Some properties of transition radiation in weakly inhomogeneous and nonstationary plasma

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The transition radiation from a charge traveling through plasma is considered for a smooth variation of plasma density in space or time. Various types of anisotropy of this radiation are considered. It is shown that there is an increase in anisotropy of the angular distribution of transition-radiation energy for frequencies in excess of the optical frequency when the characteristics scales of either time or space variation in the plasma density are increased.

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INTRODUCTION

Ginzburg and Frank[1] were the first to point out that a charge moving through an inhomogeneous dielectric can emit electromagnetic waves. This is now referred to as transition radiation. The emission of transition radiation in inhomogeneous media was subsequently investigated in a large number of papers, most of which have been reviewed, for example, by Bass and Yakovenko.[2] Nevertheless, a series of new results has been obtained in recent years, including the discovery of the anisotropy of transition radiation produced when the charge passes through a resonance layer of weakly inhomogeneous plasma in which the permittivity becomes zero,[3,4] and a new type of transition radiation due to the nonstationary character of a medium,[5-7] which has a number of important properties.[1)] One of the properties of transition radiation emerging from the resonance region of a hot, isotropic, weakly inhomogeneous plasma is the cascade character of the wave generation mechanism in which Langmuir oscillations are excited during an intermediate stage which is followed by the transformation of these oscillations into electromagnetic waves.[3,4] Transition radiation produced in this way has a well-defined anisotropy. The intensity of the radiation is a maximum when the charge moves in the direction of increasing plasma density, and decreases exponentially in the reverse case, i.e., when the charge moves in the direction of decreasing density. This anisotropy is also found to arise in magnetized plasma[9] which, in turn, leads to anisotropy in the emission of transverse waves in the beam-plasma discharge.[10]

In this paper, we investigate various types of anisotropy of transition radiation due to a moving relativistic charge and to the resonant properties of a plasma layer in which the permittivity becomes zero. In the first part, we consider the transition radiation from a charge in homogeneous nonstationary plasma, the density of which varies smoothly with time.[2)] An exact solution of the problem is obtained for this case. Analysis of this solution shows that the angular distribution of the emitted radiation is highly anisotropic for a relativistic charge moving through plasma. This is analogous to the anisotropy investigated by Ginzburg and Tsytovich.[6] The anisotropy in the angular distribution is enhanced when the characteristic time τ of the density variation is increased. The dependence of the radiation intensity I on the difference Δn in plasma density is determined by the parameter $\tau\Delta\omega(k)$, where $\Delta\omega(k)$ is the change in the wave frequency $\omega_k = (\omega_{pe}^2 + k^2c^2)^{1/2}$ due to a change in the plasma density. When $\tau\Delta\omega(k) \ll 1$, the radiation intensity is proportional to the square of the density difference Δn. If, however, $\tau\Delta\omega(k) \gg 1$, the character of this dependence is modified, and it is found that $I \sim \Delta n$. The total intensity falls exponentially with increasing τ. In the limit of a sharp jump in density ($\tau \to 0$), the expressions for the field are the same as those obtained in[7] but differ from the corresponding formulas in[5,6]. This is connected, in particular, with the fact that the frequency dispersion must be properly taken into account in the case of transition radiation in nonstationary plasma (see Appendix for further details). It is important to note, however, that, in the asymptotic limit of an ultrarelativistic charge $\gamma = (1-\beta^2)^{-1/2} \gg 1$, and the results reported in[5-7] do, in fact, agree within the cone $\Delta\theta \sim 1/\gamma$.

In the second part of this paper, we investigate the anisotropy of transition radiation in stationary plasma with density varying slowly in space. For $\omega > \max\omega_{pe} \equiv \omega_{pm}$, the anisotropy of the transition radiation is found to be quite similar to that described above (the angular distribution of the radiated energy is anisotropic, especially for a relativistic charge).

The essential role of the complex point of synchronism is also elucidated. This can be summarized as follows. Suppose L is the characteristic scale of plasma density inhomogeneity on which the component $k_z(z)$ of the wave vector along the density gradient changes by an amount Δk_z. For frequencies $\omega \gg \omega_{pm}$ and provided $L\Delta k_z \ll 1$, the increase in the phase of the electromagnetic wave due to the plasma inhomogeneity is small, so that the inhomogeneity can be looked upon as a perturbation by analogy with the theory of scattering of high-energy particles in quantum mechanics.[11] The field amplitude of the transition radiation is then determined by perturbations close to the real axis of z, which is in full agreement with the ideology of Galeev.[12] However, in the other limiting case, when $L\Delta k_z \gg 1$, the main contribution to the field amplitude is due to the complex point of synchronism, z_s, at which the Cerenkov resonance condition $\omega = k_z(z_s, \omega)v$ is satisfied (v is the velocity of the charge). For frequencies $\omega < \omega_{pm}$, the anisotropy is analogous to that investigated in[3,4].

