Weak Langmuir turbulence of an isothermal plasma

V. E. Zakharov, S. L. Musher, and A. M. Rubenchik

Institute of Automation and Electric Measurements, Siberian Division, USSR Academy of Sciences
(Submitted November 15, 1974)
Zh. Eksp. Teor. Fiz. 69, 155–168 (July 1975)

The fine structure of weak Langmuir turbulence spectra produced by induced scattering by ions is studied. The spectrum is found to be stationary only at a small turbulence level ($W/nT < (kr_D/N)(M/m)^{1/2}$, where N is the number of particles in the Debye sphere). Otherwise the spectrum is nonstationary and consists of solitons moving in k-space and absorbed in the small-wave-number region via a Langmuir-collapse mechanism. Depending on the excitation conditions, the spectrum can be regular or can consist of discrete harmonics. The analytic deductions are confirmed by numerical calculations.

PACS numbers: 52.35.Js

INTRODUCTION

The study of Langmuir turbulence, namely a plasma state characterized by a high level of excited Langmuir waves, is one of the principal problems of plasma physics (see[1,2]). The reason is that in many important methods of plasma heating, for example by a relativistic beam or a laser pulse, the energy is initially transferred to the Lagnmuir waves, and only then is it converted into heat. At the same time, the physics of Langmuir turbulence of the plasma involves many various phenomena the study of which is of considerable general interest.

The present paper is devoted to the study of Langmuir turbulence in the case when the main mechanism of the nonlinear interaction of the Langmuir oscillations is their scattering by plasma ions. This mechanism plays the decisive role in an isothermal plasma at not too large turbulence intensities (see[1-4]).

Recent numerical experiments[4,5] have shown that the evolution of weakly turbulent spectra due to induced scattering by ions do not correspond to the traditional concepts of relaxation within the framework of the kinetic equation with a collision term. The turbulence spectra turned out to be singular, concentrated in **k**-space, on surfaces, lines, or even in individual points. In many cases the spectra turn out to be essentially nonstationary. The theory of singular spectra concentrated on lines and surfaces (jet theory) was developed in[4].

In the present paper we show that the one-dimensional jets in **k**-space, which are produced in most methods of Langmuir-wave excitation, have as a rule a fine structure in the form of intensity oscillations. If the instability increment is spectrally narrow, as is the case in parametric excitation of the waves, then the jets break up into chains of discrete peaks. Localized perturbations—solitons—can propagate both along the jet and along the chain of peaks. At not too small turbulence intensities, no stationary spectrum is produced, and the solitons are periodically detached from the instability region. In this case, the energy absorption in the plasma has an oscillatory character.

1. FUNDAMENTAL EQUATIONS

The kinetic equation describing the evolution of the spectrum of the Langmuir turbulence produced in the plasma by induced scattering by ions is of the form[1,2,4]

$$\frac{\partial n_{\mathbf{k}}}{\partial t}=2\left(\gamma_{\mathbf{k}}+\int T_{\mathbf{kk}'}n_{\mathbf{k}'}d\mathbf{k}'\right)n_{\mathbf{k}}+f_{\mathbf{k}},$$

$$T_{\mathbf{kk}'}=-T_{\mathbf{k}'\mathbf{k}}=\frac{\omega_p^2}{2n_0T}\frac{(\mathbf{kk}')^2}{k^2k'^2}\operatorname{Im}G(\mathbf{k}-\mathbf{k}',\omega_{\mathbf{k}}-\omega_{\mathbf{k}'}),$$
$$G(\mathbf{k},\omega)=\frac{L(\mathbf{k},\omega)}{1-L(\mathbf{k},\omega)},\quad L(\mathbf{k},\omega)=\frac{T_e}{Mn_0}\int\left(\mathbf{k}\frac{\partial f_0(\mathbf{v})}{\partial\mathbf{v}}\right)\frac{d\mathbf{v}}{\mathbf{kv}-\omega}. \quad (1)$$

Here $f_0(\mathbf{v})$ is the ion distribution function; γ_k includes the linear damping and the growth increment of the Langmuir waves; the term f_k takes into account the influence of the thermal noise and of four-plasmon scattering, while ω_k describes the law of dispersion of the Langmuir waves. We shall henceforth assume for simplicity that $f_k = f = \text{const}$ and $\gamma_k = -\gamma_0$ outside the finite region of **k**-space (γ_0 is the collision damping). Then as $|\mathbf{k}| \to \infty$ we have $n_k \to n_0 = f/2\gamma_0$.

In the derivation of (1), the ion-sound oscillations were assumed to be "static," namely it was assumed that the time of the nonlinear processes greatly exceeds the damping time of the ion sound. This is correct if[6]

$$\omega_p\int n_{\mathbf{k}}\,d\mathbf{k}/nT\ll kr_D(m/M)^{1/2}(\gamma_s/\omega_s)^2. \quad (2)$$

Here γ_s and ω_s are the damping and the frequency of the ion sound.

