The oscillator representation in Landau's problem of the motion of a particle in a uniform field

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We propose a method for solving the problem of the motion of a particle in a uniform electromagnetic field, which uses a representation of dynamical quantities by Bose operators. We show the usefulness of the method by deriving the nonlinear Heisenberg–Euler Lagrangian and the Landau level radiation width as examples.

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L. D. Landau considered 45 years ago the quantum mechanical problem of the motion of an electron in a uniform magnetic field.[1] He reduced it to the linear oscillator problem, using the analogy between the commutator of the generalized momenta

$$[\Pi_x,\Pi_y]=ieH \tag{1}$$

and the commutator $[q, p] = i$. One could say that Landau applied the oscillator representation of the generalized momenta

$$\Pi_x=(a+a^+)\sqrt{eH/2},\quad \Pi_y=-i(a-a^+)\sqrt{eH/2}, \tag{2}$$

to diagonalize the square of the transverse momentum

$$\Pi_\perp^2=\Pi_x^2+\Pi_y^2=eH(a^+a+aa^+). \tag{3}$$

An important property of these relations is that they remain valid in the relativistic theory. Because of this the calculation of a variety of physical effects can be appreciably simplified. We demonstrate this by the example of two problems of which the solutions have been known for some time but which have remained of interest up to the present.

1. QUANTAL NON-LINEARITY IN ELECTRODYNAMICS

It is well known that electrodynamics becomes a nonlinear theory due to the phenomenon of pair creation. This manifests itself in that the action function of the electromagnetic field W acquires a correction[2]

$$W'=-i\,\mathrm{Sp}\ln G, \tag{4}$$

which contains G which is the propagator of a particle imbedded in an external electromagnetic field. We assume that the field is constant and uniform, and we use a system of coordinates in which the field strengths E and H are parallel to the z-axis. For the transverse components of the generalized momenta the Landau representation (2) then remains valid, and for the longitudinal ones its analogue

$$\Pi_z=-i(b-b^+)\sqrt{eE/2},\quad \Pi_0=(b+b^+)\sqrt{eE/2} \tag{5}$$

(b^+, b are, like a^+, a, Bose creation and annihilation operators). Thus

$$W'=-i\,\mathrm{Sp}\ln\left[\frac{1}{\mu^2-\Pi^2}\right]=-i\,\mathrm{Sp}\int_0^\infty\frac{ds}{s}\exp[-s(\mu^2-\Pi^2)] \tag{6}$$

$$=-i\int_0^\infty\frac{ds}{s}e^{-s\mu^2}\,\mathrm{Sp}\exp[seE(bb+b^+b^+)-seH(a^+a+aa^+)].$$

The trace is taken over all quantum numbers of the particle, and the Hamiltonian depends explicitly only on the number eigenvalues a^+a and b^+b. The summation over the other quantum numbers thus reduces to multiplying by the degree of degeneracy of the Landau levels. For the transverse components this equals $eHL_xL_y/2\pi$.[3] We recall its derivation: the energy is independent of the magnitude of the momentum p_x which leads to a degree of degeneracy $L_x\Delta p_x/2\pi$. The total change in momentum is caused by the Lorentz force and equals eHL_y. Similarly the degree of degeneracy of the longitudinal components equals $L_z\Delta p_z/2\pi$. The change in p_z is caused by the electrical field $\Delta p_z = eET$, which leads to the formula $eEL_zT/2\pi$.

The action thus turns out to be proportional to the 4-dimensional volume, as should be the case in a constant and uniform field. The coefficient of the volume is the required correction to the Lagrangian

$$\mathscr{L}'=-i\frac{eH}{2\pi}\frac{eE}{2\pi}\int_0^\infty\frac{ds}{s}e^{-s\mu^2}\,\mathrm{sp}\exp[seE(bb+b^+b^+)-seH(a^+a+aa^+)], \tag{7}$$

where tr indicates the trace over the explicitly appearing variables only. As these are separated we can find each trace separately. Thus

$$\mathrm{sp}\exp[-seH(a^+a+aa^+)]=\sum_{n=0}^\infty\exp[-seH(2n+1)]={}^1/_2\,\mathrm{sh}\,(seH). \tag{8}$$

