Quadrupole moment of a localized Langmuir perturbation in an electron plasma

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It is shown that a quadrupole-type static electric field appears in the space surrounding a three-dimensional localized Langmuir perturbation at distances greatly exceeding the dimension of the perturbation. The quadrupole moment corresponding to this "long-range" field is found.

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This paper is devoted to the structure of the electric field of a localized Langmuir perturbation in an electron[1)] plasma. This question was elucidated earlier[1] in the one-dimensional case, and it was shown that at large distances from the perturbation-localization region there is a static electric field that falls off with increasing distance from the perturbation region only in power-law fashion ($\propto 1/x^2$). The corresponding electric field was called long-range in [1]. In this study it was possible to obtain a long-range electric field in the three-dimensional case. What was principally new here was that high-frequency force acting on the electron was no longer cancelled out by the space-charge field (this cancellation greatly decreased the effect in the one-dimensional case). At the same time, the power-law decrease of the electric field was preserved: the long-range field turned out to be quadrupole and static.

Just as in [1], we consider perturbations that decrease exponentially at infinity; these can be characterized by a single spatial scale L which is much larger than the Debye radius ($L \gg v_T/\omega_p$). The characteristic time of spreading away of such a perturbation, τ, is large because the group velocity of the Langmuir oscillations is small in comparison with the time of flight of the electron through the perturbation:

$$\tau \sim \omega_p L^2/v_T^2 \gg L/v_T.$$

The long-range field is brought about by the high-frequency pressure force (which is proportional to the square of the electric field amplitude), which distorts the distribution function of the electrons that pass through the perturbation region. These distortions are transported at the electron thermal velocity over large distances, and this leads to the appearance of an electric field in the region $|\mathbf{r}| \gg L$. Our problem is to find this field at distances that are large in comparison with L. At the same time, we assume that these distances are $\lesssim v_T\tau$. Since the electrons negotiate these distances within a time much shorter than the time of restructuring of the Langmuir perturbation, the distortions of the distribution function over such distances are quasistatic. It is precisely this circumstance that makes it possible to solve the problem completely. The sought effect arises in second-order approximation in the amplitude of the Langmuir perturbation.

The initial equations are the equation for the electron distribution function f and the Poisson equation

$$\frac{\partial f}{\partial t} + \mathbf{v}\frac{\partial f}{\partial \mathbf{r}} - \frac{e\mathbf{E}}{m}\frac{\partial f}{\partial \mathbf{v}} = 0, \quad \mathbf{E} = -\nabla\varphi, \tag{1}$$

$$\Delta\varphi = 4\pi e\left[\int f d^3v - n_0\right]. \tag{2}$$

Here n_0 is the density of the neutralizing background, and e and m are the charge and mass of the electron.

We use the method of successive approximations and put

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 + \ldots, \quad f = f_0 + f_1 + f_2 + \ldots.$$

Here $f_0 = n_0(m/2\pi T)^{3/2}\exp(-mv^2/2T)$, and

$$\mathbf{E}_1 = \boldsymbol{\varepsilon}\exp(-i\omega_p t) + \boldsymbol{\varepsilon}^*\exp(i\omega_p t) = -\nabla\varphi\exp(-i\omega_p t) - \nabla\varphi^*\exp(i\omega_p t); \tag{3}$$

φ is a slowly varying function of time, satisfying the equation

$$\partial\varphi/\partial t = i(3T/2m\omega_p)\Delta\varphi. \tag{4}$$

In the linear approximation we obtain for the distribution function the expression

$$f_1(\mathbf{r}, \mathbf{v}, t) = \frac{e}{m}\frac{\partial f_0}{\partial \mathbf{v}} e^{-i\omega_p t}\left[\frac{i}{\omega_p}\boldsymbol{\varepsilon} + \frac{1}{\omega_p^2}\frac{d\boldsymbol{\varepsilon}}{dt} - \frac{1}{\omega_p^3}\frac{d^2\boldsymbol{\varepsilon}}{dt^2} - \frac{1}{\omega_p^4}\frac{d^3\boldsymbol{\varepsilon}}{dt^3} + \ldots\right] + \text{c.c.}, \tag{5}$$

where $d/dt = \partial/\partial t + (\mathbf{v}\cdot\nabla)$. Expression (5) was obtained by iteration with respect to $1/\omega_p$, retaining the number of terms required to find the long-range electric field.