Equation (1) is valid also for a plasma in a magnetic field, if $\omega_H/\omega_p \ll 1$. In this case the magnetic field can be taken into account only in the wave dispersion law ω_k:

$$\omega_{\mathbf{k}}=\omega_p\left[1+\frac{3}{2}(kr_D)^2+\frac{1}{2}\frac{\omega_H^2}{\omega_p^2}\frac{k_\perp^2}{k^2}\left(1-\frac{\omega_p^2}{k^2c^2}\right)\right]. \quad (3)$$

Equation (1) can have a stationary solution $n_0(k) > 0$ for which

$$2n_0(\mathbf{k})\left(\gamma_{\mathbf{k}}+\int T_{\mathbf{kk}'}n_0(\mathbf{k}')d\mathbf{k}'\right)+f_{\mathbf{k}}=0. \quad (4)$$

We put

$$n(\mathbf{k},t)=n_0(\mathbf{k})+\delta n(\mathbf{k},t)+\ldots,\quad \delta n_{\mathbf{k}}/n_0(\mathbf{k})\ll 1.$$

From the linearized equation (1) follows the relation[7]

$$\frac{d}{dt}\int\frac{\delta n^2(\mathbf{k},t)}{n_0(\mathbf{k})}d\mathbf{k}+2f\int\frac{\delta n^2(\mathbf{k})}{n_0^2(\mathbf{k})}d\mathbf{k}=0. \quad (5)$$

which shows that $\int d\mathbf{k}\,\delta n^2(\mathbf{k}, t)/n_0(\mathbf{k})$ relaxes with a reciprocal time $2f/n_0 \sim \gamma_0$.

At $f_k = 0$, the relaxation time is infinite. This unexpected fact shows that the stationary solutions of (1), if they do exist, have a small stability margin. A numerical calculation carried out within the framework of jet theory has shown[4] that the spectra of the Langmuir turbulence are quasi-one-dimensional in a number of important cases. This fact can be simply explained. As seen from (1), the most strongly interacting are waves with parallel **k** and **k**'. Therefore, if initially the exciting spectrum of the oscillations is narrow with respect to angle, then this property is preserved also with time. In addition, many important mechanisms of nonlinear damping of waves, primarily the process of coalescence of two Langmuir waves into one electromagnetic wave, do not exist for one-dimensional spectra, so that additional absorption of the waves that disturb the one-dimensionality takes place. The same property is possessed also by induced scattering by electrons. In addition, a strong factor that leads to one-dimensionality is the magnetic field. As seen from (4), at $\omega_H/\omega_p \sim kr_D$ the corrections to the dispersion law, necessitated by the magnetic field, become equal to the thermal corrections, and at $1 \gg \omega_H/\omega_p \gg kr_D \gg v_T/c$ the entire kinetics of the Langmuir waves becomes one-dimensional.

We present the explicit form of the one-dimensional equations:

$$\frac{\partial n(k)}{\partial t}=2\left(\gamma_k+\int_{-\infty}^{\infty}T(k+k')n(k')dk'\right)n(k)+f_k,$$
$$T(q)=\frac{\omega_p^2}{2n_0T}\operatorname{Im}\left\{L\left(\frac{3}{2}\sqrt{\frac{M}{m}}r_Dq\right)\Big/\left[1-L\left(\frac{3}{2}\sqrt{\frac{M}{m}}r_Dq\right)\right]\right\}, \tag{6}$$

where $L(\xi)$ is the function $L(\mathbf{k}, \omega)$ from (1) for the one-dimensional case. The function P(q) differs from zero if $k + k' \approx k_{diff} \sim (m/M)^{1/2}/r_D$. Therefore at $k \gg k_{diff}$ only waves with opposite signs of k interact. If furthermore the spectrum is symmetrical, i.e., $n(k) = n(-k)$, then Eq. (6) takes the simpler form

$$\frac{\partial n(k)}{\partial t}=2\left(\gamma_k+\int_{-\infty}^{\infty}T(k-k')n(k')dk'\right)n(k)+f_k. \tag{6'}$$

Another important approximation is the "peak kinetics" approximation. Assume that initially the waves lie in a small region of k-space near $\mathbf{k} \sim \mathbf{k}_0 \gg k_{diff}$. An analysis of the matrix element shows that the waves interacting most strongly with the initial ones are those located along the vector $\mathbf{k}_0$ near the points $k_0 \pm k_{diff}$. In this case the waves in the vicinity of the point $-k_0 + k_{diff}$ will grow, and those in the vicinity of the point $\pm k_0 - k_{diff}$ will attenuate. If the initial noise level is small, then a peak with width $\Delta k \ll k_{diff}$ s produced near the point $-k_0 + k_{diff}$. Repeating this reasoning, we arrive at the conclusion that the spectrum in **k**-space will comprise after a certain time a linear sequence of peaks located near the points $\pm(k_0 - nk_{diff})$.

Assuming that the intensities of all the peaks differ from the noise, we represent the distribution n_k in the form

$$n_k=\sum_{n=0}^{n_1}N_n\delta(k_0-nk_{diff})+\sum_{n=0}^{n_2}M_n\delta(-k_0+nk_{diff}).$$

Eq. (1) now takes the form

$$\partial N_n/\partial t=N_n[\gamma_n+T(M_{n-1}-M_{n+1})],$$
$$\partial M_n/\partial t=M_n[\gamma_n+T(N_{n-1}-N_{n+1})], \tag{7}$$

where T is the value of $T_{kk'}$ with the largest absolute value. In the symmetrical case we have $M_n = N_n$ and (7) reduces to the single equation

$$\partial N_n/\partial t=N_n[\gamma_n+T(N_{n-1}-N_{n+1})]. \tag{8}$$

Equations (7) and (8) describe the kinetics of the peaks; the actual peak widths $\Delta k \ll k_{diff}$ do not enter in these equations. Nonetheless, these peaks must not be too narrow $(\Delta k r_D)^2 \gg W/nT_e$, for otherwise the quasi-monochromatic peaks will experience automodulation instability.