1. ANISOTROPY OF TRANSITION RADIATION DUE TO A CHARGE IN A NONSTATIONARY HOMOGENEOUS PLASMA

We shall consider the transition radiation emitted by uniformly moving charge in cold homogeneous plasma whose density is a function of time.

In the Fourier representation

$$\mathbf{H}(\mathbf{r},t)=\int d\mathbf{k}\,\mathbf{H}_k(t)\exp(i\mathbf{kr})$$

the equation for the magnetic field $\mathbf{H}_k$ is [7]

$$\frac{d^2}{dt^2}\mathbf{H}_k+[k^2c^2+\omega_{pe}^2(t)]\mathbf{H}_k=\frac{iqc}{2\pi^2}[\mathbf{kv}]\exp(-i\mathbf{kv}t), \quad (1.1)$$

where q is the charge and $\mathbf{v} = \mathbf{e}_z v$ is the velocity. If, however, the plasma density n(t) increases with time, then, using the formulas given in [7], we can show that the current density of the "created" plasma particles contains the term

$$\delta\mathbf{j}_k(t)=-\frac{ie^2}{mck^2}\int_{-\infty}^{t}dt'[\mathbf{k}\times\mathbf{H}_k(t')]\frac{dn(t')}{dt'}.$$

As a result, in plasmas with density increasing with time, the right-hand side of (1.1) acquires the additional term

$$\int_{-\infty}^{t}dt'\mathbf{H}_k(t')\frac{d\omega_{pe}^2(t')}{dt'}.$$

However, if we differentiate with respect to time and substitute $d\mathbf{H}_k/dt = (-i\mathbf{k}\cdot\mathbf{v})\mathbf{F}_k$, we obtain (1.1). Hence, it follows that the expression for the transition-radiation intensity emitted by a charge in plasma whose density increases with time will contain the additional factor $(\mathbf{k}\cdot\mathbf{v}/\omega_k)^2$ as compared with the case of decaying plasma.

Suppose that the Langmuir frequency of the plasma is

$$\omega_{pe}^2(t)=\omega_{p1}^2+(\omega_{p2}^2-\omega_{p1}^2)(1+e^{-t/\tau})^{-1}. \quad (1.2)$$

In this expression, τ is the characteristic time of the plasma density variation; in a decaying plasma, $\omega_{p1} > \omega_{p2}$. For the time dependence specified by (1.2), the solution of (1.1) can be expressed in terms of hypergeometric functions. However, for convenience of comparison of the various limiting cases, we shall divide our analysis into two stages.

Let us begin with a sharp ($\tau \to 0$) change in the plasma density. In the frequency region in which we are interested, we must, in fact, have $\omega_k\tau \ll 1$. When this condition is satisfied, the exact solution of (1.1) yields the following expression for the transition-radiation field, which is identical to that obtained in [7]:

$$\mathbf{H}_k=\frac{iqc(\omega_{p2}^2-\omega_{p1}^2)[\mathbf{k}\times\mathbf{v}]}{4\pi^2(\omega_k^2-k_z^2v^2)(\Omega_k^2-k_z^2v^2)}\left(\frac{\mathbf{kv}}{\omega_k}\right)^s\left[\left(1+\frac{\mathbf{kv}}{\omega_k}\right)\exp\{-i\omega_kt\}+(-1)^s\left(1-\frac{\mathbf{kv}}{\omega_k}\right)\exp\{i\omega_kt\}\right], \quad (1.3)$$

where $\omega_k^2 = \omega_{p2}^2 + k^2c^2$ and $\Omega_k^2 = \omega_{p1}^2 + k^2c^2$. The parameter s can assume two values, namely, s = 0 and s = 1, which correspond, respectively, to decreasing and increasing plasma density. Using (1.3), we obtain the following formula for the radiation energy:

$$w=\int d\Omega\int_0^\infty dk\frac{q^2v^2k^2(\omega_{p2}^2-\omega_{p1}^2)^2\sin^2\theta}{4\pi^2(\omega_k-\mathbf{kv})^2(\Omega_k^2-k_z^2v^2)^2}\left(\frac{\mathbf{kv}}{\omega_k}\right)^{2s}. \quad (1.4)$$

In this expression, θ is the angle between the velocity of the charge and the wave vector $\mathbf{k}$ and $d\Omega = 2\pi\sin\theta d\theta$ is the solid angle element. It is important to note that the magnetic field has only the azimuthal component $H_\varphi(\mathbf{r}, t)$.

