting that under the conditions (3.11) the term containing the normalization integrals tends to zero, we get the relation[1)]

$$K_1K_3+2(K_1+K_2-K_1K_2)=0, \quad (3.13)$$

which is the necessary condition in order that solutions exist which are finite as $r \to 0$. The meaning of Eq. (3.13) consists in the fact that equilibrium solutions do not exist for all relations between the electron and ion concentrations and their temperatures, but only for those when the energy of the collective interaction between the particles, compressing the beam, exactly compensates the thermal energy spread of the particles in the transverse direction. Together with the requirement that the number of particles be finite the limitation (3.13) is even further strengthened by the inequalities (3.11) and the physical condition $K_3 > 0$. We are thus led to the conclusion that equilibrium relativistic beams which are uniform in z and φ with a finite particle density do not exist with arbitrarily a priori given electron and ion numbers and their temperatures. Such beams are possible only for peculiar cases when conditions (3.11) and (3.13) are satisfied. An example of a solution which is peculiar in that sense is the Bennett distribution

$$\psi_\alpha(r)=2\ln[1+(r/r_0)^2]+\text{const},\quad \alpha=i,e,$$

which corresponds to the parameter values $K_1 = K_2 - K_3 = 3$, which—as one can easily check—satisfy condition (3.13); r_0 is an arbitrary scale parameter.

Do stationary configurations of relativistic beams exist for arbitrary values of K_1, K_2, and K_3 (of course, satisfying conditions (3.11))? It turns out that such configurations exist, if we drop the requirement that the potentials ψ_α and the densities n_α are finite at r = 0. Let

$$\lim_{r\to+0} r\frac{d\psi_\alpha}{dr}=2\lambda_\alpha,\quad \alpha=i,e.$$

The particle densities as $r \to +\infty$ have an integrable singularity, if $0 < \lambda_\alpha < 1$, α = i, e. The integration constant C in (3.12) equals in that case

$$C=\frac{K_3}{2K_1}(2\lambda_e-2)^2+4(\lambda_e-1)(\lambda_i-1).$$

Substituting this value of C into (3.12) and letting $r \to \infty$ we get instead of the earlier relation (3.13) between the constants K_1, K_2, and K_3 the following equation:

$$\frac{K_3}{2K_1}[(K_1-1)^2-(\lambda_e-1)^2]+(K_1-1)(K_2-K_3-1)-(\lambda_e-1)(\lambda_i-1)=0. \quad (3.14)$$

When $\lambda_i = \lambda_e = 0$ Eq. (3.14) is the same as (3.13). Relation (3.14) and with it the balance between the energy of the collective compression of the beam and the energy of the thermal spread of the particles can be satisfied for a wide range of variation in the parameters K_1, K_2, and K_3 by suitably choosing the quantities λ_i and λ_e.

We integrate Eqs. (3.8) and (3.9) from r_1 to r and then let $r_1 \to 0$ and let r tend to infinity. Writing

$$\tilde{\nu}_\alpha=\lim_{r_1\to+0}\int_{r_1}^{\infty} e^{-\psi_\alpha(r)}r\,dr,\quad \alpha=i,e,$$

we find the potentials ψ_i and ψ_e for large r in the form

$$\psi_e\approx2\left(\lambda_e+K_1\frac{\tilde{\nu}_i}{\nu_i}\right)\ln r,\quad \psi_i\approx2\left(\lambda_i+K_2\frac{\tilde{\nu}_e}{\nu_e}-K_3\frac{\tilde{\nu}_i}{\nu_i}\right)\ln r,\quad r\to\infty. \quad (3.15)$$

When $\lambda_\alpha \neq 0$ Eqs. (3.15) and (3.10) do not contradict one another and are the same provided[2)]

$$\lambda_e+K_1\frac{\tilde{\nu}_i}{\nu_i}=K_1,\quad \lambda_i+K_2\frac{\tilde{\nu}_e}{\nu_e}-K_3\frac{\tilde{\nu}_i}{\nu_i}=K_2-K_3,$$

whence

$$\tilde{\nu}_i/\nu_i=1-\lambda_e/K_1,\quad \tilde{\nu}_e/\nu_e=1-(K_1\lambda_i+K_3\lambda_e)/K_1K_2.$$

The number of particles in the region $r > 0$ is thus smaller than the total number of particles given earlier by us. Physically this means that when $0 < \lambda_\alpha < 1$ there exist stationary equilibrium beam configurations such that only a fraction $(\tilde{\nu}_\alpha/\nu_\alpha)$ of the particles are in the region $r > 0$, while the remaining fraction $(1-\tilde{\nu}_\alpha/\nu_\alpha)$ of the particles is condensed at r = 0. We emphasize that in this situation the differential Eqs. (3.8) and (3.9) have a sense only in the range $r > 0$, while Eqs. (3.5) and (3.6) describe the beam for all r, that is, also for r = 0.