To evaluate the second trace we use the formula

$$bb+b^+b^+=iU(b^+b+bb^+)U^{-1},\quad U=\exp[{}^1/_8 i\pi(b^+b^+-bb)], \tag{9}$$

which reduces it to the preceding one

$$\mathrm{sp}\exp[seE(bb+b^+b^+)]=\mathrm{sp}\,U\exp[iseE(b^+b+bb^+)]U^{-1}=i/2\sin(seE). \tag{10}$$

Hence

$$\mathscr{L}'=\frac{1}{16\pi^2}\int_0^\infty\frac{ds}{s}\frac{eH}{\mathrm{sh}\,seH}\frac{eE}{\sin seE}e^{-s\mu^2}. \tag{11}$$

It is necessary to regularize this expression, subtracting the first terms of the Taylor series:[4].

$$\mathscr{L}'=\frac{1}{16\pi^2}\int_0^\infty\frac{ds}{s^3}\left[\frac{seH}{\mathrm{sh}\,seH}\frac{seE}{\sin seE}-1-\frac{e^2s^2}{6}(E^2-H^2)\right]e^{-s\mu^2}. \tag{12}$$

For spinor particles the changes in the calculation are small: firstly, by virtue of the Fermi statistics the sign in Eq. (4) is changed; secondly, when quadrating the propagator a factor ½ appears:

$$W_{sp}'=+i\,\mathrm{Sp}\ln\frac{1}{m-\gamma\Pi}=i\,\mathrm{Sp}\ln\frac{1}{m+\gamma\Pi}=\frac{i}{2}\mathrm{Sp}\ln\frac{1}{m^2-(\gamma\Pi)^2}. \tag{13}$$

The square $(\gamma\Pi)^2 = \Pi^2 + e\sigma_{\mu\nu}F_{\mu\nu}/2$ contains a spin term which commutes with Π^2. The trace over the spin components can thus be split off as a factor which for the magnetic part equals

$$e^{seH}+e^{-seH}=2\,\mathrm{ch}\,seH, \tag{14}$$

and for the electrical part 2 cos seE. We thus get the Heisenberg-Euler Lagrangian[4]

$$\mathscr{L}_{sp}' = -\frac{1}{8\pi^2}\int_0^\infty \frac{ds}{s^3}\left[\frac{seH}{\operatorname{th} seH}\frac{seE}{\operatorname{tg} seE} - 1 + \frac{e^2s^2}{3}(E^2-H^2)\right]e^{-sm^2}. \quad (15)$$

In the case of weak fields the expression within the square brackets equals $-(es)^4(E^4 + H^4 + 5E^2H^2)/45$ and after evaluating the integral we get the well known expression

$$\mathscr{L}_{sp}^{(4)} = \frac{e^4}{8\pi^2}\frac{(E^2-H^2)^2+7(EH)^2}{45m^4}. \quad (16)$$

A similar fourth order correction, caused by the contribution of scalar particles, follows from Eq. (12) and equals

$$\mathscr{L}_{sc}^{(4)} = \frac{e^4}{16\pi^2}\frac{7(E^2-H^2)^2+4(EH)^2}{360\mu^4}. \quad (17)$$

Expressions (12) and (15) contain poles at $s = n\pi/eE$, which produce an imaginary part of the Lagrangian and lead to the instability of the vacuum with respect to pair formation (see, e.g.,[3]).

2. SYNCHROTRON RADIATION

Synchrotron radiation (or magnetic bremsstrahlung) is nothing but the radiative decay of a Landau level. The most straightforward method to evaluate it is therefore to calculate the mass operator of a particle in a magnetic field (see[5]). Unfortunately in the cited papers a calculational technique was used which complicated the issue without justification. The Landau representation reduces the problem to a trivial one.

As there is no electrical field the longitudinal momentum components remain unchanged. We replace them by their eigenvalues

$$\Pi_\parallel = (\Pi_0, \Pi_z) = (\varepsilon, 0), \quad (18)$$

which were chosen in this way to exclude spiral motion; Π_x and Π_y are, as before, given by Eqs. (2).