In the second order in the field amplitude, the system (1) and (2) takes the form

$$\frac{df_2}{dt} - \frac{e}{m}\mathbf{E}_2\frac{\partial f_0}{\partial \mathbf{v}} = \frac{e}{m}\mathbf{E}_1\frac{\partial f_1}{\partial \mathbf{v}} = \frac{e^2}{m^2\omega_p^2}\frac{\partial}{\partial v_\alpha}\left\{\frac{\partial f_0}{\partial v_\beta}\left[\left(\varepsilon_\alpha^*\frac{d\varepsilon_\beta}{dt} + \varepsilon_\alpha\frac{d\varepsilon_\beta^*}{dt}\right) - \frac{i}{\omega_p}\left(\varepsilon_\alpha^*\frac{d^2\varepsilon_\beta}{dt^2} - \varepsilon_\alpha\frac{d^2\varepsilon_\beta^*}{dt^2}\right) - \frac{1}{\omega_p^2}\left(\varepsilon_\alpha^*\frac{d^3\varepsilon_\beta}{dt^3} + \varepsilon_\alpha\frac{d^3\varepsilon_\beta^*}{dt^3}\right)\right]\right\}, \tag{6}$$

$$\operatorname{div}\mathbf{E}_2 = -4\pi e\int f_2 d^3v. \tag{7}$$

In the right-hand side of (6) we retained only the terms that vary slowly with time; the terms containing the factors $\exp(\pm 2i\omega_p t)$, in view of their exponentially small contribution to the function f_2 in the region $|\mathbf{r}| \gg L$ ($\mathbf{r}$ is the distance from the perturbation-localization region) have been omitted.

We represent $\mathbf{E}_2$ in the form $\mathbf{E}_2 = -\nabla\Phi_2$ and separate in the right-hand side of (6) the terms in the form of the total derivative with respect to time. As a result we obtain

$$\frac{df_2}{dt} + \frac{e}{m}\frac{\partial f_0}{d\mathbf{v}}\nabla\Phi_2 = \frac{e^2}{m^2\omega_p^2}\left\{\frac{\partial^2 f_0}{\partial v_\alpha \partial v_\beta}\frac{d}{dt}\left[\varepsilon_\alpha^*\varepsilon_\beta - \frac{i}{\omega_p}\left(\varepsilon_\alpha^*\frac{d\varepsilon_\beta}{dt} - \varepsilon_\beta\frac{d\varepsilon_\alpha^*}{dt}\right) - \frac{1}{\omega_p^2}\left(\varepsilon_\alpha^*\frac{d^2\varepsilon_\beta}{dt^2} + \varepsilon_\beta\frac{d^2\varepsilon_\alpha^*}{dt^2}\right) + \frac{1}{\omega_p^2}\frac{d\varepsilon_\alpha}{dt}\frac{d\varepsilon_\beta}{dt}\right] - \frac{\partial f_0}{\partial v_\beta}\frac{d}{dt}\left[\frac{2i}{\omega_p}(\varepsilon_\gamma^*\nabla_\gamma\varepsilon_\beta - \varepsilon_\gamma\nabla_\gamma\varepsilon_\beta^*) + \frac{3}{\omega_p^2}\left(\varepsilon_\gamma^*\nabla_\gamma\frac{d\varepsilon_\beta}{dt} + \varepsilon_\gamma\nabla_\gamma\frac{d\varepsilon_\beta^*}{dt}\right)\right] - \frac{mf_0}{T}(\mathbf{v}\nabla)\left[|\boldsymbol{\varepsilon}|^2 - \frac{3T}{m\omega_p^2}|\operatorname{div}\boldsymbol{\varepsilon}|^2 + \frac{3}{\omega_p^2}\left|\frac{d\boldsymbol{\varepsilon}}{dt}\right|^2\right] + \frac{3T}{m\omega_p^2}\frac{\partial f_0}{\partial v_\beta}\nabla_\alpha[\nabla_\gamma\varepsilon_\gamma^*\nabla_\alpha\varepsilon_\beta + \nabla_\gamma\varepsilon_\gamma\nabla_\alpha\varepsilon_\beta^*]\right\}. \tag{8}$$

In the derivation of (8) we have used Eq. (4).

