Anomalous diffusion of a low-density plasma in an adiabatic trap

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It is shown that if flute oscillations are excited in a plasma an appreciable diffusion of the latter can be caused even by rare Coulomb collisions. The diffusion mechanism under consideration may resemble that of neoclassical diffusion. In both cases, the particles are transported across the magnetic field along determined trajectories, whereas Coulomb collisions simply transfer the particles from one trajectory to another.

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INTRODUCTION

Adiabatic traps frequently operate with constant and prolonged (several seconds) plasma injection. Under these conditions, the stationary level of the plasma density sets in as a result of balance between the entry of the particles into the trap via injection and their departure through charge exchange with the residual gas. If the plasma density exceeds a certain critical value, then the so-called flute instability is excited in traps with a simple mirror field. The development of this instability leads quite frequently to establishment of a new stationary state.[1] Regular constant-amplitude oscillations are then present in the plasma, and the plasma itself goes off quite rapidly to the chamber walls. The processes that lead to the plasma ejection have remained unclear so far. Indeed, the regular periodic oscillations cannot cause a directed motion of the particles, and the plasma densities typical of the discussed experiments are so low that the diffusion flux calculated in accord with the classical theory should be quite negligible.

We show in this paper how ordered oscillations can lead at a low plasma density to the loss of the particles from an adiabatic trap. The proposed loss mechanism takes into account essentially certain peculiarities of adiabatic traps. The ions are contained in the traps by magnetic mirrors, and the electrons by an electric potential that is produced spontaneously. The potential reaches a maximum at the center of the trap and falls off both along the magnetic field and in the transverse direction. Since the chamber walls are kept at a constant (zero) potential, the radial electric field should depend on the coordinate along the magnetic field (the z coordinate). The frequency of the oscillations of the electron in the potential well along the magnetic field greatly exceeds the frequency of the flute oscillations. Therefore when the latter are considered it is necessary to use the value of the radial electric field averaged over OZ. The result of the averaging depends on the distance to which the electron penetrates in the traps, i.e., in final analysis, on the energy of its longitudinal motion.

The radial electric field leads to a drift of the electrons in azimuth. The flute oscillations also travel in this direction. In the flute oscillations, the electrons oscillate along the magnetic field, and the amplitude of the displacement depends on the mismatch between the phase velocity of the flute oscillations and the electron drift velocity. Since the drift velocity depends in turn on the energy, the Coulomb collisions should lead to random changes of the electron displacement amplitude in the flute oscillations. This process, as shown in the present paper, causes an anomalously rapid diffusion of the electrons across the magnetic field. Its mechanism has much in common with the mechanism of neoclassical diffusion (see, e.g., [2]).

We consider in this paper electron diffusion, but the ions can diffuse in similar fashion. Indeed, the oscillation amplitude of an ion along the magnetic field, and consequently also the average radial electric field acting on the ion, depends on the ratio of the ion energies in the directions along and across the magnetic field. In addition, in adiabatic traps, drift due to inhomogeneity of the magnetic field, with a velocity proportional to the ion energy, is superimposed on the drift of the ions in the crossed fields. The frequency of the ion-ion Coulomb collisions in adiabatic traps is usually very low, but the ion energy can vary, for example, under the influence of cyclotron oscillations, which are very frequently excited spontaneously in such systems.

1. BASIC EQUATIONS

An adiabatic trap with a simple mirror field is an axially-symmetrical system. To simplify the calculations we replace the axial symmetry by a planar symmetry. We introduce a Cartesian coordinate system with the OZ axis along the magnetic field. We set the OY axis in correspondence with the azimuth, and OX to the radius. We consider the motion of an electron in an electric field with a potential

$$\varphi(\mathbf{r}, t)=\varphi_0(x, z)+\varphi_1(y, t).$$

Here $\varphi_0(x, z) = \varphi_0(x)(1 - (z/L)^2)$ simulates the constant potential that retains the electrons in the trap, and $\varphi_1(y, t) = \varphi_1\cos(ky - \omega t)$ is the potential of the flute oscillations. The equations of motion of the electron take the form

$$\dot{x}=\frac{ck\varphi_1}{H}\sin(ky-\omega t), \tag{1}$$

$$\dot{y}=\frac{c}{H}\frac{d\varphi_0}{dx}\left(1-\left(\frac{z}{L}\right)^2\right), \tag{2}$$

$$\ddot{z}=-\frac{2e}{m}\varphi_0(x)\frac{z}{L^2}. \tag{3}$$

Since the frequency of the flute oscillations is small in comparison with the electron-cyclotron frequency, and the wavelength is large in comparison with the electron Larmor radius, we have assumed for the velocity of the electron motion across the magnetic field the expression $\mathbf{v} = cH^{-2}[\mathbf{H}\times\nabla\varphi]$.