Ignoring the nonrelativistic case $v \ll c$, which was investigated in [7], we shall consider (1.4) in the ultrarelativistic limit $\gamma \gg 1$, where $\gamma = (1 - v^2/c^2)^{-1/2}$ is the relativistic factor. In the latter limit, (1.4) shows that the spectral energy density $w_{\omega,\Omega}$ per unit interval of ω_k and of the solid angle Ω has a maximum near $\omega_k \sim \gamma\omega_{p2}$ and $\theta \sim \gamma^{-1}$. When this is taken into account and we consider the predominant part of the radiation in the cone $\theta \sim 1/\gamma \ll 1$, the expression given by (1.4) can be simplified and this leads to the following result:

$$w_{\omega,\Omega}\approx\frac{q^2\theta^2}{\pi^2c}\left(\frac{\omega_{pe}}{\omega}\right)^4\left(\frac{\Delta n}{n}\right)^2\left(\theta^2+\gamma^{-2}+\frac{\omega_{p1}^2}{\omega^2}\right)^2\left(\theta^2+\gamma^{-2}+\frac{\omega_{p2}^2}{\omega^2}\right)^2, \quad (1.5)$$

which was previously obtained by Ginzburg and Tsytovich [6] for an ultrarelativistic charge in a dielectric. Therefore, in the ultrarelativistic limit and for a sharp variation in the plasma density with time, the angular distribution of transition radiation is highly anisotropic and most of it is localized within the cone $\theta \sim 1/\gamma$ drawn around the direction of motion.[3)]

Let us now suppose that the plasma density is a smooth function of time. Substituting (1.2) in (1.1), we obtain the following solution of the latter with the right-hand side equal to zero:

$$H_1(t)=F(ia,\ ib;\ \sigma;-e^{t/\tau})\exp(i\Omega_kt),$$
$$H_2(t)=F(-ia,\ -ib;\ 2-\sigma;\ -e^{t/\tau})\exp(-i\Omega_kt), \quad (1.6)$$

where

$$a=(\Omega_k+\omega_k)\tau,\quad b=(\Omega_k-\omega_k)\tau,\quad \sigma=1+ia+ib,$$

and F is the hypergeometric function. We note that this solution describes the physical situation where, for $t \to -\infty$, a monochromatic wave with energy flux $S_f^{(-)}$ propagates in the plasma. Because of the variation in plasma density with time, this is followed by the appearance of a wave traveling in the reverse direction, in which the energy flux for $t \to +\infty$ is $S_b^{(+)}$. Askar'yan and Pogosyan [13] have considered a similar effect for a dielectric. If $S_f^{(+)}$ is the energy flux in the original wave for $t \to +\infty$, then (1.6) readily yields the expression

$$S_f^{(+)}=S_f^{(-)}+S_b^{(+)},\quad \frac{S_f^{(+)}}{S_f^{(-)}}=\frac{\operatorname{sh}^2\pi(\Omega_k+\omega_k)\tau}{\operatorname{sh}(2\pi\omega_k\tau)\operatorname{sh}(2\pi\Omega_k\tau)}>1.$$

We note that, when the plasma density is a function of time, the energy flux S_f in the "test" wave increases and, consequently, the original wave is amplified. However, for an adiabatic change of density, when $2\pi\omega_k\tau \gg 1$, this effect is exponentially small. In the opposite limiting case, when $2\pi\Omega_k\tau \ll 1$, the gain in the energy flux of the "test" wave is $(\Omega_k + \omega_k)^2/4\omega_k\Omega_k$ and increases with decreasing $(\omega_k/\Omega_k) < 1$.

To solve the inhomogeneous equation (1.1), we shall use the solutions given by (1.6) and the method developed by Amatuni and Korkhmazyan [14] except that, in formulating the boundary conditions, we shall assume that there is no radiation for $t \to -\infty$. Basically simple analysis which, however, must be carried out with some care, yields the following expression for the field of the transition radiation for $t \to +\infty$:

$$H_k(t)=A(\omega_k,\ \Omega_k)e^{-i\omega_kt}+A(-\omega_k,\ -\Omega_k)e^{i\omega_kt}, \quad (1.7)$$

where

$$A(\omega_k,\Omega_k)=i\frac{qvc}{2\pi^2}k_\perp\tau^2B(ib,-ia)\Gamma(i\mathbf{kv}\tau-i\omega_k\tau)\times\Gamma(i\Omega_k\tau-i\mathbf{kv}\tau)\Gamma(-i\Omega_k\tau-i\mathbf{kv}\tau)\Gamma^{-1}(1-i\omega_k\tau-i\Omega_k\tau). \quad (1.8)$$