In [1] we have noted a general property of the system (7) and (8), that if the right-hand side of (8) contains a certain function in the form of a total derivative with respect to time, $dF(\mathbf{r}, \mathbf{v}, t)/dt$, with

$$\int F(\mathbf{r},\mathbf{v},t)\,d^3v=0, \tag{9}$$

then this system has a solution $f_Z = f$, $\mathbf{E}_2 = 0$. By virtue of this property, the terms that are total derivatives with respect to time and satisfy the condition (9) make no contribution to the perturbation of the distribution function at large $|\mathbf{r}|$ and can be omitted in our problem. Thus, Eq. (8) can be reduced to the form

$$\frac{df_2}{dt}+\frac{e}{m}\frac{\partial f_0}{\partial \mathbf{v}}\nabla\Phi_2=\frac{e^2}{m^2\omega_p^2}\Big\{-\frac{1}{\omega_p^2}\frac{\partial^2 f_0}{\partial v_\alpha \partial v_\beta}(\mathbf{v}\nabla)\,[\varepsilon_\alpha^*(\mathbf{v}\nabla)^2\varepsilon^\beta+\varepsilon^\beta(\mathbf{v}\nabla)^2\varepsilon_\alpha^*$$
$$-(\mathbf{v}\nabla)\varepsilon_\alpha^*(\mathbf{v}\nabla)\varepsilon_\beta]-\frac{3}{\omega_p^2}\frac{\partial f_0}{\partial v_\beta}(\mathbf{v}\nabla)\,[\varepsilon_\gamma^*\nabla_\gamma(\mathbf{v}\nabla)\varepsilon_\beta+\varepsilon_\gamma\nabla_\gamma(\mathbf{v}\nabla)\varepsilon_\beta^*]$$
$$-\frac{mf_0}{T}(\mathbf{v}\nabla)\left[|\varepsilon|^2+\frac{3}{\omega_p^2}|(\mathbf{v}\nabla)\varepsilon|^2-\frac{3T}{m\omega_p^2}|\operatorname{div}\varepsilon|^2\right] \tag{10}$$
$$+\frac{3T}{m\omega_p^2}\frac{\partial f_0}{\partial v_\beta}\nabla_\alpha[\nabla_\gamma\varepsilon_\gamma^*\nabla_\alpha\varepsilon_\beta+\nabla_\gamma\varepsilon_\gamma\nabla_\alpha\varepsilon_\beta^*]\Big\}.$$

After simple transformations this equation can be written in the form

$$\frac{\partial f_2}{\partial t}+(\mathbf{v}\nabla)f_2+\frac{e}{m}\frac{\partial f_0}{\partial \mathbf{v}}\nabla\Phi_2=\frac{1}{m}\frac{\partial f_0}{\partial v}\nabla U+\frac{1}{m}\frac{\partial f_0}{\partial v_\beta}\nabla_\alpha Q_{\alpha\beta}, \tag{11}$$

where

$$U=\frac{e^2}{m^2\omega_p^2}\left[|\varepsilon|^2+\frac{3T}{m\omega_p^2}|\operatorname{div}\varepsilon|^2-\frac{1}{\omega_p^2}(\mathbf{v}\nabla)^2|\varepsilon|^2+\frac{3}{\omega_p^2}|(\mathbf{v}\nabla)\varepsilon|^2\right],$$
$$Q_{\alpha\beta}=\frac{3e^2T}{m^2\omega_p^4}[\nabla_\gamma\varepsilon_\gamma^*\nabla_\alpha\varepsilon_\beta+\nabla_\gamma\varepsilon_\gamma\nabla_\alpha\varepsilon_\beta^*-2\delta_{\alpha\beta}|\operatorname{div}\varepsilon|^2].$$

Thus, the three-dimensional problem considered now differs fundamentally from the one-dimensional problem in that the high-frequency force acquires a nonpotential part (the last term in the right-hand side of (11)), whereas in the one-dimensional case it is obvious that the force is potential and the right-hand side of (11) reduces simply to the form

$$\frac{1}{m}\frac{\partial f_0}{\partial v}\frac{\partial U(x)}{\partial x}$$

Therefore the right-hand side of (11) and the last term in the left-hand side of this equation were cancelled out in the one-dimensional case, so that the effective high-frequency force was much smaller than simply[2)] $-\partial U/\partial x$. The presence of a nonpotential term in the right-hand side of (11) in the three-dimensional case excludes the possibility of such a cancellation, and the effect appears in lower order in r_D/L.