An example of such a configuration is the so-called "generalized Bennett distribution":[2]

$$\psi_\alpha(r)=2\ln r[(r/r_0)^\eta+(r_0/r)^\eta]+\text{const},\quad \alpha=i,e,$$

which occurs when $K_1 = K_2 - K_3 = 1 + \eta$, where η can be chosen arbitrarily within the range (0, 1). In that case $\lambda_e = \lambda_i = 1 - \eta$. There is a fraction $(1-\eta)/(1+\eta)$ of the particles in the state which is condensed at r = 0 while the remaining fraction $2\eta/(1+\eta)$ is in the region $r > 0$. The difficulties connected with the shortage of particles in the region $r > 0$ in the "generalized Bennett distribution" have in the literature[2, 5] been interpreted as due to the fact that it is inadequate to pose the problem of a relativistic beam free of external fields. The insufficient current and charge at r = 0 must not be considered to be imposed from outside (as would be the case in the problem of a beam around a thin wire) but as being part of the beam itself. The condensation of an appreciable current and charge (i.e., of electrons and ions) on the beam axis is the result of its self-compression (pinch effect) when the energy of the collective interaction is not compensated by the energy of the thermal spread of the particles.

In the present section we considered the electrons and ions in the beam in the classical perfect gas approximation. However, when the particle density tends to infinity this approximation loses its meaning, and the more so as in the condensed state with an infinite density there is, generally speaking, an appreciable fraction of the total number of particles. The classical equations are applicable only for a description of the "external," non-condensed electrons and ions. Of course, we cannot use them to find the size of the condensation region, which in the classical perfect gas approximation vanishes.

4. STRUCTURE OF RELATIVISTIC BEAMS WHEN ELECTRON DEGENERACY IS TAKEN INTO ACCOUNT

When the densities of the particles interacting through the Coulomb forces n_α, α = i, e, increase the electrons and ions cease to be perfect gases (in the frame of reference comoving with the particles) when

$$n_\alpha\sim n_{\alpha\,\text{imperf}}\equiv(T_\alpha/e^2)^3$$

(see [8], Sec. 75), while for densities

$$n_\alpha\sim n_{\alpha\,\text{deg}}\equiv(m_\alpha T_\alpha/\hbar^2)^{3/2}$$

they become degenerate ([8], Sec. 56). From these relations it follows that if the electron temperature is sufficiently large, $T_e \gg E_a$ ($E_a = m_e e^4/\hbar^2 = 27.2$ eV is the unit of atomic energy), $n_{e\,\text{deg}} \ll n_{e\,\text{imperf}}$ and with increasing concentration the electrons become degenerate, while still remaining a perfect gas. This fact enables us to make a detailed study of the structure of a relativistic beam in the region where there is condensation of charge and current. In view of the fact that the ion mass is appreciably larger than the electron mass, we can neglect the ion degeneracy.

The equilibrium distribution function of the electrons in the beam has in the l.f. in quantum statistics the form

$$f_e=\{1+\exp[\alpha+(\varepsilon_e(\mathbf{p})-V_0p_z+U_e(r))/T_e(1-\beta_0^2)^{1/2}]\}^{-1}, \quad (4.1)$$

while the volume element in momentum space equals $d\tau_{\mathbf{p}} = 2d^3p/(2\pi\hbar)^3$. The scalar α in (4.1) plays the role of a normalization constant (A_e in the classical statistics) and is connected with the number of particles N_e per unit length of the beam as follows:

$$N_e=2\pi\int_0^\infty r\,dr\frac{2}{(2\pi\hbar)^3}\int f_e\,d^3p.$$

As in classical statistics we use the expansion (3.2). For non-relativistic temperatures as before the main contribution to the integrals (2.8) comes from the immediate vicinity near the minimum of the index of the exponential.[3] We introduce a dimensionless potential for the forces acting on the electron:

$$\psi_e(r)=\alpha+m_e c^2/T_e+U_e(r)/T_\perp.$$

After changing the integration variable,

$$x=(p_z-p_0)^2/2m^*T_\parallel+p_\perp^2/2m^*T_\perp,$$

we find an expression for the electron density:

$$n_e(r)=\frac{8\pi m^*T_\perp(2m^*T_\parallel)^{1/2}}{(2\pi\hbar)^3}\int_0^\infty\frac{x^{1/2}\,dx}{1+\exp\{x+\psi_e(r)\}}.$$