2. STATIONARY REGIMES

In the analysis of the solutions of (4) it is easy to establish the possibility of realizing, depending on the form of γ_k, two principally different regimes—singular and regular. We rewrite (4) in the form

$$\gamma_k+\int_{-\infty}^{\infty}T(k-k')n_{k'}dk'=-f_k/2n_k. \tag{9}$$

Assuming the thermal noise to be small, $f_k \to 0$, we simplify (9) to

$$\gamma_k+\int_{-\infty}^{\infty}T(k-k')n_{k'}dk'=0, \tag{10}$$

which is legitimate if n_k does not vanish.

Equation (10) is a Fredholm equation of the first kind, and its solution is extremely sensitive to the detailed structure of the functions γ_k and T(k) and may not exist at all. We consider one particular example. We choose γ_k and T(k) in the form

$$\gamma_k=\gamma_0\frac{d}{dk}\exp\left\{-\frac{(k-k_0)^2}{\delta_1^2}\right\},\quad T(k)=-T_0\frac{d}{dk}\exp\left\{-\frac{k^2}{\delta_2^2}\right\}.$$

Solving (10) by the Fourier method, we obtain

$$n_k=\frac{\gamma_0}{2\sqrt{\pi}T_0}\frac{\delta_2}{\delta_1}\frac{1}{(\delta_1^2-\delta_2^2)^{1/2}}\exp\left\{-\frac{(k-k_0)^2}{\delta_1^2-\delta_2^2}\right\}. \tag{11}$$

The solution (11) is meaningful only at $\delta_1 > \delta_2$, i.e., in the case when the width of the increment γ_k in **k**-space exceeds the characteristic scale of the interaction T(k). Formula (11) describes the peak in k-space concentrated near $k = k_0$ and having a width $(\delta_1^2 - \delta_2^2)^{1/2}$. At $\delta_1 > \delta_2$ we have the regular regime. As $\delta_1 \to \delta_2$ we get $n_k \to \gamma_0\delta(k - k_0)/T_0$ and a transition takes place to the singular regime in the form of a single δ-like peak.

At $\delta_1 < \delta_2$ we can no longer use Eq. (10). In this case the distribution consists of several δ-like peaks (see also [7,8]).

Let us examine the regular regime in greater detail. By virtue of the anti-symmetry of the kernel T(k), the following relations hold:

$$T(\lambda=0)=\int_{-\infty}^{\infty}T(k)dk=0,\quad T(\lambda)=\frac{1}{2\pi}\int_{-\infty}^{\infty}T(k)e^{i\lambda k}dk.$$

For a regular solution (10) to exist, we must stipulate

$$\gamma(\lambda=0)=\int_{-\infty}^{\infty}\gamma(k)dk=0,\quad \gamma(\lambda)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\gamma_ke^{i\lambda k}dk. \tag{12}$$

This condition was satisfied in the case considered in Sec. 2. Generally speaking, in a real situation this is not so. Actually, the distribution n_k differs from zero only in a certain interval $k_{min} < k < k_{max}$. Therefore Eq. (10) must be replaced by an equation of the Wiener-Hopf type

$$\gamma_k+\int_{k_{min}}^{k_{max}}T(k-k')n(k')dk'=0 \tag{13}$$

with the additional conditions

$$n(k_{min})=n(k_{max})=0$$

and the requirement $n(k) > 0$ at $k_{min} < k < k_{max}$. If the condition of the regular approximation is satisfied with a margin and if the characteristic scale of the variation of γ_k is large ($\Delta k \gg k_{diff}$), then Eq. (13) can be simplified to the form

$$\gamma_k + q\frac{\partial n_k}{\partial k} = 0, \quad q = \int_{-\infty}^{\infty} T(k)\,k\,dk, \tag{14}$$

which corresponds to the differential approximation. In this case

$$n_k = -\frac{1}{q}\int_{k_{min}}^{k_{max}} \gamma_k\,dk. \tag{15}$$

The point k_{max} corresponds to the upper instability limit $\gamma(k_{max}) = 0$. Obviously, the point k_{min} is now determined from the relation

$$\int_{k_{min}}^{k_{max}} \gamma_k\,dk = 0. \tag{16}$$

The entire reasoning given above is valid provided only that the homogeneous equation

$$\int_{k_{min}}^{k_{max}} T(k-k')\,n_{k'}\,dk' = 0 \tag{17}$$

has no nontrivial solutions. In the case when the differential approximation is valid, this means that the function $T(\lambda)$ must have no zeroes. If such zeroes do exist, $T(\pm\lambda_n) = 0$, it is possible to add to the solution oscillating terms

$$n_k = n_{0k} + \sum_n \left(A_n e^{i\lambda_n k} + A_n^* e^{-i\lambda_n k}\right), \tag{18}$$

where n_{0k} is given by (15). To calculate the constants A_n, we multiply (14) by $\exp\{\pm i\lambda_n k\}$ and integrate with respect to k. As a result we obtain a set of integral conditions