In these expressions, B(ib, −ia) and $\Gamma(\xi)$ are, respectively, the beta and gamma functions. Let $A_0(\mathbf{k})$ be the value of the amplitude $A(\omega_k, \Omega_k)$ for $\tau = 0$. From (1.7) and (1.8), we then have

$$\left|\frac{A(\omega_k,\Omega_k)}{A_0(\mathbf{k})}\right|^2=\frac{2\pi\omega_k\tau(\omega_k-\mathbf{kv})(\Omega_k^2-k_z^2v^2)}{(\omega_k+\mathbf{kv})(\Omega_k^2-\omega_k^2)\operatorname{sh}2\pi\omega_k\tau}$$

$$\times\frac{\operatorname{sh}\pi(\omega_k+\Omega_k)\tau\operatorname{sh}\pi(\Omega_k-\omega_k)\tau}{\operatorname{sh}\pi(\Omega_k+kv)\tau\operatorname{sh}\pi(\Omega_k-kv)\tau}\frac{\operatorname{sh}\pi(\omega_k+kv)\tau}{\operatorname{sh}\pi(\omega_k-kv)\tau},\qquad(1.9)$$

$$\left|\frac{A(\omega_k,\Omega_k)}{A(-\omega_k,-\Omega_k)}\right|^2=\left[\frac{\operatorname{sh}\pi(\omega_k+kv)\tau}{\operatorname{sh}\pi(\omega_k-kv)\tau}\right]^2.$$

We now consider the dependence of the radiation-field amplitude on the velocity v of the charge and the characteristic time τ of the variation in density. The effect of the "width" of the density jump on the amplitude of the Fourier harmonic of the radiation field is characterized by the function

$$f(k,\ kv;\ \tau)=|A(\omega_k,\ \Omega_k)/A_0(k)|^2,$$

whose dependence on the parameters of the problem is defined by (1.9). We now evaluate the derivatives $d\ln f/d\ln\tau$ and, recalling that for positive $\xi_{1,2,s}$ and $\xi_1\leq\xi_2+\xi_3$, we have

$$\frac{\xi_1}{\operatorname{th}\xi_1}<\frac{\xi_2}{\operatorname{th}\xi_2}+\frac{\xi_3}{\operatorname{th}\xi_3},$$

we find that $df/d\tau<0$. Therefore, the transition-radiation amplitude decreases with increasing characteristic time τ. For an adiabatic density variation, when $2\pi\omega_{p2}\tau>\gamma$, the above function is a decreasing exponential: $f\sim\exp[-2\pi(\omega_k-\mathbf{k}\cdot\mathbf{v})\tau]$. Secondly, it is clear from (1.9) that, as τ increases, there is an increase in the anisotropy of the radiation relative to the forward (in the direction of motion of the charge) and backward directions. Thus, for the adiabatic change in density, the probability of emission of waves in "backward directions" is lower than the forward probability in the ratio of $\exp(-4\pi\mathbf{k}\cdot\mathbf{v}\tau\cos\theta)$.

In the case of the relativistic charge, the spectral density of the radiation energy has a maximum at high frequencies $\omega_k\sim\gamma\omega_{p2}$, as before (in the system attached to the moving charge, this corresponds to frequencies of the order of ω_{p2}/γ because of the Doppler shift). If, moreover, the time $\gamma/2\pi\omega_{p2}$ necessary to generate the radiation is less than τ, the radiation is localized within a cone of angle $\Delta\theta\sim1/\gamma$ around the direction of motion of the charge, and its parameters are close to those for the radiation emitted in the case of a sharp jump in the plasma density. In the other limiting case, when $2\pi\omega_{p2}\tau>\gamma$, the "width" of the density jump affects the entire frequency spectrum of the radiation. At the same time, the radiation cone contracts: $\Delta\theta\sim(2\pi\omega_{p2}\tau\gamma)^{-1/2}<1/\gamma$. Outside the cone, the spectral density decreases exponentially with increasing angle θ. In the limiting case when $2\pi\omega_{p2}\tau>\gamma\gg1$,

$$f\approx\frac{4\pi\omega_k\tau(\omega_k-kv)(\Omega_k^2-k_z^2v^2)}{(\omega_k+kv)(\omega_{p1}^2-\omega_{p2}^2)}\{1-\exp[-2\pi(\Omega_k-\omega_k)\tau]\}\times\exp[-2\pi(\omega_k-kv)\tau].\qquad(1.10)$$