If we take the quantum statistics of the electrons into account Eq. (3.9) becomes

$$\frac{1}{r}\frac{d}{dr}r\frac{d\psi_i}{dr}=\frac{1}{r_0^2}\int_0^\infty\frac{x^{1/2}\,dx}{1+\exp\{x+\psi_e(r)\}}-\frac{2K_3}{\nu_i}e^{-\psi_i},\tag{4.2}$$

where we have denoted the factor in front of the integral, which has the dimensions of cm^{-2}, by $1/r_0^2$. Expressed in terms of atomic constants

$$r_0^2=\frac{\pi}{4\sqrt{2}}a^2\left(\frac{E_a}{T_e}\right)^{1/2}\frac{T_i}{T_e}(1-\beta_0^2)^{1/2},$$

$a=\hbar^2/m_e e^2$ is the Bohr radius. As the integral in (4.2) can not be evaluated analytically, we cannot, as in classical statistics, write down Eq. (4.2) in such a way that it already contains information about the a priori fixed number of electrons N_e. $\psi_e(r)$ must therefore additionally satisfy the normalization condition

$$2K_2=\frac{1}{r_0^2}\int_0^\infty r\,dr\int_0^\infty\frac{x^{1/2}\,dx}{1+\exp\{x+\psi_e(r)\}}.\tag{4.3}$$

Equation (4.2) together with (4.3) go over into the classical Eq. (3.9) when $\psi_e(r)$ is positive and large for all $r\geq 0$. In the opposite limiting case when $\psi_e(0)$ has a large absolute magnitude, but is negative, the electrons close to the beam axis are strongly degenerate. In contrast to (3.9) Eq. (4.2) is not scale invariant. This means that the transverse dimension of the beam is of the order

$$r_0=0.394\cdot10^{-8}\left(\frac{E_a}{T_e}\right)^{1/4}\left(\frac{T_i}{T_e}\right)^{1/2}(1-\beta_0^2)^{1/4}\ [\mathrm{cm}],$$

i.e., of the order of atomic dimensions when the combination of dimensionless parameters is of the order of unity. The transverse size of the beam is determined by the balance between the energy of the collective compression and the energy of the electron Fermi interaction. The size of the beam is thus determined by the quantum properties of the electrons and, hence, can not be obtained in the framework of the classical equations.

Introducing the dimensionless coordinate $\zeta=r/r_0$ and the function $w_\alpha(\zeta)=\psi_\alpha(r_0\zeta)$, $\alpha=i, e$, we can write the equations which determine the beam structure, when we take electron degeneracy into account, in the form

$$\frac{1}{\zeta}\frac{d}{d\zeta}\zeta\frac{dw_e}{d\zeta}=\frac{2K_1}{\nu}e^{-w_i},\tag{4.4}$$

$$-\frac{1}{\zeta}\frac{d}{d\zeta}\zeta\frac{dw_i}{d\zeta}=\int_0^\infty\frac{x^{1/2}\,dx}{1+\exp\{x+w_e(\zeta)\}}-\frac{2K_3}{\nu}e^{-w_i},\tag{4.5}$$

$$2K_2=\int_0^\infty\zeta\,d\zeta\int_0^\infty\frac{x^{1/2}\,dx}{1+\exp\{x+w_e(\zeta)\}},\tag{4.6}$$

where $\nu=\int_0^\infty\zeta\exp(-w_i(\zeta))d\zeta$. The same considerations as for the classical Eqs. (3.5) and (3.6) show that the inequalities (3.11), which are the necessary conditions for the self-containment of a relativistic beam with a finite number of particles, follow from Eqs. (4.4) to (4.6).

The Fermi repulsion of the electrons prevents the beam from compressing itself to zero size so that the electron and ion concentrations, and also the potentials w_i and w_e are finite as $\zeta\to 0$. The potential of the forces acting on the ions, w_i, is determined apart from an arbitrary additive constant so that we can put $w_i(0)=0$. The value of the potential $w_e(0)\equiv w_0$ determines the degree of degeneracy of the electrons and is uniquely determined by giving the parameters K_1, K_2, and K_3, i.e., by giving the number of electrons, N_e, of ions, N_i, and their temperatures T_e and T_i. The expansion of the potentials $w_i(\zeta)$ and $w_e(\zeta)-w_0$ near $\zeta=0$ can be seen from Eqs. (4.4) and (4.5) to start with terms quadratic in ζ. The boundary conditions for Eqs. (4.4) and (4.5) thus have the form

$$w_i=w_i'=w_e'=0,\quad w_e=w_0\ \text{when}\ \zeta=0.$$

Giving as parameters K_1, K_3, and w_0 we can after integrating Eqs. (4.4) and (4.5) and using the normalization condition (4.6) find the value of the parameter K_2 which corresponds to a given value of w_0.