$$\int_{k_{min}}^{k_{max}} dk \exp\{\pm i\lambda_n k\}\left\{\gamma_k + 2f_k\left[n_{0k} + \sum_n \left(A_n \exp\{i\lambda_n k\} + A_n^* \exp\{-i\lambda_n k\}\right)\right]^{-1}\right\} = 0, \tag{19}$$

which determine A_n and A_n^*. In spite of the smallness of f_k, these quantities may turn out to be comparable with n_k. In the presence of additional zeroes of $T(\lambda)$, the "jet" designed by (18) has an additional oscillatory structure. Analyzing expression (6) for $T(\xi)$, we can show that the function $T(\lambda)$ has no additional zeroes in a plasma with hot ions, at $T_i \gg T_e$. A zero arises at $T_i/T_e \sim 1$; the exact value of this ratio depends on the detailed structure of the ions distribution function. At $T_i \ll T_e$ there are always additional zeroes, and the jet has an oscillatory structure.

For the usual arrangement of the increments $\gamma(0) < 0$, the condition (12) can be satisfied only if

$$\int_0^{\gamma_{max}} \gamma_k\,dk < 0.$$

This leads to the condition for the existence of the above-described stationary regime in the form of an inequality for the instability increment:

$$\gamma_{inst}/\gamma_0 < k_0/\Delta k. \tag{20}$$

If the inequality (20) is not satisfied, we get an accumulation of the Langmuir energy in a state with k = 0, meaning the formation of a Langmuir condensate. As soon as the condensate greatly exceeds the level of the thermal noise, it becomes unstable and this leads to the appearance of Langmuir collapse and to establishment of strong turbulence in the region of small wave numbers.

Thus, condition (20) is simultaneously the condition for the realization of a "purely weakly-turbulent" regime. When the absolute condition is satisfied, strong turbulence takes place in the range of small numbers and weak turbulence in the range of large wave numbers. The latter is now essentially nonstationary.

3. NONSTATIONARY REGIMES

Singular and regular regimes are possible also in the nonstationary case. We consider first the regular regime. Assume that initially there are in **k**-space region where we can neglect the damping and the thermal noise. Equation (1) takes in this region the form

$$\frac{\partial n_k}{\partial t} = 2n_k \int_{-\infty} T(k-k')\,n_{k'}\,dk'. \tag{21}$$

We consider solutions of (21) in the form of waves traveling with constant velocity v towards the smaller wave numbers. We note that the kernel in (21) can be represented in the form

$$T(k-k') = \frac{\partial}{\partial k} S(k-k'), \quad \int_{-\infty}^{\infty} S(k)\,dk = q, \tag{22}$$

where $S(\xi)$ is a positive function that decreases as $|\xi| \to \infty$. We assume that $n_k \to n_0$ as $|k| \to \infty$. Making the substitution $\partial/\partial t \to v\partial/\partial k$, we integrate (22) ($x = n/n_0 - 1$):

$$\ln(1+x_k) = \frac{2n_0}{v}\int_{-\infty}^{\infty} S(k-k')\,x_{k'}\,dk'. \tag{23}$$

If $x_{max} \ll 1$, the characteristic dimension of the solution in **k**-space is $\Delta k \gg k_{diff}$, then

$$S(k-k') = q[\delta(k-k') + \alpha k_{diff}^2 \delta''(k-k')].$$

Here α is a dimensionless parameter. Equation (23) now takes the form

$$\frac{\alpha\beta}{1-\beta} k_{diff}^2 \frac{d^2x}{dk^2} - x + \frac{x^2}{2(1-\beta)} = 0, \quad \beta = \frac{2qn_0}{v}. \tag{24}$$

Equation (24) has at $0 < \beta < 1$ a solution that decreases on both sides

$$x_k = 3(1-\beta)\,\mathrm{ch}^{-2}\left(\frac{1-\beta}{4\alpha\beta}\right)^{1/2} \frac{k}{k_{diff}}. \tag{25}$$

The solution (25) is a solitary wave—a soliton—and is valid at $1 - \beta \ll 1$. In the same approximation, the nonstationary equation (21) reduces to the well known Korteweg—de Vries equation

$$\frac{1}{v}\frac{\partial x_k}{\partial t} = (1-\beta)\frac{\partial x_k}{\partial k} - \beta x_k \frac{\partial x_k}{\partial k} - \alpha\beta k_{diff}^2 \frac{\partial^3 x_k}{\partial k^3}.$$

The characteristic scale of the soliton $k_{diff}[4\alpha\beta/(1-\beta)]^{1/2} \gg k_{diff}$ decreases with decreasing β. We consider Eq. (23) in the limiting case as $\beta \to 0$. The characteristic scale of the soliton should now be small in comparison with k_{diff}, and Eq. (23) can be simplified to

$$x_k = \exp(\beta \tilde{S}(k) N) - 1.$$

Here

$$\tilde{S}(k) = \frac{k_{diff}}{q} S(k) \sim 1, \quad N = \frac{1}{k_{diff}}\int_{-\infty}^{\infty} x_k\,dk.$$

The dimensionless parameter N is the ratio of the number of quasiparticles in the soliton to the number of "background" particles over the dimension k_{diff}.