The mass operator has the form

$$\mathfrak{M} = \frac{e^2}{(2\pi)^4}\int\frac{d^4k}{k^2}\int_0^\infty ds(2\Pi-k)\exp[is(\Pi-k)^2-is\mu^2](2\Pi-k). \quad (19)$$

We take the exponential through to the right; only the operators Π_x, Π_y change then, as they are just the ones which do not commute with the index. We have

$$\Pi_x \to \Pi_x(s) = (\Pi_x-k_x)\cos 2seH - (\Pi_y-k_y)\sin 2seH + k_x,$$
$$\Pi_y \to \Pi_y(s) = (\Pi_y-k_y)\cos 2seH + (\Pi_x-k_x)\sin 2seH + k_y. \quad (20)$$

We restrict ourselves to evaluating the leading terms in k/ϵ. This gives

$$\mathfrak{M} = \frac{4e^2}{(2\pi)^4}\int\frac{d^4k}{k^2}\int_0^\infty ds\Pi\cdot\Pi(s)\exp[is(\Pi-k)^2-is\mu^2], \quad (21)$$

$$\Pi\cdot\Pi(s) = \Pi^2+\Pi_\perp^2(1-\cos 2seH)+ieH\sin 2seH. \quad (22)$$

On the mass shell $\Pi^2 = \mu^2$, $\Pi_\perp^2 = \epsilon^2 - \mu^2$. Moreover, we shall assume that $eH \ll \mu^2 \ll \epsilon^2$. In that case

$$\mathfrak{M} = \frac{4e^2\mu^2}{(2\pi)^4}\int\frac{d^4k}{k^2}\int_0^\infty ds\left(1+2\frac{\varepsilon^2}{\mu^2}\sin^2 seH\right)\exp[is(\Pi-k)^2-is\mu^2]. \quad (23)$$

The matrix elements of the operator $\mathfrak{M}$ have the simplest form in the system of coherent states $|\beta\rangle$, defined by the equation $a|\beta\rangle = \beta|\beta\rangle$. In that base (see Appendix)

$$\langle\beta|\exp[is(\Pi-k)^2-is\mu^2]|\beta\rangle = \exp(-2is\varepsilon k_0+2i\mathbf{N}\mathbf{k}), \quad (24)$$

while

$$|\mathbf{N}| = \frac{(\varepsilon^2-\mu^2)^{1/2}}{eH}\sin seH. \quad (25)$$

Thus

$$\mathfrak{M}_{\beta\beta} = \frac{4e^2\mu^2}{(2\pi)^4}\int\frac{d^4k}{k^2}\int_0^\infty ds\left(1+2\frac{\varepsilon^2}{\mu^2}\sin^2 seH\right)\exp[2i(\mathbf{N}\mathbf{k}-s\varepsilon k_0)]. \quad (26)$$

By virtue of the optical theorem the imaginary part of the mass operator is connected with the decay probability

$$W_\beta = -\frac{1}{\varepsilon}\operatorname{Im}\mathfrak{M}_{\beta\beta}, \quad (27)$$

which enables us to write

$$W_\beta = -\frac{4e^2\mu^2}{\varepsilon}\int\frac{d\mathbf{k}}{2\omega(2\pi)^3}\int_0^\infty ds\left(1+\frac{\varepsilon^2}{\mu^2}\sin^2 seH\right)\cos(2\mathbf{N}\mathbf{k}-2s\varepsilon\omega). \quad (28)$$

We can easily find the integral over the angles of the emitted photons:

$$\int d\Omega\cos(2\mathbf{N}\mathbf{k}-2\varepsilon s\omega) = \frac{\pi}{N\omega}[\sin(2\omega\varepsilon s+2N\omega)-\sin(2\omega\varepsilon s-2N\omega)]. \quad (29)$$

The argument of the second sine equals

$$2\omega\varepsilon s\left[1-\frac{\sin seH}{seH}\left(1-\frac{\mu^2}{\varepsilon^2}\right)^{1/2}\right] \approx \omega\varepsilon s\left[\frac{\mu^2}{\varepsilon^2}+\frac{(seH)^2}{3}\right], \quad (30)$$

and in the first one we can put $N \approx \epsilon s$. (The possibility of expanding in powers of seH is connected with the fact that the parametric integral is produced in the region $seH \sim \mu/\epsilon \ll 1$.)