It is convenient to perform the numerical integration of Eqs. (4.4) and (4.5) by giving instead of the three constants K_1, K_2, and K_3 three other constants: w_0, $\kappa_1=K_1/\nu$, and $\kappa_2=K_3/\nu$, and after the solution determine to which K_1, K_2, and K_3 the solution we have found corresponds. In accordance with the inequalities (3.11) we can not obtain the solutions we need, which satisfy the conditions that the number of particles is finite and that the beam is self-contained, for arbitrary a priori given κ_1 and κ_2, which must also be restricted to a certain region. The value of w_0 can be given arbitrarily. As an example we show in Fig. 1 the region of the allowed values of the parameter $\kappa=\kappa_1=\kappa_2$ (for the case when $T_i=T_\perp$) as function of w_0. When the degree of degeneracy increases ($w_0\to-\infty$) the region of the allowed values of κ increases, while when we go over to the region of classical statistics for the electrons ($w_0\to+\infty$) this region steeply narrows and in the limit becomes a line corresponding to the classical relation (3.13). To illustrate this we have shown the distribution of the potentials w_e and w_i and also the electron and ion concentrations for the parameter values $K_1=K_3=2.8$, $K_2=4$ ($w_0=-1$, $T_i=T_\perp$) in Fig. 2, and for the values $K_1=K_3=1.01$, $K_2=8$ ($w_0=-1$, $T_i=T_\perp$) in Fig. 3.

In the region of large negative w_0, i.e., in the region of strong electron degeneracy, Eqs. (4.4) and (4.5) can be studied analytically. Using (4.4) we can write Eq. (4.5) in the form

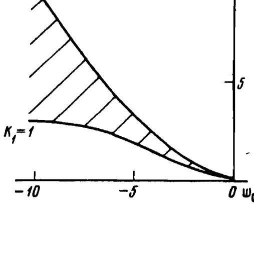

FIG. 1. The region of allowed values of the parameter κ as function of the degree of degeneracy of the electrons; $T_i=T_\perp$.

$$\frac{1}{\zeta}\frac{d}{d\zeta}\zeta\frac{d}{d\zeta}\left[w_i(\zeta)+\frac{K_3}{K_1}w_e(\zeta)\right]=\int_0^\infty\frac{x^{1/2}\,dx}{1+\exp\{x+w_e(\zeta)\}}. \quad (4.7)$$

We shall assume that $|w_0| \gg 1$, $w_0 < 0$, and that the ion temperature T_i is much higher than effective transverse electron temperature $T_\perp$:

$$T_i \gg T_\perp = T_e(1-\beta_0^2)^{1/2}. \quad (4.8)$$

Under condition (4.8) $K_3 \ll K_1$ and we can neglect the second term in the square brackets in (4.7). As long as the potential $w_e(\zeta)$ is still negative and has a large absolute magnitude, the integral on the right-hand side of (4.7) equals $\frac{2}{3}|w_e(\zeta)|^{3/2}$, and the potential $w_i(\zeta)$ satisfies the equation

$$\frac{1}{\zeta}\frac{d}{d\zeta}\zeta\frac{dw_i}{d\zeta}=\frac{2}{3}|w_e(\zeta)|^{3/2}, \quad -w_e(\zeta)\gg 1. \quad (4.9)$$

It is sufficient for us to find the potential $w_i(\zeta)$ only in the region $w_i(\zeta) \sim 1$ as with increasing w_i the ion concentration decreases fast. Expanding $w_e(\zeta)$ in (4.9) in a power series in ζ and integrating Eq. (4.9) we find

$$w_i(\zeta)=|w_0|^{3/2}\zeta^2/6. \quad (4.10)$$

The potential $w_i(\zeta)$ is of order unity when $\zeta \sim |w_0|^{-3/4} \ll 1$ so that the remaining terms in the expansion of $w_e(\zeta)$ in powers of ζ can be neglected when determining $w_i(\zeta)$.