To determine N it is necessary to solve the transcendental equation

$$N = \frac{1}{k_{diff}}\int_{-\infty}^{\infty} [\exp(\beta\tilde{S}(k)N) - 1]\,dk. \tag{26}$$

Using the narrowness of the soliton, we can put in (26)

$$S(k)=S_0(1-k^2/k_0^2), \quad (27)$$

where $k_0 \sim k_{diff}$. Calculating the integral, we obtain

$$N\approx\frac{1}{(2\pi\beta N)^{1/2}}\frac{k_0}{k_{diff}}e^{\beta N},$$

from which it follows, with logarithmic accuracy, that $N \sim \beta^{-1}\ln\beta^{-1}$ as $\beta \to 0$, and the characteristic scale of the soliton is

$$\Delta k\sim k_{diff}\ln^{-1/2}(1/\beta)<k_{diff}.$$

Since the real parameter of this approximation is the quantity $\ln^{-1/2}(1/\beta)$, it is valid only at very small β. Nonetheless, a comparison of the two limiting cases allows us to assume that the solution of the soliton type exists in the entire interval $0 < \beta < 1$. Then in practically the entire interval, with the exception of the vicinity of its ends, the soliton dimension is $\Delta k \sim k_{diff}$. Inasmuch as $\beta \sim 1/v$, we can state that the soliton dimension depends little on the velocity. However, the soliton intensity does depend on the velocity in an essential manner. There exists a minimal velocity $v_0 = 2qn_0$; as $v \to v_0$ we have $N \sim (v/v_0 - 1)^{1/2}$, and at $v \gg v_0$ we get $N \sim (v/v_0)\ln(v/v_0)$. Expressing the velocity in terms of the maximum soliton amplitude and the noise amplitude, we obtain

$$v\sim 2q\frac{n_{max}}{\ln(n_{max}/n_0)}.$$

Actually the dependence of the velocity on the thermal noise is weaker, since n_{max} (if $n_{max} \gg n_0$) is also proportional to $\ln n_0$. Therefore in fact the velocity is determined only by the parameters of the increment: $v \sim \gamma_{inst}k$. The soliton can manage to become attenuated by the collisions before reaching the region of small k, if $\gamma_0 k_0 > v$, which coincides with the criterion for the existence of a stationary solution of (21). When the soliton is damped, it is slowed down and ultimately the soliton is stopped. Within the framework of the Korteweg–de Vries equation, the solitons are repelled (see, e.g., [9]); this probably takes place also within the framework of the more exact equation (22). The establishment of the stationary state can therefore be represented as a result of slowing down of solitons.

In the opposite case $\gamma_{inst} > \gamma_0 k_0/\Delta k$, the soliton does not have time to slow down and is absorbed only in the collapse region. Then there are no grounds for expecting a stationary state to be established; this conclusion is confirmed (see below) by results of numerical experiments. In the case of a narrow instability increment and when a condition inverse to (21) is satisfied, a nonstationary singular regime described by Eq. (10) or (11) is realized.

It was shown in[10] that Eq. (8) has at $\gamma_{inst} = 0$ an exact solution in the form of a soliton that travels along a chain of peaks:

$$N_n(t)=F(Tt-n/s-\tau_0), \quad (28)$$

where

$$F(\xi)=N_0\left(1+\frac{a}{1-b+b\,\mathrm{ch}\,\delta\xi}\right).$$

The quantities δ, b, and s are obtained from the equations

$$\delta[(1-b)^2+b^2\mathrm{sh}^2(\delta/s)]=2N_0(1-b)(1-b+a)\,\mathrm{sh}(\delta/s),$$
$$\delta(1-b)=N_0(2-2b+a)\,\mathrm{sh}(\delta/s),\quad \delta=2N_0\,\mathrm{sh}(\delta/s).$$

At $a \gg 1$ we have approximately

$$\delta=N_0 a,\quad b^2=1/2a,\quad s=\delta/\ln a.$$

The soliton has a velocity

$$v=TN_0ak_{diff}/\ln a,\quad N_0\sim n_0k_{diff}.$$

We note that, as shown by Manakov,[11] Eq. (8) with $\gamma_{inst} = 0$ is a perfectly integrable dynamic system, within the framework of which one can obtain exact formulas describing soliton conditions. It is shown by the same token that the solitons are repelled from one another. The same (at $\gamma = 0$) is true of the system (7). At $\gamma_{inst} \gg \gamma_0 k/k_{diff}$, the nonstationary regime constitutes a successive detachment of the solitons from the instability region.

Let us estimate the parameters of this process. To this end we consider the interaction of two peaks,

$$\partial N_1/\partial t=N_1(\gamma_{inst}-TN_2),\quad \partial N_2/\partial t=TN_2N_1, \quad (29)$$

of which the first is in the instability region and the second is at a distance k_{diff} from this region. At t = 0 we have $N_1 = N_2 = N_0$. The system (29) has an integral

$$N_1+N_2-2N_0=\gamma_{inst}N_2/TN_0.$$

The maximum amplitude is reached when the intensities of both peaks are comparable; at $N_{1,2}/N_0 \gg 1$ we obtain with logarithmic accuracy

$$N_{max}\sim\frac{1}{2}\frac{\gamma_{inst}}{T}\ln\frac{\gamma_{inst}}{2TN_0}. \quad (30)$$

The characteristic time of the process is

$$\tau\sim\gamma_{inst}^{-1}\ln(\gamma_{inst}/2TN_0).$$

The average energy flux to the plasma

$$N_{max}/\tau\sim\gamma_{inst}^2\omega_p/2T\sim\gamma_{inst}^2 nT_e/\omega_p \quad (31)$$

does not depend on the thermal-noise level. Strictly speaking, formula (30) contains also factors of the type $\ln\ln(\gamma_{inst}/2TF_0)$, which are set equal to unity in order of magnitude.