The spectral distribution of the radiation is given by the formula

$$dW = \frac{e^2}{4\pi^2}\left(\frac{\mu}{\varepsilon}\right)^2 d\omega\int_0^\infty\frac{ds}{s}\left[1+2\frac{\varepsilon^2}{\mu^2}(seH)^2\right]\left[\sin\omega\varepsilon s\left(\frac{\mu^2}{\varepsilon^2}+\frac{s^2e^2H^2}{3}\right)-\sin 4\omega\varepsilon s\right] \quad (31)$$

or

$$dW = \frac{\alpha}{\pi}\left(\frac{\mu}{\varepsilon}\right)^2 d\omega\left[\int_0^\infty\frac{dx}{x}(1+2x^2)\sin\xi\left(x+\frac{x^3}{3}\right)-\frac{\pi}{2}\right], \quad (32)$$

where $\xi = \omega\mu^3/eH\epsilon^2$. This formula is the basic result of the cited paper by Schwinger.[5] The mass operator was found in that paper by a considerably more complicated method. Retaining terms $\propto\omega$ we get corrections of the order $\chi = eH\epsilon/\mu^3$ which are quantum mechanical in character.

The achieved simplification is caused mainly by the application of the coherent representation of states which, in turn, is connected with the introduction of the Landau boson operators. Finally, the differential form of the optical theorem turned out to be convenient.

APPENDIX

We introduce the shift operator for the transverse components of the generalized momentum

$$D^+(\eta)\Pi_\perp D(\eta) = \Pi_\perp - k_\perp.$$

In explicit form it is

$$D(\eta) = \exp(\eta a^+ - \eta^* a), \quad \eta = (-k_x - ik_y)/\sqrt{2eH}.$$

In that case

$$\exp[is(\Pi-k)^2-is\mu^2] = \exp[is(\Pi_\parallel-k_\parallel)^2-is\mu^2]D^+(\eta)\exp(-is\Pi_\perp^2)D(\eta)$$
$$= \exp[is(\Pi_\parallel-k_\parallel)^2-is\mu^2]\exp(-is\Pi_\perp^2)D^+(\eta_s)D(\eta) =$$
$$= \exp[is(k_\parallel^2-2k_\parallel\Pi_\parallel)]D^+(\eta_s)D(\eta),$$

where $\eta_s = \eta e^{2iseH}$.

The shift operators are multiplied, using the Baker-Hausdorff formula

$$e^X e^Y = e^{X+Y} \exp {}^1/_2 [X, Y],$$

provided [X, Y] is a c-number. We get

$$D^+(\eta_s) D(\eta) = D(u) \exp(i \operatorname{Im} \eta_s^* \eta), \quad u = \eta - \eta_s.$$

Finally, the matrix element of the shift operator equals, in the coherent base,

$$\langle\beta|D(u)|\beta\rangle = \langle\beta|\exp(ua^+)\exp(-u^* a)\exp(-|u|^2/2)|\beta\rangle = \exp(\beta^* u - \beta u^* - |u|^2/2).$$

Retaining only the linear terms in the exponent we find

$$\langle\beta|\exp[is(\Pi - k)^2 - is\mu^2]|\beta\rangle = \exp(-2isk_0\varepsilon + 2i\mathbf{N}\mathbf{k}).$$

The vector **N** has only the transverse components

$$\mathbf{N} = \frac{|\beta|}{\sqrt{2eH}} (\sin 2seH; 1 - \cos 2seH; 0).$$

Bearing in mind that $|\beta|^2 = \langle\beta|a^+a|\beta\rangle = (\epsilon^2 - \mu^2 - eH)/2eH$, we get

$$|\mathbf{N}| = \frac{(\varepsilon^2 - \mu^2)^{1/2}}{eH} \sin seH.$$

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