Substituting the value (4.10) for $w_i(\zeta)$ into Eq. (4.4) and integrating, we find the potential $w_e(\zeta)$:

$$\nu = 3|w_0|^{-3/2},\quad w_e(\zeta)=w_0+2K_1\int_0^{|w_0|^{3/2}\zeta^2/6}(1-e^{-z^2})\frac{dz}{z}$$
$$= w_0+K_1[C+\ln(|w_0|^{3/2}\zeta^2/6)-Ei(-|w_0|^{3/2}\zeta^2/6)], \quad (4.11)$$

where C = 0.577 ... is Euler's constant, and Ei(x) the exponential integral. In the region $\zeta \sim |w_0|^{-3/4} \ll 1$ the potential w changes by an amount much less than $|w_0|$ if $K_1 \gtrsim 1$ so that our assumptions are justified. In the limit of small and large ζ we have

$$w_e(\zeta)-w_0=K_1|w_0|^{3/2}\zeta^2/6, \quad \zeta \ll |w_0|^{-3/4},$$
$$w_e(\zeta)-w_0=K_1\ln(e^C|w_0|^{3/2}\zeta^2/6), \quad \zeta \gg |w_0|^{-3/4}. \quad (4.12)$$

When ζ increases the ion concentration practically vanishes even for a relatively small decrease in the electron concentration. The characteristic size of the region containing the ions is

$$r_i \sim r_0|w_0|^{-3/4} \ll r_0,$$

while the region occupied by the electrons increases exponentially when $|w_0|$ increases with $w_0 < 0$:

$$r_e \sim r_0\exp(|w_0|/2K_1) \gg r_0.$$

In order to connect the degeneracy parameter w_0 with the number of electrons N_e, i.e., with the parameter K_2

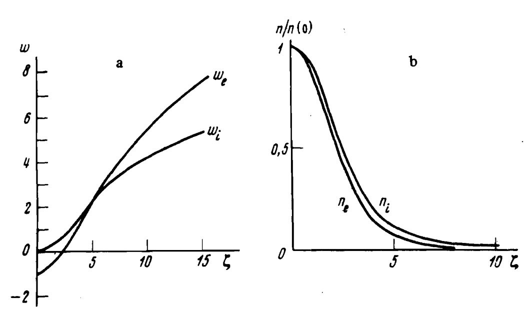

FIG. 2. The potentials (a) and the normalized electron and ion concentrations (b) for $K_1 = K_3 = 2.8$; $K_2 = 4$ (here $T_i = T_\perp$, $w_0 = -1$).

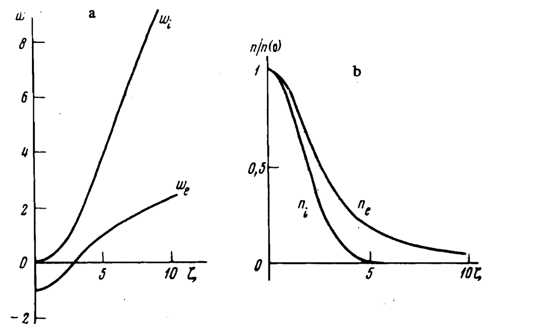

FIG. 3. The potentials (a) and the normalized electron and ion concentrations (b) for $K_1 = K_3 = 1.01$; $K_2 = 8$ (where $T_i = T_\perp$, $w_0 = -1$).

we substitute the potential (4.11) into the normalization integral (4.6). The main contribution to the integral comes from the region $\zeta \gtrsim 1 \gg |w_0|^{-3/4}$ so that we can use the asymptotic formula (4.12). As a result of the integration we get

$$K_2=\frac{3}{4}\frac{\pi/K_1}{\sin\pi/K_1}\left(\frac{\pi K_1^3}{|w_0|^3}\right)^{1/2}\exp\left\{\frac{|w_0|}{K_1}-C\right\}, \quad -w_0\gg 1. \quad (4.13)$$

It follows from Eq. (4.13) that when the electron number N_e increases, the degree of degeneracy w_0 increases proportional to the logarithm of K_2.

The analytical solution obtained here describes the quantal structure of a relativistic beam which is condensed onto the axis.

In the theory presented here there does not appear a direct limitation on the limiting currents. The total current is in an ultrarelativistic beam equal to $I = eN_ec$, i.e., it is exclusively determined by the number of electrons per unit length of the beam which can be given arbitrarily.

The problem of the equilibrium structure of beams with a narrow spread in energy when $T_e \lesssim E_a = 27.2$ eV requires that imperfection is taken into account. This problem was not considered in our paper and remains unsolved.

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1) We are grateful to V. N. Lyakhovitskiĭ for drawing our attention to this relation.

2) When deriving Eqs. (3.10) we assumed that the number of particles N_e was given, but we made no assumptions whatever about the behavior of the densities n_α, α = i. e as $r \to 0$.

3) In quantum statistics this approximation is valid for not too strong a degeneracy of the electron gas. The appropriate condition has the form $|w_0| \ll m_ec^2/T_e$ and is a weak one. The definition and meaning of w_0 are given below, after Eq. (4.6).

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192