When the plasma is heated by a high-frequency field or by relativistic electron beams, the condition (20) is easily violated. In parametric heating the inertia interval is usually quite large ($k_0/k_{diff} \sim 10$ in experiments with hydrogen plasma; for the ionosphere $k/k_{diff} \sim 15-20$), and the instability increment

$$\gamma(k)=\frac{\omega_p^2}{2n_0T}\,\mathrm{Im}\,G\left(\frac{\omega_0-\omega_k}{kv_{Ti}}\right),$$

where G is given by (1), has half the width of the diffusion interval. Therefore under parametric instability a singular regime is always realized, and is stationary or nonstationary, depending on the excess over the threshold.

When the plasma is heated by a relativistic electron beam, the instability increment is maximal on the line $\omega_p = k_z c$ (the z axis is directed along the beam axis)[4]:

$$\gamma_{max}\approx\omega_p\frac{n'}{n}\frac{mc^2}{E}\frac{1}{(\Delta\theta)^2}\frac{\omega_p^2}{k^2c^2},$$

where n′ is the density, E is the energy, and $\Delta\theta$ is the angle scatter of the beam electrons. The bulk of the energy is released at $k \sim \omega_p/c$, and the inertia interval is small even in a thermonuclear plasma. The characteristic width of the increment has a modulus $k \sim \omega_p(\Delta\theta)^2/c$. Therefore a regular regime of the turbulence spectra is possible only for beams with very large angle scatter:

$$(\Delta\theta)^2>(m/M)^{1/2}c/vT.$$

In the opposite case, a regular distribution of the oscillations takes place near the line of the resonance of the waves with the beam particles $\omega/k_z \sim c$, and a singular spectrum is obtained in the region $k < \omega_p/c$.

Owing to the relative smallness of k_0/k_{diff}, the turbulence is as a rule nonstationary in the case of beam heating. The presence of oscillations, in the region of large angles, on the resonance line $\omega/k_z = c$ does not make it possible to apply to it the one-dimensional model, and a complete investigation of this case is possible only with the aid of a computer.

4. ROLE OF LAGNMUIR COLLAPSE

As already mentioned, in those cases when there is a nonstationary regime and the Langmuir plasmons do not have time to be damped by the collisions, condensation of the plasmons takes place in the states with k = 0, followed by development of collapse.

Two collapse regimes are possible (see [6]). If the intensity of the condensate is $W/nT_e < m/M$, then a subsonic collapse develops. The effective reciprocal time of the condensate damping due to the collapse is

$$\gamma_{eff} \approx \omega_p (M/m)^{1/2} (W/nT_e)^{3/2}.$$

Comparing the energy flux into the plasma (31) with the energy dissipation in the collapse, we obtain

$$W/nT_e \sim (\gamma_{inst}/\omega_p)^{2/3} (m/M)^{1/3}. \qquad (32)$$

This regime takes place if

$$\gamma_{inst}/\omega_p < m/M. \qquad (33)$$

The characteristic width δk of the collapsing condensate in **k**-space is determined from the condition

$$W/nT_e \sim (\delta k r_D)^2$$

and is of the order of

$$\delta k \sim k_D (\gamma_{inst}/\omega_p)^{2/3} (m/M)^{1/10} < k_{diff}.$$

In this regime, the collapse has no significant influence on the weak-turbulence region $k_{diff} < k < k_0$.

At $W/nT_e > m/M$, the collapse is supersonic. The reciprocal time of its development now takes the form

$$\gamma_{eff} \sim \omega_p (m/M)^{1/2} (\gamma/\omega_p)^{4/3}$$

For the intensity of the condensate we now have

$$W/nT \sim (M/m)^{1/2} (\gamma_{inst}/\omega_p)^{4/3},$$

and for its spectral width

$$\delta k \sim k_D (M/m)^{1/4} (\gamma_{inst}/\omega_p)^{2/3} > k_{diff}.$$

In this case the collapse zone is bounded from below by the region of weak turbulence.

If $\gamma_{inst}/\omega_p > k r_D (m/M)^{1/2}$, then the turbulence is strong and its character is independent of the temperature ratio in the plasma.