We note that the frequency $\omega_k-\mathbf{k}\cdot\mathbf{v}$ has a minimum for $k=(\omega_{p2}/v)(1-\beta^2\cos^2\theta)^{-1/2}\cos\theta$ for which it is equal to $\omega_{p2}(1-\beta^2\cos^2\theta)^{1/2}$, and the frequency $\Omega_k-\omega_k$ decreases monotonically with increasing k ($\beta=v/c$). It is clear from (1.10) that, when $2\pi|\omega_{p1}-\omega_{p2}|\tau\ll\gamma$, the small denominator $\omega_{p1}^2-\omega_{p2}^2$ is absent from the function f. Consequently, similarly to the case of a sharp jump in the plasma density, the intensity of the radiation is proportional to the square of the small density jump Δn. In the other limiting case, when $2\pi|\omega_{p1}-\omega_{p2}|\tau>\gamma$, the function f is inversely proportional to Δn, and the radiation intensity is proportional to the first power of the density jump. We note that, when $2\pi(\Omega_k-\omega_k)\tau\ll1$, the variable part of the plasma density may be looked upon as a perturbation and, consequently, the transition-radiation field can be derived from perturbation theory by analogy with the theory of scattering of high-energy particles in quantum mechanics.[11] The radiation field amplitude will, of course, then be proportional to the change in the plasma density.

It follows from (1.10) that, when $\tau>\gamma/2\pi\omega_{p2}$, there is a reduction in the width of the frequency band corresponding to the maximum energy density, which is described by $\delta\omega_k/\omega_k\sim(\gamma/2\pi\omega_{p2}\tau)^{1/2}$.

Therefore, the foregoing results lead to the conclusion that, as the characteristic time τ of the plasma density variation increases, there is an enhancement of the anisotropy of the angular distribution of the radiated energy. In the ultrarelativistic case, when $2\pi\omega_{p2}\tau>\gamma\gg1$, the radiation cone contracts with increasing τ, and the transition radiation energy becomes exponentially small in proportion to the factor $\exp(-2\pi\omega_{p2}\tau\gamma^{-1})$.

2. ANISOTROPY OF TRANSITION RADIATION IN WEAKLY INHOMOGENEOUS PLASMA

Let us now consider the transition radiation emitted by a charge moving in the direction of the density gradient in inhomogeneous plasma, which is parallel to the z axis. In the Fourier representation

$$\mathbf{H}(\mathbf{r},t)=\int d\omega\int d\mathbf{k}_\perp\left[\mathbf{e}_z\frac{\mathbf{k}_\perp}{k_\perp}\right]H_\omega(k_\perp,z)\exp(i\omega t-i\mathbf{k}_\perp\mathbf{r})$$

the magnetic field H_ω is a solution of the equation[12]

$$\frac{d^2}{dz^2}H_\omega-\frac{d\ln\varepsilon}{dz}\frac{d}{dz}H_\omega+\frac{\omega^2}{c^2}(\varepsilon-\sin^2\theta)H_\omega=\frac{qk_\perp}{2i\pi^2c}\exp\left(-i\frac{\omega}{v}z\right),\qquad(2.1)$$

where $k_\perp=(\omega/c)\sin\theta$, ϵ is the permittivity for which the dependence on z is

$$\varepsilon(z)=1-(\omega_{pm}/\omega)^2[1+\exp(-2z/L)]^{-1}.\qquad(2.2)$$

and L is the length of a plasma density inhomogeneity. We note that (2.2) simulates the profile of the vacuum-plasma boundary for a monotonically increasing plasma density, so that $0<\omega_{pe}<\omega$. The frequency spectrum of the transition radiation can then be naturally divided into two regions: $\omega>\omega_{pm}$ (optical region) and $\omega<\omega_{pm}$ for which there is a very different structure and a different method of investigation must be employed.

1. Consider the optical region first. For a smoothly-varying plasma inhomogeneity $\omega_{pm}L\gg c$, and the efficiency of generation of transition radiation in the case of a relativistic charge is of practical interest. We shall therefore suppose that the relativistic factor γ is large.

We note, to begin with, that the solution of (2.1) has singularities at the zeros z_0 and poles z_∞ of the permittivity $\epsilon(z)$. When the permittivity is given by (2.2), these points are defined as follows:

$$z_\infty=i\frac{\pi}{2}L(1+2n),\quad z_0=z_\infty-\frac{L}{2}\ln\left(1-\frac{\omega_{pm}^2}{\omega^2}\right),$$

where n = 0, ±1, ... In the problem we are considering, the synchronism points

$$z_s=z_\infty-\frac{L}{2}\ln\left(1+\frac{\omega_{pm}^2\beta^2/\omega^2}{1-\beta^2\cos^2\theta}\right),$$

at which the Cerenkov resonance condition $\omega=k_z(z_s)v$ is satisfied, play an important role; $k_z=(\omega/c)(\epsilon-\sin^2\theta)^{1/2}$ is the component of the wave vector along the inhomogeneity. We note that all the "special" points z_0, z_∞, and z_S are located in the complex plane of z in a symmetric fashion relative to the real axis of z. It will be convenient to introduce the frequencies $\omega_\gamma = \gamma\omega_{pm}$, $\omega_* = \omega_{pm}^2 L/c$, which characterize relativistic and inhomogeneity effects in plasma, respectively.