To determine the long-range part of the high-frequency potential with the required accuracy it suffices to solve Eq. (11) relative to f_2, leaving out the derivative with respect to time in its left-hand side. The latter is possible because we seek the electric field at distances $\lesssim v_T\tau$, where the distribution function, as already noted above, is quasistatic. As a result of integration of this equation over the trajectories, we obtain for f_2

$$f_2=\frac{e}{T}f_0\Phi_2-\frac{1}{T}f_0U+\frac{1}{v}\int_{-\infty}^{0}Q(\mathbf{r}+\mathbf{n}\xi)\,d\xi, \tag{12}$$

where $\mathbf{n} = \mathbf{v}/v$, and $Q(\mathbf{r}, \mathbf{v})$ denotes the last term in (11).

An expression for Φ_2 can be obtained from the plasma quasineutrality condition

$$\int f_2\,d^3v=0.$$

As a result we obtain

$$\Phi_2=\frac{e}{m\omega_p^2}\left\{|\varepsilon|^2-\frac{2T}{m\omega_p^2}\nabla_\alpha[\varepsilon_\alpha\nabla_\beta\varepsilon_\beta^*-\varepsilon_\beta\nabla_\beta\varepsilon_\alpha^*]-\frac{3T^2}{nm^2\omega_p^2}I\right\}, \tag{13}$$

where

$$I=\int\frac{d^3v}{n}\frac{\partial f_0}{\partial v_\beta}\int_{-\infty}^{0}\nabla_\alpha q_{\alpha\beta}(\mathbf{r}+\mathbf{n}\xi)\,d\xi, \tag{14}$$

and

$$q_{\alpha\beta}=\nabla_\gamma\varepsilon_\gamma^*\nabla_\alpha\varepsilon_\beta+\nabla_\gamma\varepsilon_\gamma\nabla_\alpha\varepsilon_\beta^*-2\delta_{\alpha\beta}|\operatorname{div}\varepsilon|^2.$$

We note that in the expression for Φ_2 all the terms except the last decrease exponentially rapidly with increasing distance from the perturbation region. We carry out all the possible integrations in this term. We change over in the integral with respect to the velocities to spherical coordinates:

$$v_x=v\sin\vartheta\cos\varphi,\quad v_y=v\sin\vartheta\sin\varphi,\quad v_z=v\cos\vartheta.$$

As a result we have

$$I=-\frac{m}{T}\int_0^\infty f_0v^2\,dv\int_0^\pi\sin\vartheta\,d\vartheta\int_0^{2\pi}d\varphi\int_{-\infty}^{0}n_\beta\nabla_\alpha q_{\alpha\beta}(\mathbf{r}+\mathbf{n}\xi)\,d\xi.$$

In the calculation of the last three integrals it is advantageous to make the following change of variables:

$$x'=x+\xi\sin\vartheta\cos\varphi,\quad y'=y+\xi\sin\vartheta\sin\varphi,\quad z'=z+\xi\cos\vartheta.$$

Recognizing that in this notation we have $n_\beta = (x'_\beta - x_\beta)/|\mathbf{r}' - \mathbf{r}|$, we obtain

$$I=-\frac{m}{T}\int_0^\infty f_0v^2\,dv\iiint_{-\infty}^{\infty}dx'\,dy'\,dz'\,\frac{x_\beta'-x_\beta}{|\mathbf{r}'-\mathbf{r}|^3}\frac{\partial}{\partial x_\alpha}q_{\alpha\beta}(\mathbf{r}')$$

or else, after integrating with respect to v, we get

$$I=-\frac{nm}{4\pi T}\int d\mathbf{r}'\left[\frac{\delta_{\alpha\beta}}{|\mathbf{r}'-\mathbf{r}|^3}-3\frac{(x_\alpha'-x_\alpha)(x_\beta'-x_\beta)}{|\mathbf{r}'-\mathbf{r}|^5}\right]q_{\alpha\beta}(\mathbf{r}'). \tag{15}$$

At large distances ($|\mathbf{r}| \gg L$) we can expand in (15) in the parameter $|\mathbf{r}'|/|\mathbf{r}|$. Confining ourselves here to the first term of the expansion, we get

$$I=-\frac{nm}{4\pi T}\left(\frac{\delta_{\alpha\beta}}{|\mathbf{r}|^3}-3\frac{x_\alpha x_\beta}{|\mathbf{r}|^5}\right)\int d\mathbf{r}'\,q_{\alpha\beta}(\mathbf{r}').$$