5. NUMERICAL EXPERIMENT

To check on the representations developed above concerning the structure of the Langmuir-turbulence spectra, we performed a series of numerical experiments. Specifically, we considered the equation

$$\partial N_\varkappa/\partial\tau = N_\varkappa\left(\gamma_\varkappa' + \int G(\varkappa,\varkappa') N_{\varkappa'} d\varkappa'\right) + f,$$

where κ, τ, N, γ, G(κ, κ'), and f are dimensionless quantities defined by the relations

$$G(\varkappa,\varkappa') = T_{kk'}/T_{max}, \quad \varkappa = 2k/3k_{diff}; \quad \tau = \gamma_{max} t;$$
$$\gamma' = \gamma_k/\gamma_{max}; \quad N_\varkappa = n_\varkappa T_{max}/\gamma_{max}.$$

In typical variants, the number of points in modulo **k** was 200 (one diffusion interval spanned 15 points), and the noise level was 10^{-2}—10^{-4}. For the increment we chose the model expression

$$\gamma_k = \gamma_0 \exp[-(k-k_0)^2/\delta^2] - 1.$$

In the first stage of experiments, δ was chosen to be much larger than k_{diff}—a "broad" increment. At large excesses above threshold ($\gamma_0 \gg 1$), a periodic detachment of the soliton takes place (see (28)), with a characteristic width on the order of k_{diff} (see Fig. 1). At small excesses above threshold, a stationary regular distribution is established. The character of the establishment coincides with that described in Sec. 3.

Figure 2 shows stationary spectra for different δ. It is seen that the envelope of these distributions is described sufficiently well by a differential approximation, and the distribution itself is deeply modulated in accordance with the results of Sec. 2.

2. In the case of a narrow increment ($\delta < k_{diff}$), the distribution of the oscillations in modulo **k** is practically independent of the exact value of δ and has a singular character. We present therefore the results of calculations only for parametric instability. Figure 3 shows the motion of the soliton (28) over a chain of peaks in the region of small k. Figure 4 shows the stationary distribution of the oscillations at $\gamma_{inst}/\gamma_0 = 4.37$. The width of the peaks is determined by the noise level and we can obtain from (1)

$$\Delta k/k_{diff} \sim (n_0/n_{max})^{1/2}.$$

The one-dimensional model investigated by us in this paper was based essentially on the stability of the "jets." This result was obtained in [4] for smooth distributions. It can be shown that the singular spectra are unstable with respect to oscillations propagating at large angles to the jet. Therefore, in order to ascertain the limits of applicability of the one-dimensional approximation, a number of numerical experiments were performed, in which the two-dimensional equation (1) was solved in an exact formulation. In typical variants, the number of points was 100—110 in modulo **k**, and 32 with respect to

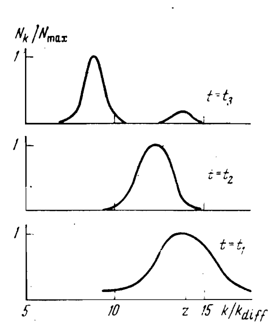

FIG. 1

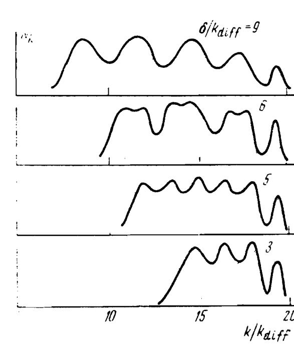

FIG. 2

FIG. 1. Distribution of N_k for the case of a "broad" increment in the one-dimensional model for an infinite excess over the instability threshold at three successive instants of time; Z is the point of the maximum increment.

FIG. 2. Distribution of N_k at small excesses over the instability thresholds for increments with different widths δ/k_{diff}.

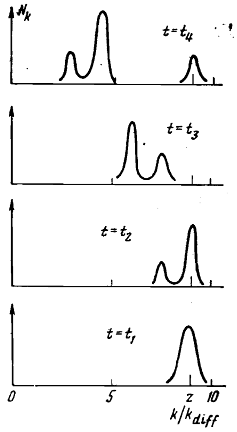

FIG. 3

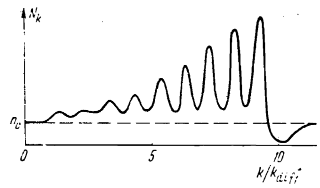

FIG. 4

FIG. 3. Distribution of N_k for the case of parametric instability in the one-dimensional model for large excesses above threshold and several successive instants of time.

FIG. 4. Stationary distribution of N_k at $\gamma_0 t = 100$ for parametric instability; the excess over threshold is 4.37.

angle. The Frank-Nicholson difference scheme of second order of accuracy in time was applied to Eq. (1). In the integration with respect to angle, we used quadratures of appropriate order of accuracy, which took into account the singularity of the spectra.

In the first series of experiments, we studied the parametric instability (33). It is seen from the results (see Fig. 5) that the one-dimensional approximation describes satisfactorily the real situation, both qualitatively and quantitatively.

The distribution of the oscillations excited by a relativistic electron beam with an angle spread $\theta = 15°$ at $\gamma_{inst} \gg \gamma_0$ is shown in Fig. 6. We see that in the inertia region, $k < \omega_p/c$, there are two "narrow jets" that are well described by the one-dimensional model. The width of the jets depends little on the angle spread of the particles in the beam and is much narrower than in the parametric excitation of the waves ($\theta \sim 10°$). On the other hand, the distribution of the oscillations at $k > \omega_p c$ is quite complicated.

We present plots of the energy flux Q_k as a function of the wave vector:

$$Q_k = \omega_p \int_{-1}^{1} \gamma_k N_k \, d\cos\theta \qquad (34)$$

—see Fig. 7. The sharp maxima at distances $\sim k_{diff}$ along the jet ($k\cos\theta \sim \omega_p/c$) show that in spite of the smooth variation of the oscillation amplitude the width of the jet pulsates strongly.