We now consider the solution of (2.1), assuming to be specific, that $v > 0$. In this case, the charge travels from the vacuum into the plasma. At very high frequencies $\omega > \omega_*\gamma^2$, the separation between the points z_0, z_∞, z_S nearest to the real axis of z is small in comparison with the characteristic wavelength c/ω. Since the magnetic field H_ω at the special points z_0, z_∞ is finite, the plasma can be looked upon as a small perturbation by analogy with the use of perturbation theory in quantum mechanics [11] for the scattering of high-energy particles. In the zero-order approximation, the solution of (2.1) is given by the field of a uniformly moving particle

$$H_\omega^{(0)}(z)=\frac{qk_\perp}{2i\pi^2 c}\left(k_0^2-\frac{\omega^2}{v^2}\right)^{-1}\exp\left(-i\frac{\omega}{v}z\right). \quad (2.3)$$

In this expression, $k_0 = (\omega/c)\cos\theta$. In the next approximation in the parameter $(\omega_{pm}/\omega)^2$, the solution of (2.1) has the form

$$H_\omega^{(1)}(z)=\frac{1}{2ik_0}\int_{-\infty}^{+\infty}dz' F_\omega(z')\exp[-ik_0|z-z'|], \quad (2.4)$$

where

$$F_\omega(z)=(1-\varepsilon)H_\omega^{(0)}(z)+\frac{d\ln\varepsilon}{dz}\frac{dH_\omega^{(0)}(z)}{dz}$$

and the function $H_\omega^{(0)}$ is given by (2.3). The asymptotic field for $z \to \pm\infty$ is found by shifting the contour of integration into the lower half-plane of z. The main contributions to the integral given by (2.4) are then connected with the residues of the function $F_\omega(z)$ at the pole and the zero of $\epsilon(z)$, which compensate one another to within the small parameter $\omega_*/2\omega$.[4)]

The field amplitude in the forward direction is given by

$$|H_\omega^{(1)}|=\frac{qL\theta}{4\pi c}\left(\frac{\omega_{pm}\gamma}{\omega}\right)^2\exp\left[-\frac{\pi\omega L}{4c}(\theta^2+\gamma^{-2})\right].$$

Hence, it is clear that, in the above frequency band, the transition radiation is localized within the narrow cone $\theta \lesssim (c/\omega L)^{1/2} < \gamma^{-1}$ drawn around the direction of motion of the charge, and the radiated energy is exponentially small.

For frequencies $\omega < \omega_*\gamma^2$, the distance between the synchronism point z_S and the pole at z_∞ is greater than the wavelength corresponding to the effective wave vector $k_{eff} = (\omega/v) - k_z(z)$, so that the dependence of k_z on the plasma density must be taken into account. The above perturbation-theory method is then invalid, but the quasiclassical approximation can be employed.

We shall now determine two linearly independent solutions of (2.1) with right-hand sides equal to zero by specifying their quasiclassical asymptotic behavior at infinity as follows:

$$H_1(z)=(i\varepsilon/2k_z)^{1/2}\exp[-i\psi(z)], \quad z\to+\infty,$$
$$H_2(z)=(i\varepsilon/2k_z)^{1/2}\exp[+i\psi(z)], \quad z\to-\infty; \quad (2.5)$$

where

$$\psi(z)=\int_0^z k_z(z')dz'.$$

When the plasma is weakly inhomogeneous, so that $k_z L \gg 1$, the Wronskian of the solutions of (2.1) may be assumed to be equal to unity. Hence, denoting the right-hand side of (2.1) by $f_\omega(z)$, we can write the solution of this equation in the following form which explicitly satisfies the radiation boundary condition:

$$H_\omega(z)=H_1(z)\int_{-\infty}^{z}dz'\frac{f_\omega(z')}{\varepsilon(z')}H_2(z')+H_2(z)\int_z^{\infty}dz'\frac{f_\omega(z')}{\varepsilon(z')}H_1(z'). \quad (2.6)$$

This expression yields the following asymptotic expression for the transition radiation field when $z \to \pm\infty$, respectively:

$$H_{(\pm)}(z)=H_{1,2}(z)\int_{-\infty}^{\pm\infty}dz' f_\omega(z')H_{2,1}(z')\varepsilon^{-1}(z'). \quad (2.7)$$

To obtain an accurate asymptotic estimate for (2.7) when $v > 0$, let us shift the integration path toward the real axis and into the lower half-plane of z, and use the quasi-classical expressions for $H_{1,2}(z)$. The integrand in (2.7) will then acquire exponential factors of the form

$$\exp\left[-i\int_{z_s}^{z}k_{\text{eff}}(z')dz'\right]=\exp[-i\psi_c(z)]. \quad (2.8)$$

The synchronism point z_S is a turning point for $k_{eff}(z)$, so that the Stokes line, defined by the condition $\operatorname{Re}\psi_c(z) = 0$, will pass through it. The function $\exp(-i\psi_c)$ decreases monotonically on the Stokes line with increasing distance from the point z_S, which is the point of absolute maximum for this function. It follows that the magnitude of the integral in (2.7) is determined by competition between contributions due to the synchronism point and the singularities of the integrand which lie between the real axis of z and the Stokes line. In general, when there are several synchronism points, the contour of integration must be deformed into segments of Stokes lines which are topologically equivalent to it and are the nearest to the real axis of z (this procedure is carried out in accordance with the conditions for the validity of the method of steepest descents; see, for example, [15]).