Taking (3) into account, we average (2) over the fast oscillations along the magnetic field:

$$\dot{y}=\frac{c}{H}\frac{d\varphi_0}{dx}\left(1-\frac{\varepsilon_\parallel}{2e\varphi_0}\right). \tag{4}$$

Here $\epsilon_{\parallel}$ is the energy of electron motion in the direction along the magnetic field, taken at the point z = 0, i.e., at the bottom of the potential well.

It is quite important in what follows that the average electric field, and with it also the average drift velocity, depends on the longitudinal electron energy. It is obvious that this dependence arises for any shape of the potential well except the square well. Since the exact form of the potential $\varphi_0(z)$ produced in the trap is unknown, we have assumed the simplest parabolic dependence.

Under the action of the Coulomb collisions, the electron energy fluctuates about a certain mean value $\epsilon_{\parallel 0}$. The simplest model equation describing these fluctuations is of the form

$$\delta\dot{\varepsilon}_{\parallel}=-\nu\delta\varepsilon_{\parallel}-\zeta(t). \tag{5}$$

Here $\delta\epsilon_{\parallel} = \epsilon_{\parallel} - \epsilon_{\parallel 0}$ is the frequency of the Coulomb collision and (t) is a random function that is δ-correlated in time:

$$\langle\zeta(t_1)\zeta(t_2)\rangle=\sigma^2\delta(t_1-t_2),\quad \sigma^2=2T_{\parallel}^2\nu,\quad T_{\parallel}\approx\varepsilon_{\parallel 0}.$$

The angle brackets denote averaging over the statistical ensemble.

It follows from (5) that[1)]

$$\delta\varepsilon_{\parallel}=\int_{-\infty}^{t}dt'e^{\nu(t'-t)}\zeta(t'). \tag{6}$$

Using (6), we reduce (4) to the form

$$\dot{y}=\frac{c}{H}\frac{d\varphi_0}{dx}\left(1-\frac{\varepsilon_{\parallel 0}}{2e\varphi_0}-\frac{1}{2e\varphi_0}\int_{-\infty}^{t}dt'e^{\nu(t'-t)}\zeta(t')\right). \tag{7}$$

It is convenient in the subsequent calculation to introduce the dimensionless variables

$$\eta=\pi+ky-\omega t,\quad \xi=x\omega H/kc\varphi_1,\quad \tau=t\omega.$$

Equations (1) and (7) then take the form

$$d\xi/d\tau=-\sin\eta, \tag{8}$$

$$\frac{d\eta}{d\tau}=w(\xi)-\int_{-\infty}^{\tau}d\tau_1e^{\alpha(\tau_1-\tau)}\chi(\tau_1). \tag{9}$$

Here

$$\alpha=\frac{\nu}{\omega},\quad w(T_{\parallel},\xi)\equiv w(\xi)=\frac{kv}{\omega}\left(1-\frac{T_{\parallel}}{2e\varphi_0}\right)-1,\quad v=\frac{c}{H}\frac{d\varphi_0}{dx},$$

$$\chi=\zeta\frac{kv}{\omega^2}\frac{1}{2e\varphi_0},$$

$$\langle\chi(\tau_1)\chi(\tau_2)\rangle=\beta^2\delta(\tau_1-\tau_2),$$

$$\beta^2=\frac{1}{\omega}\left(\frac{kv}{\omega}\right)^2\left(\frac{\sigma_0}{2e\varphi_0}\right)^2.$$

2. HOMOGENEOUS ELECTRIC FIELD

A. Diffusion Mechanism

If the stationary electric field is homogeneous and consequently the drift velocity along OY does not depend on ξ ($w(\xi,\epsilon) = w(\epsilon)$, $w(T_{\parallel},\xi) \equiv w$), then Eqs. (8) and (9) can be integrated:

$$\xi(\tau)=\xi(0)-\int_0^{\tau}d\tau'\sin\eta(\tau'), \tag{10}$$

$$\eta(\tau)=\eta(0)+w\tau-\int_0^{\tau}d\tau'\int_{-\infty}^{\tau'}d\tau''e^{\alpha(\tau''-\tau')}\chi(\tau''). \tag{11}$$