CONCLUSION

As seen from all the foregoing, the main Langmuir turbulence properties governed by the induced scattering by the ions are determined primarily by the anti-symmetry of the kernel $T_{kk'}$; an important role is played by the condition $k_0 \gg k_{diff}$. Equation (1) with anti-symmetrical kernel $T_{kk'}$ describes, besides the Langmuir turbulence, a large class of plasma-turbulence problems, primarily the problem of electromagnetic-wave energy transport on account of scattering by electrons, to which a number of papers have been devoted recently[3,12],

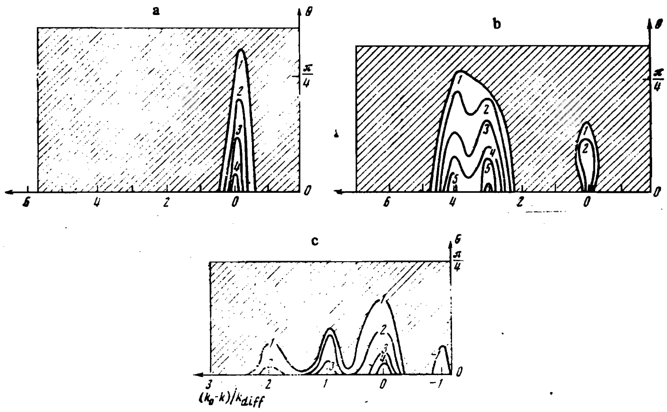

FIG. 5. Level lines of the function $\ln[N(k, \cos\theta)/N_0]$ for parametric excitation of waves: a, b—at large excesses above threshold at two successive instants of time; c—the same level lines for a twofold excess above the instability threshold ($\gamma_0 t = 60$).

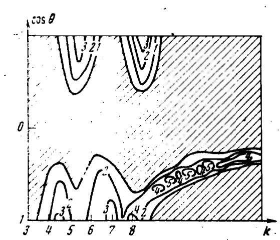

FIG. 6. Level lines of the function $\ln[N(k, \cos\theta)/N_0]$ for the case of pumping with a relativistic electron beam ($\gamma_0 t = 20$); $\Delta\theta = 15°$, $\gamma_{inst}/\gamma_0 \gg 1$.

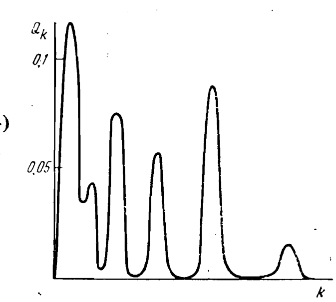

FIG. 7. Plot of the function (34) ($k \geq \omega_p/c$) for the case of pumping with a beam ($\gamma_0 t = 20$).

and also a number of problems on the kinetics of turbulence in a magnetoactive plasma.

In light of the foregoing, the study of other problems of Langmuir turbulence are also of general physical interest. Foremost among these is the problem of turbulence in non-isothermal plasma. An important role in the problem of beam heating is played by the study of the turbulence in the case $k_{diff} \gtrsim k_0$, when the principal role is played by the interaction of one or several spectral peaks with collapsing condensate. Finally, the study of strong turbulence due to modified decay instability is of fundamental interest.

[1] B. B. Kadomtsev, in: Voprosy teorii plazmy (Problems of Plasma Theory), Vol. 4, Gosatomizdat, 1964.

[2] A. A. Galeev and R. Z. Sagdeev, ibid.

[3] A. A. Galeev and R. A. Syunyaev, Zh. Eksp. Teor. Fiz. **63**, 1266 (1972) [Sov. Phys.-JETP **36**, 669 (1973)].

[4] B. N. Breĭzman, V. E. Zakharov, and S. L. Musher, Zh. Eksp. Teor. Fiz. **64**, 1297 (1973) [Sov. Phys.-JETP **37**, 658 (1973)].

[5] E. Valeo, C. Oberman, and F. W. Perkins, Phys. Rev. Lett. **28**, 340, 1972.

[6] V. E. Zakharov, Zh. Eksp. Teor. Fiz. **62**, 1745 (1972) [Sov. Phys.-JETP **35**, 908 (1973)].
[7] E. Valeo and W. Cruer, Phys. Fluids **16**, 675, 1973.
[8] F. W. Perkins, C. Oberman, and E. Valeo. Princeton University, PPL-AP69, 1973 (preprint).
[9] V. E. Zakharov, Zh. Eksp. Teor. Fiz. **60**, 993 (1971) [Sov. Phys.-JETP **33**, 538 (1971)].
[10] V. E. Zakharov, S. L. Musher, and A. M. Rubenchik, ZhETF Pis. Red. **19**, 249 (1974) [JETP Lett. **19**, 151 (1974)].
[11] S. V. Manakov, Zh. Eksp. Teor. Fiz. **67**, 543 (1974) [Sov. Phys.-JETP **40**, 269 (1975)].
[12] Ya. B. Zel'dovich and R. A. Syunyaev, Zh. Eksp. Teor. Fiz. **62**, 153 (1972) [Sov. Phys.-JETP **35**, 81 (1972)]; Ya. B. Zel'dovich, E. V. Levich, and R. A. Syunyaev, Zh. Eksp. Teor. Fiz. **62**, 1392 (1972) [Sov. Phys.-JETP **35**, 733 (1972)].

Translated by J. G. Adashko
17