For frequencies $\omega < \omega_*\gamma^2$ in which we are interested, the Stokes line runs around and above the singularity of $H_{1,2}(z)$ in the lower half-plane. It therefore provides the main contribution. Using the method of steepest descents, we find that the radiation-field amplitude for $z \to \infty$ is given by

$$|H_\omega(z)|^2=\frac{q^2(\omega_{pm}/\omega)^2(L\theta^2/8\pi^3\omega c)}{(\theta^2+\gamma^{-2})(\theta^2+\gamma^{-2}+\omega_{pm}^2/\omega^2)}\exp\left[-\frac{\pi\omega L}{2c}(\theta^2+\gamma^{-2})\right] \quad (2.9)$$

where we have taken into account the fact that, in the ultrarelativistic case, the waves are preferentially emitted at angles $\theta \ll 1$ to the direction of motion of the charge.

Let us consider (2.9) in greater detail. For an ultrarelativistic charge, the diffuseness of the plasma boundary has little effect on the radiation parameters in the cone $\theta \sim \gamma^{-1}$ if $\gamma\omega_{pm} < \omega \ll c\gamma^2/L$. Hence, it is clear that the condition $\omega_{pm}L < c\gamma$ should be satisfied. Such estimates show that, for frequencies $\omega \sim \gamma\omega_{pm}$ corresponding to the maximum of the spectral energy density and for angles $\theta \lesssim \gamma^{-1}$, the inhomogeneity length L must be less than the length $c\gamma/\omega_{pm}$ which characterizes the generation of the transition radiation so that the diffuseness of the plasma boundary is not "seen." This estimate was obtained by Amatuni and Korkhmazyan [14] by a different method. It is clear from (2.9) that, as the inhomogeneity length increases, the anisotropy of the angular distribution of the transition radiation is reduced. For $L > c\gamma/\omega_{pm}$ and frequencies $\omega \sim \gamma\omega_{pm}$, the radiation cone contracts in accordance with the formula

$\theta \sim (c/\gamma\omega_{pm}L)^{1/2}$, and the transition radiation energy decreases exponentially.

Analogous results are obtained when the direction of motion of the charge is reversed (taking place from plasma into vacuum). In the optical region, therefore, only the anisotropy of the angular energy distribution of the transition radiation is possible.

2. Let us now consider the transition radiation for frequencies $\omega < \omega_{pm}$. In contrast to the foregoing, in this region the problem has the particular feature that the above "special" points have an asymmetric disposition, i.e., the zero of $\epsilon(z)$ is shifted out of the complex plane onto the real axis of z (when dissipation is taken into account, the zero of ϵ lies at a distance $\Delta z = L_\omega \nu_{eff}/\omega$ from the real axis of z, where L_ω is the inhomogeneity length at the plasma resonance point $\omega_{pe} = \omega$ and ν_{eff} is the effective collision frequency). A consequence of this is the appearance of a new type of anisotropy, namely, the intensity of the transition radiation is a maximum when the charge moves in the direction of increasing plasma density, and decreases exponentially in the opposite case. The physical mechanism responsible for this type of anisotropy is discussed in detail in [3]. Since the transition radiation due to a single charge moving along the density gradient was previously discussed in [3,4], we shall consider a different case, namely, the transition radiation due to a modulated current flowing at an angle to the density gradient. We note that for a source moving at an angle to the direction of inhomogeneity, the effective velocity along the inhomogeneity may exceed the velocity of light in vacuum (for further details see [9]; other methods of producing motion with such velocity are discussed by Bolotovskiĭ and Ginzburg [16]).