Assume that the Coulomb collisions have been turned off ($\chi \equiv 0$), and then the electron trajectories on the (ξ, η) plane become sinusoidal (see Fig. 1). The smaller the difference between the phase velocity of the oscillations and the drift velocity of the electron, i.e., the smaller $w(\epsilon)$, the larger the amplitude of the sinusoid. Assume that initially $w(\epsilon)$ is large—the electron is on the trajectory labeled 1. Under the influence of the Coulomb collisions its energy can change in such a way that $w(\epsilon)$ decreases, and then the electron goes over to trajectory 2. If the electron energy again increases at the instant of time when it is located in the lower part of trajectory 2, then it goes over to trajectory 3. The end result of these processes is a displacement of the electron along the ξ axis, i.e., across the magnetic field. It is easily seen that the diffusion mechanism considered by us is similar in character to the neoclassical diffusion mechanism (e.g.,[2]). Indeed, in both cases the particle moves across the magnetic field along definite trajectories, and the Coulomb collisions only transfer the particles over from one trajectory to another.

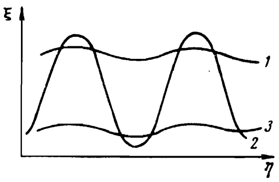

FIG. 1. Electron trajectories in a homogeneous electric field at various values of the energy $w(\epsilon_2) < w(\epsilon_1)$, $w(\epsilon_3)$.

B. The Diffusion Coefficient

Let us determine with the aid of (10) and (11) the variance of the values of the coordinate η:

$$d_\eta(\tau)=\langle\eta^2\rangle-(\langle\eta\rangle)^2=\frac{\beta^2}{2\alpha^3}(e^{-\alpha\tau}-1+\alpha\tau), \tag{12}$$

and also the rate at which the variance of the coordinate ξ changes:

$$D_\xi(\tau)=\frac{d}{d\tau}d_\xi(\tau)=2\left\langle\sin\eta(\tau)\int_0^{\tau}d\tau'\sin\eta(\tau')\right\rangle$$

$$-2\langle\sin\eta(\tau)\rangle\left\langle\int_0^{\tau}d\tau'\sin\eta(\tau')\right\rangle. \tag{13}$$

It follows from (12) that $d_\eta(\tau)$ increases with time. We consider time intervals τ satisfying the condition $\tau \gg \tau_1^*$, where τ_1^* is defined by the equation $d_\eta(\tau_1^*) = \pi$. In this case we can leave out the second term of (13). We shall see below that for sufficiently long time intervals $D_\xi(\tau)$ tends to a finite limit, $D_\xi(\tau) \to D_\xi$, as $\tau \to \infty$. This limit has the meaning of the diffusion coefficient

$$D_\xi=\frac{1}{2}\operatorname{Re}\int_0^{\infty}d\tau'\langle\exp(i\eta(\tau)-i\eta(\tau'))\rangle. \tag{14}$$

The integral in (14) is a certain modification of the integral, introduced by Wiener, over random trajectories. We calculate it by a standard procedure (see, e.g.,[3]). We replace the integrals in (11) and (14) by sums. In particular, we represent the difference $\eta(\tau) - \eta(\tau')$ in the form

$$\eta(\tau)-\eta(\tau')=w(\tau-\tau')+\sum_{n=-\infty}^{\tau/\Delta\tau}a_n\chi_n\Delta\tau, \tag{15}$$

where

$$a_n=\alpha^{-1}(e^{-\alpha\tau}-e^{-\alpha\tau'})\ \text{as}\ n<\tau'/\Delta\tau,$$

$$a_n=\alpha^{-1}(e^{-\alpha(\tau-n\Delta\tau)}-1)\ \text{as}\ n>\tau'/\Delta\tau.$$

We assume that $\chi_n = \chi(n\Delta\tau)$ has a normal distribution

$$p(\chi_n)=\left(\frac{\Delta\tau}{2\pi\beta^2}\right)^{1/2}\exp\left(-\frac{\chi_n^2\Delta\tau}{2\beta^2}\right) \tag{16}$$

and is δ-correlated in the index n:

$$\langle \chi_n \chi_m \rangle = \begin{cases} \beta^2/\Delta\tau, & n=m \\ 0, & n \neq m \end{cases}. \tag{17}$$