We can ultimately write for Φ_2

$$\Phi_2=\frac{e}{m\omega_p^2}\Big\{|\varepsilon|^2-\frac{2T}{m\omega_p^2}\nabla_\alpha[\varepsilon_\alpha\nabla_\beta\varepsilon_\beta^*-\varepsilon_\beta\nabla_\beta\varepsilon_\alpha^*] \tag{16}$$
$$+\frac{3T}{4\pi m\omega_p^2}\left(\frac{\delta_{\alpha\beta}}{|\mathbf{r}|^3}-3\frac{x_\alpha x_\beta}{|\mathbf{r}|^5}\right)\int d\mathbf{r}'\,(\nabla_\gamma\varepsilon_\gamma^*\nabla_\alpha\varepsilon_\beta+\nabla_\gamma\varepsilon_\gamma\nabla_\alpha\varepsilon_\beta^*-2\delta_{\alpha\beta}|\operatorname{div}\varepsilon|^2]\Big\}$$

Thus, the long-range part of the electric potential has a quadrupole character.[3)] The corresponding quadrupole moment is

$$D_{\alpha\beta}=\frac{9eT}{2\pi m^2\omega_p^4}\int d\mathbf{r}\left[\Delta\varphi^*\frac{\partial^2\varphi}{\partial x_\alpha\,\partial x_\beta}+\Delta\varphi\frac{\partial^2\varphi^*}{\partial x_\alpha\,\partial x_\beta}-2\delta_{\alpha\beta}|\Delta\varphi|^2\right]. \tag{17}$$

It is easy to show that the quantity $D_{\alpha\beta}$ is conserved. Indeed, let us take the time derivative of $D_{\alpha\beta}$ and use Eq. (4) in the integrand. As a result we get

$$\frac{\partial D_{\alpha\beta}}{\partial t}=i\frac{27eT^2}{4\pi m^3\omega_p^5}\int d\mathbf{r}\Big\{\Delta\varphi^*\frac{\partial^2}{\partial x_\alpha\,\partial x_\beta}\Delta\varphi-\Delta\Delta\varphi^*\frac{\partial^2\varphi}{\partial x_\alpha\,\partial x_\beta}+\Delta\Delta\varphi\frac{\partial^2\varphi^*}{\partial x_\alpha\,\partial x_\beta}$$
$$-\Delta\varphi\frac{\partial^2}{\partial x_\alpha\,\partial x_\beta}\Delta\varphi^*-2\delta_{\alpha\beta}[\Delta\varphi^*\Delta\Delta\varphi-\Delta\varphi\Delta\Delta\varphi^*]\Big\}.$$

Integrating now twice by parts, say in the second, third, and fifth terms of the integrand, we find that $\partial D_{\alpha\beta}/\partial t \equiv 0$.

Thus, the tensor $D_{\alpha\beta}$ can be expressed in terms of the characteristics of the perturbation at the initial instant of time:

$$D_{\alpha\beta}=\frac{9eT}{2\pi m^2\omega_p^4}\int d\mathbf{r}\left[\Delta\varphi_0^*\frac{\partial^2\varphi_0}{\partial x_\alpha\,\partial x_\beta}+\Delta\varphi_0\frac{\partial^2\varphi_0^*}{\partial x_\alpha\,\partial x_\beta}-2\delta_{\alpha\beta}|\Delta\varphi_0|^2\right],$$
$$\varphi_0=\varphi(\mathbf{r},0).$$

In order of magnitude, the quadrupole moment is obviously equal to $eL^2N_D(W/n_0T)(r_D/L)$, where N_D is the Debye number and $W \sim \epsilon^2/8\pi$ is the energy density of the Langmuir perturbation.

The presence of a quadrupole electric field means that the region of the influence of the localized Langmuir perturbation greatly exceeds the dimension L of the perturbation.

In conclusion, the author thanks D. D. Ryutov for a discussion of the results.

[1)] It is assumed that the infinitely heavy ions form a homogeneous neutralizing background.

[2)] Actually the effective HF force was equal to $-\partial U_{eff}/\partial x$, where $U_{eff} = U - e\Phi_2$ differs from zero only in second order in the parameter r_D/L.

[3)] We note that the presence of a nonpotential term in the high-frequency force produces also in the perturbation region eddy currents and a corresponding magnetic field. This effect can in principle be of independent interest, but it is immaterial in our problem, since it contains the small quantity $\sim v_{Te}/c$.

[1] M. P. Ryutova, Zh. Eksp. Teor. Fiz. 67, 2161 (1974) [Sov. Phys.-JETP 40, 1072 (1975)].

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