Suppose that the extraneous current density is given by

$$\mathbf{j}(\mathbf{r},\, t) = j_0(\xi)\mathbf{e}_\xi \cos\omega(t-\xi/v), \quad v>0, \tag{2.10}$$

where $\mathbf{e}_\xi = \mathbf{e}_z \cos\theta_0 + \mathbf{e}_y \sin\theta_0$ defines the direction of motion of the current, $\xi = y\cos\theta_0 - z\sin\theta_0$, and the modulation frequency ω is less than the plasma frequency ω_{pm}. For the source defined by (2.10), the right-hand side of (2.1) is

$$f_\omega(z) = \frac{4\pi}{c}\left[ik_\perp j_\omega(z)\cos\theta_0 + \varepsilon\frac{d}{dz}\left(\frac{j_\omega(z)}{\varepsilon}\right)\sin\theta_0\right]. \tag{2.11}$$

The solution of (2.1) with the right-hand side defined by (2.11) can be found by a method analogous to that used in [4]. Consider the transition-radiation power emitted in the case of a current ribbon of thickness 2a in the ξ direction and infinite in the x direction. When the charges travel in the direction of increasing plasma density $|\theta_0| < \pi/2$

$$\frac{\partial^2 W}{\partial t\,\partial x} \approx \frac{2L_\omega}{\pi c} j_0^2 a^2 \int_0^{\theta_m} d\theta \left(\frac{\sin\sigma}{\sigma}\right)^2 \exp\left(-\frac{4\omega L_\omega}{3c}\theta^3\right). \tag{2.12}$$

In this expression,

$$L_\omega = \frac{L/2}{1-(\omega/\omega_{pm})^2}, \quad \beta = \frac{v}{c}, \quad \sigma = \frac{\omega a(\sin\theta_0 - \beta\sin\theta)}{v\cos\theta_0},$$

θ is the angle of emission of radiation into the vacuum, θ_m is a limiting angle of the order of $(3c/4\omega L_\omega)^{1/2}$, and the inhomogeneity length L_ω is evaluated at the plasma resonance point $\omega_{pe} = \omega$, using (2.2). It is clear from (2.12) that, for given total current, which corresponds to $j_0 a = \text{const}$, the generation optimum in the region of the plasma resonance is determined by the conditions

$$\theta_0 \leqslant \beta\theta_m, \quad \omega a\theta_m/c \leqslant 1,$$

where the main part of the radiation is localized within the cone $\theta \sim \theta_m$. When $\theta \lesssim \beta\theta_m$ but $\omega a\theta_m/c \gg 1$, the directivity of the emission is improved:

$$\text{for } \theta = \theta_0/\beta, \quad \Delta\theta/\theta \sim c/a\omega \ll 1,$$

but the intensity is reduced (in proportion to $c/\omega a$). Therefore, for frequencies $\omega < \omega_{pm}$, the transition radiation has an anisotropic angular energy distribution, and is also anisotropic relative to the direction of variation of the plasma density in the system attached to the moving charge.

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APPENDIX

Let us consider in greater detail the reason for the difference between the results reported here and those given by Ginzburg and Tsytovich. [5,6] In the present paper, we use a somewhat different method for the preparation of states in the nonstationary plasma. In particular, the ionization of an individual atom and the recombination of an individual plasma particle occur rapidly, so that the interaction between the field due to the moving charge and the plasma particles during the ionization and recombination time can be neglected. The oscillator velocity of the particles in the decaying plasma is then given by

$$v_k(t) = -\frac{e}{m}\int_{-\infty}^{t} dt' E_k(t'). \tag{A.1}$$

At the same time, while the plasma density is increased, we have the following expression for particles "created" at time t_0:

$$v_k(t) = -\frac{e}{m}\int_{t_0}^{t} dt' E_k(t'). \tag{A.2}$$

In contrast to the results in [5,6], the difference between (A.1) and (A.2) leads to a time dependence of the transition-radiation intensity as a function of the direction of variation of the plasma density, which is important in the nonrelativistic case. [7] As the density is reduced, the transition radiation in nonstationary plasma has a dipole character, i.e., $W_{\omega,\Omega} \sim v^2$, whereas, when the plasma density increases, it has the quadrupole character, so that $W_{\omega,\Omega} \sim v^4$.

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1) We note additionally that, as shown in [8], radiation can be emitted as a result of a time variation in charge density even in the case of a charge moving uniformly in a homogeneous stationary medium.

2) It is important to note that transition radiation was investigated in [5-7] for a rapid variation of permittivity with time. On the other hand, under the usual conditions obtaining in decaying laboratory plasma, the characteristic time τ of the variation in plasma density due to volume recombination is several orders greater than the time t_f for the generation of the transition radiation, so that the density-jump approximation is invalid under these conditions.

3) See [6] for further analysis of (1.5); the force on the charge and the energy associated with the macroscopic renormalization of its mass are also calculated in [6].

4) Hence, it follows that the inclusion in (2.1) of the term $(d\ln\epsilon/dz)\, dH_\omega/dz$ which was neglected in [14] is, in fact, fundamental. On the real axis of z, this term is, of course, small but, in the present problem, the radiation field is determined by the contribution of points located in the complex plane of z, where it is not at all small.

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