As $\Delta\tau \to 0$, the last equality goes over into

$$\langle \chi(\tau_1)\chi(\tau_2) \rangle = \beta^2 \delta(\tau_1 - \tau_2)$$

(cf. (5)). Averaging in (14) with the aid of (16) and (17) and returning from summation to integration, we get

$$\left\langle \exp\left(i \sum_{n=-\infty}^{\tau/\Delta\tau} a_n \chi_n \Delta\tau \right) \right\rangle = \sum_{n=-\infty}^{\tau/\Delta\tau} \langle \exp(i a_n \chi_n) \rangle$$

$$= \exp\left\{ -\frac{\beta^2}{2\alpha^2} \left[(e^{-\alpha\tau} - e^{-\alpha\tau'})^2 \int_{-\infty}^{\tau'} d\tau'' e^{2\alpha\tau''} + \int_{\tau'}^{\tau} (1 - e^{\alpha(\tau''-\tau')})^2 d\tau'' \right] \right\}$$

$$= \exp\left\{ -\frac{\beta^2}{2\alpha^3} (\alpha(\tau-\tau') - 1 + e^{\alpha(\tau'-\tau)}) \right\}. \tag{18}$$

As a result, the expression for the diffusion coefficient D_ξ takes the form

$$D_\xi = \frac{1}{2} \int_0^\infty d\tau \cos(w\tau) \exp\left(-\frac{\beta^2}{2\alpha^3} (\alpha\tau - 1 + e^{-\alpha\tau}) \right). \tag{19}$$

We have succeeded in evaluating the integral (19) only in the simplest limiting cases. Assume that the condition $\beta \gg \max(2^{1/2}\alpha^{3/2}, 2\alpha w^{1/2})$ is satisfied; then we can put $\cos(w\tau) = 1$ in (19) and expand the argument of the exponential in a series up to second order inclusive:

$$D_{\xi 1} \approx \frac{1}{2} \int_0^\infty d\tau \exp\left(-\frac{\beta^2}{4\alpha} \tau^2 \right) = \frac{(\pi\alpha)^{1/2}}{2\beta}. \tag{20}$$

At $2\alpha^{1/2} w \gg \beta \gg 2^{1/2}\alpha^{3/2}$ the integrand in (19) oscillates rapidly and asymptotic methods must be used to calculate the integral (see, e.g., [4]):

$$D_{\xi 2} \approx \beta^2 / 4w^4. \tag{21}$$

Finally, at $\beta \ll 2^{1/3}\alpha^{2/3}$ we can retain in the argument of the exponential in the integrand only the first term

$$D_{\xi 3} \approx \frac{1}{2} \int_0^\infty d\tau \cos(w\tau) \exp\left(-\frac{\beta^2 \tau}{2\alpha^2} \right) = \frac{\beta^2}{4\alpha^2 (w^2 + \beta^4/4\alpha^4)}. \tag{22}$$

The dependence of the diffusion coefficient on the collision frequency is shown schematically in Fig. 2 (cf. the analogous dependence of the coefficient of neoclassical diffusion[2]). It must be recalled here that to calculate the diffusion coefficient we used the condition $\tau \gg \alpha^{-1}$ (see above), and therefore the expressions obtained by us are suitable only for the description of slow processes with a characteristic time scale t_0 much larger than ν^{-1}.

C. The Diffusion Equation

It is known (see, e.g., [5]) that random processes whose duration greatly exceeds the "correlation distribution" time τ^* can be described by a diffusion equation of the Fokker-Planck type. We assume $\tau^* = \max(\tau_i^*)$, where i = 1, 2, 3. The time τ_i^* was defined in the preceding section as the time of mixing over the coordinate η ($d_\eta(\tau_1^*) = \pi$).

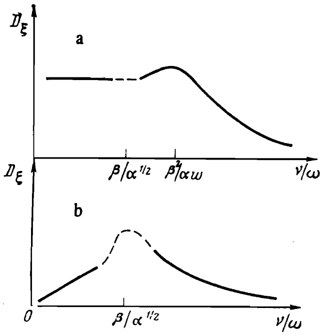

FIG. 2. Qualitative dependence of the diffusion coefficient D_ξ on the collision frequency v: a) $w < \beta/\alpha^{1/2}$, b) $w > \beta/\alpha^{1/2}$.

At $\tau \gg \tau_1^*$ it is possible to disregard the dependence on this coordinate in the Fokker-Planck equation. In addition, at $\tau \gg \tau_1^*$ the dynamic friction coefficient $F_\xi(\tau) = \langle \xi \rangle = \langle \sin \eta \rangle$ vanishes (see (8)). As a result, the diffusion equation takes the form

$$\frac{\partial f}{\partial \tau} - D_\xi \frac{\partial^2 f}{\partial \xi^2} = Q(\xi, \tau). \tag{23}$$

Here $f = f(\xi, \tau)$ is the electron distribution function and $Q(\xi, \tau)$ is the source function.

The time τ_2^* properly speaking has also the meaning of the time of uncoupling of the correlations in the values of the coordinate ξ, viz., $D_\xi(\tau) = D_\xi$ = const; here $D_\xi(\tau)$ is defined by (13). Finally, the time τ_3^* is equal to α^{-1}. At $\tau \gg \tau_3^*$ the electron "forgets" the initial energy value, so that the diffusion process can be considered by using a distribution function that does not depend on the electron energy. This approximation is reasonable, since usually the time of containment of the electrons in adiabatic traps greatly exceeds the time ν^{-1} between collisions.

The analysis shows that the expression for τ^* can be reduced to the form

$$\tau^* = \max(\alpha^{-1}, \alpha^2/\beta^2). \tag{24}$$

Let us determine also the characteristic space scale over which the decoupling of the correlations ξ^* takes place. The quantity ξ^* is equivalent to the mean free path in ordinary hydrodynamics. At sufficiently small values of w it is equal to the path traversed by the electron during the time τ^* between collisions $\xi^* = \tau^*$, and at large w it is equal to the electron displacement amplitude in the field of the flute oscillations. Using these arguments, we obtain

$$\begin{aligned} \xi^* &= \alpha^{-1}, \quad \beta \gg \max(2^{1/2}\alpha^{3/2}, 2\alpha w^{1/2}), \\ \xi^* &= \alpha^2/\beta^2, \quad 2^{1/2}\alpha^{3/2} \gg \beta \gg 2^{1/2}\alpha w^{1/2}, \\ \xi^* &= w^{-1}, \quad 2^{1/2}\alpha w^{1/2} \gg \beta. \end{aligned} \tag{25}$$

We note in conclusion that the expression for the dimensional diffusion coefficient $D_X = D_\xi (dx/d\xi)^2 d\tau/dt$ contains the oscillation frequency ω only in the combination $\omega - kv$. This diffusion coefficient differs from zero at $\omega = 0$, and consequently electron diffusion can be brought about even by static perturbations of the electric potential.

3. INHOMOGENEOUS ELECTRIC FIELD

A. Motion in the Absence of Collisions

The inhomogeneity of the stationary electric field influences particularly strongly the electron motion if somewhere within the confines of the system the phase velocity of the oscillations coincides with the drift velocity, i.e., if $w(\xi)$ vanishes. We consider the motion in the vicinity of the resonance point. We expand $w(\xi)$ in this region in a series, and retain only the first term of the expansion, $w(\xi) \approx a\xi$; here $a = dw/d\xi|_{\xi=0}$ and the origin is at the resonance point. If there are no Coulomb collisions, then the system (8), (9) takes the standard form

$$d\xi/d\tau = -\sin\eta, \quad d\eta/d\tau = a\xi. \tag{26}$$

It is precisely equations of this type that describe, for example, the motion of charged particles in the problem of nonlinear Landau damping.[6] This problem has by now been investigated in sufficient detail, so that it is useful to compare it with our problem. In nonlinear Landau damping it turned out to be convenient to subdivide all particles into those trapped by the wave and the untrapped ones. On the $\xi\eta$ plane, the trapped particles lie in the vicinity of the line $\xi = 0$. Since the resonance condition is approximately satisfied for these particles. They are acted upon particularly strongly by the oscillations. In our case the trapping phenomenon consists in the fact that the flute oscillations cause the particles to move along OY with an average velocity equal to the phase velocity of the oscillations. In Fig. 3, the trajectories of the trapped particles are shown by closed lines. The trapping region is bounded by the values $|\xi| < a^{-1/2}$. The untrapped particles move relative to the wave, and the corresponding trajectories are open in Fig. 3.

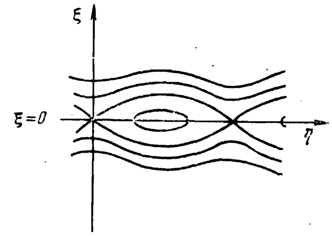

FIG. 3. Trajectories of electrons having equal energies in an inhomogeneous electric field. The electron drift velocity in crossed fields is equal to the phase velocity of the oscillations on the line $\xi = 0$.

B. The Diffusion Coefficient

The Coulomb collisions cause a random variation of the electron energy. The energy variation is accompanied by fluctuations of the position of the resonance point, which we choose to be the origin, and consequently the entire phase-trajectory picture in Fig. 3 vibrates as a unit. This vibration limits the phase memory of the electron and causes it to be ultimately knocked out of the trajectory. If the characteristic time τ^* of the decoupling of the phase correlations is much less than the period $2\pi/\Omega$ of the revolution on the trajectory, then the electron has no time to "feel" the inhomogeneity of the system. In this case we can use the results obtained in Sec. 2, taking the dependence of w on ξ into account parametrically. This will be called the frequent-collision regime. It must be noted, however, that the frequency Ω varies on going from one trajectory to the other. For this reason, collisions that are frequent in some region of the phase space may turn out to be rare in other regions.

If the condition $\Omega\tau \gg 1$ is satisfied (the rare-collision regime), then the phase memory of the electron spans many periods of the oscillations. In this case it can be assumed that the electron moves along the orbits shown in Fig. 3, going over slowly from one orbit to the other. In the rare-collision regime, the effect of the collisions on the electron motion can be taken into account within the framework of a successive approximation method, as will be done below.

At $w(\xi) = a\xi$, Eqs. (8) and (9) can be obtained from the Hamiltonian

$$H = \frac{1}{2}a\xi^2 - \cos\eta - \xi\int_{-\infty}^{\tau} d\tau'\chi(\tau')e^{\alpha(\tau'-\tau)}. \tag{27}$$

Motion that is not perturbed by Coulomb collisions (see (26)) is conveniently described by the action variables

$$I^{(0)} = \frac{1}{2\pi}\int_0^{2\pi} d\eta\,\xi = \frac{1}{a^{1/2}}\frac{4k}{\pi}\mathbf{E}\left(\frac{1}{k}\right) = \text{const}, \tag{28}$$

$$\theta^{(0)} = \Omega\tau + \text{const}. \tag{29}$$

Here

$$\Omega(I) = \pi k a^{1/2}\mathbf{K}^{-1}\left(\frac{1}{k}\right), \quad k = \left(\frac{1+H}{2}\right)^{1/2}, \tag{30}$$

H_0 is the unperturbed Hamiltonian, and **K** and **E** are complete elliptic integrals of the first and second kind, respectively.

Equations (28) and (29) describe the trajectories of the untrapped particles. Analogous expressions can be derived also for the trapped particles. We shall show, however, that the trapped particles play a rather minor role in the processes of interest to us.

The Coulomb collisions alter the action and the phase in accord with the equations

$$I^{(1)} = \frac{\partial\xi^{(0)}}{\partial\theta}\int_{-\infty}^{\tau} d\tau'\chi(\tau')e^{\alpha(\tau'-\tau)}, \tag{31}$$

$$\dot{\theta}^{(1)} = \frac{\partial\Omega}{\partial I}I^{(1)} - \frac{\partial\xi^{(0)}}{\partial I}\int_{-\infty}^{\tau} d\tau'\chi(\tau')e^{\alpha(\tau'-\tau)}. \tag{32}$$

Here $\xi^{(0)}$ is the unperturbed value of the coordinate

$$\xi^{(0)} = \frac{2k}{\sqrt{a}}\,dn\left(\frac{\theta}{\pi}\mathbf{K}\left(\frac{1}{k}\right)\right) = \frac{1}{a^{1/2}}\sum_{n=0}^{\infty} A_n\cos n\theta, \tag{33}$$

where

$$A_0 = \frac{\Omega}{a^{1/2}}, \quad A_{n\geq 1} = \frac{4\Omega}{a^{1/2}}\frac{q^n}{1+q^{2n}}, \quad q = \exp(-\pi K'/K).$$

With the aid of (31) we find the diffusion coefficient

$$D_I = \langle I^{(1)}I^{(1)}\rangle = \frac{\beta^2}{2\alpha}\int_0^{\infty} d\tau'\left\langle\frac{\partial\xi^{(0)}}{\partial\theta}(\tau)\frac{\partial\xi^{(0)}}{\partial\theta}(\tau')\right\rangle_\theta e^{\alpha(\tau'-\tau)}$$

$$= \frac{\beta^2}{4a}\sum_{n=1}^{\infty}\frac{(nA_n)^2}{\alpha^2+(n\Omega)^2}. \tag{34}$$

The angle brackets labeled θ denote here averaging over the initial phase.

Far from the resonance region ($\xi \gg 1/a^{1/2}$) we have $I \to \xi$ and accordingly D_I goes over into D_ξ. This can be easily verified by recognizing that $A_n \sim (a\xi^2)^{-n+1/2}$, and therefore at $\xi \gg 1/a^{1/2}$ we need retain in (34) only the first term of the series

$$D_\xi \approx \frac{\beta^2}{4w^2(\xi)}\frac{1}{\alpha^2+w^2(\xi)}. \tag{35}$$

We have used here the equalities $\Omega \approx 2ka^{1/2} \approx a\xi \approx w(\xi)$.

Depending on the ratio of α to $w(\xi)$, Eq. (35) goes over into (21) or (22). The latter occurs only if $\beta \ll 2^{1/2}\alpha w^{1/2}$. This restriction is quite natural, since the successive-approximation method used in this section is valid at sufficiently small values of β.

Expression (35) is the first term of the expansion in terms of the square of the ratio of the amplitude of the electron displacement $\delta\xi \sim 1/a\xi$ in the flute-oscillation field, to the distance ξ from the resonance point. At $\delta\xi \ll \xi$ the inhomogeneity of the system can be regarded as weak and quite naturally it is possible to use in this case the expressions obtained in Sec. 2B, taking parametrically into account the dependence of the electric field on the coordinate.

It was assumed in this analysis that the Coulomb collisions have little effect on the electron motion. This assumption does not hold in the vicinity of the separatrix between the trapped and untrapped particles, where $\Omega(I) \to 0$. We shall show later on, however, (see Sec. 3D) that this region makes small contributions to the quantities of real significance in the experiment.

C. Dynamic Friction Coefficient. Diffusion Equation

The Fokker-Planck equation contains, besides the diffusion coefficient, also the dynamic friction coefficient. We have shown above that if the electric field is homogeneous then the dynamic friction coefficient vanishes. Let us determine this coefficient for the case when the inhomogeneity of the electric field can be taken into account parametrically, i.e., for the frequent-collision regime, and also for the rare-collision regime far from the resonance region.

In the zeroth approximation in the inhomogeneity we have $\xi_0 = \text{const}$ and $\eta_0(\tau)$ is given by (11). The next approximation is determined by the equations

$$\dot{\xi}_1 = -\sin\eta_0(\tau), \quad (36)$$

$$\dot{\eta}_1 = a\xi_1. \quad (37)$$

In the second approximation we shall need only the equation

$$\dot{\xi}_2 = -\eta_1(\tau)\cos\eta_0(\tau). \quad (38)$$

Averaging (38) over the statistical ensemble we obtain

$$F_\xi \approx \langle \dot{\xi}_2 \rangle = -\frac{a}{2}\int_{-\infty}^{\tau} a\tau'(\tau'-\tau)\langle \sin(\eta_0(\tau')-\eta_0(\tau))\rangle. \quad (39)$$

On the other hand, the expression for the diffusion coefficient can be represented in the form

$$D_\xi = \langle \xi_1 \dot{\xi}_1 \rangle = \frac{1}{2}\int_{-\infty}^{\tau} d\tau' \langle \cos(\eta_0(\tau')-\eta_0(\tau))\rangle. \quad (40)$$

Since we are considering large time intervals $\tau \gg \tau^*$, the lower limit of integration in (39) and (40) was made infinite.

It follows from (39) and (40) that the diffusion coefficient and the dynamic-friction coefficient satisfy the relation $F_\xi = dD_\xi/d\xi$, and consequently the Fokker-Planck equation takes the form (cf. (23))

$$\frac{\partial f}{\partial t} - \frac{\partial}{\partial \xi} D_\xi \frac{\partial f}{\partial \xi} = Q(\xi, t). \quad (41)$$

In the rare-collision regimes it is convenient to use the action variables in the vicinity of the resonance point. Simple but much lengthier calculations lead to the relation $F_I = dD_I dI$, and consequently the diffusion equation in terms of the action variables takes a form analogous to (41). The use of a diffusion equation that contains a single variable is possible if rapid averaging over the phase takes place. It can be shown that the equation for the variance in θ contains a term $(\Omega\tau)^2 d_I$, where d_I is the variance in I. Thus, in the rare-collision regime $\Omega\tau^* \gg 1$ of interest to us the distribution function should indeed become very rapidly smeared out with respect to the angle θ.

D. Averaged Diffusion Coefficient

The experimental data usually include information on a certain averaged diffusion coefficient. We shall calculate it for the following model problem: We consider a plasma layer $\xi_1 \le \xi \le \xi_2$. We assume that there is no flux of particles through the left-hand boundary $\xi = \xi_1$, which corresponds to the center of an axially-symmetrical system, and that all the particles are absorbed on the right-hand boundary $\xi = \xi_2$ (the wall). We consider a stationary process $\partial f/\partial t = 0$ at constant $Q(\xi, t)$. We introduce an average diffusion coefficient $\bar{D}_\xi$ defined in such a way that the correct value of the difference $\Delta f = f(\xi_2) - f(\xi_1)$ between the distribution functions is obtained in an equivalent homogeneous system having the same dimension $\Delta\xi = \xi_2 - \xi_1$ at the same value of the source function. Simple calculations yield

$$\bar{D}_\xi = \frac{1}{2}(\xi_2-\xi_1)^2 \left(\int_{\xi_1}^{\xi_2} \frac{\xi\, d\xi}{D_\xi}\right)^{-1}. \quad (42)$$

It follows from (42) that the smaller the value of $D_\xi(\xi)$ in a given region the larger the contribution made by this region to $\bar{D}_\xi$. This result is perfectly natural, since regions with the smallest values of D_ξ offer the greatest resistance to the flow—the particles are delayed in them for the longest time.

If the resonance point falls in the interval (ξ_1, ξ_2), then expression (42) is valid only for the frequent-collision regime. The rare-collision regime calls for a special investigation. Assume that the zone occupied by the trapped particles constitutes a small fraction of the system $\xi \gg a^{-1/2}$ and is located inside the system. In this case the electron trajectories at the boundary ($\xi = \xi_{1,2}$) are close to straight lines. We hope that the results obtained in the investigation of problem describe, at least qualitatively, the greater part of the possible experimental situations. Indeed, if the plasma is bounded by a metallic wall, then, no matter how large the resonance zone and no matter where it is located, the electric potential still vanishes at the edge of the plasma, and consequently the trajectories of the electrons are close to straight lines. In this region, the electron displacements transverse to the magnetic field are minimal, and this is why it is precisely this region which determines the average diffusion coefficient.

Two circumstances play an important role in what follows: first, at the plasma boundary ($|\xi| \gg a^{-1/2}$) we have the approximate equality $\xi \approx I$, and consequently $f(\xi) \approx f(I)$; second, since the transformation $\xi, \eta \to I, \theta$ is canonical, we have $Q(\xi, \eta) = Q(I, \theta)$.

To calculate the average diffusion coefficient, just as in the frequent-collision regime (see above), we obtain the difference $\Delta f = f(\xi_2) - f(\xi_1)$. In the stationary case it is determined by the untrapped particles, whereas in the trapping region we have $f = \text{const}$. We explain this situation in Fig. 4. It shows the action I as a function of the value $\overline{\xi}^{(0)} = \Omega$ average on the trajectory (see (33)). For the trapped particles $\xi^{(0)} = 0$ and therefore they lie on the ordinate-axis segment $0 < I < 8/\pi$. It follows from Fig. 4 that a particle can go over from the region $\overline{\xi}^{(0)} < 0$ into the region $\overline{\xi}^{(0)} > 0$, bypassing the trapping region. On the other hand, if it is trapped by the wave, then simultaneously another particle must go from the trapped particles to the untrapped ones. For this reason, the total flux through the trapping region is zero, and f(I) is constant in this region. The contribution of the trapped particles to Δf is obtained from an equation similar to (41):

$$\Delta f = -2Q\int_{8/\pi}^{\xi_2} \frac{I\, dI}{D_I}. \quad (43)$$

We consider here for simplicity the symmetrical system $\xi_1 = -\xi_2$. At the same time, in the homogeneous case we would obviously have $\Delta f = 2Q\xi_2^2/\bar{D}$. Comparing this expression with (42), we get

$$\bar{D} = \xi_2^2 \left(\int_{8/\pi}^{\xi_2} \frac{I\, dI}{D_I}\right)^{-1}. \quad (42')$$

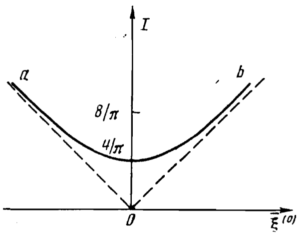

FIG. 4. Dependence of the action I on the average coordinate $\xi^{(0)}$ on the trajectory. The ordinate segment (0, 8/π) corresponds to the trapped particles, and curve ab to the untrapped ones.

This expression is similar to (42). The difference in the numerical coefficient is due to the fact that by virtue of the symmetry of the system the integration is only over the region $\xi > 0$.

The diffusion coefficient D_I decreases quite abruptly with increasing distance from the region occupied by the trapped particles (see (35)). Therefore the integral (43), just like (42′) is determined by the region at the plasma boundary $\xi \approx \xi_2$:

$$\bar{D} \approx \frac{\beta^2}{w^2(\xi_2)} \frac{1}{\alpha^2 + {}^2/_3 w^2(\xi)}. \qquad (44)$$

CONCLUSION

We have thus shown that in the presence of flute oscillations the Coulomb collisions can lead to an anomalously fast diffusion of the plasma. Expressions have been obtained for the diffusion coefficient (see (19)—(22), (35), (42), (44)) and can be used for comparison with the experimental data.

In particular, an estimate of the diffusion coefficient with the aid of (21), which should be used under the experimental conditions, yields $D = D_{cl}(10^2-10^4)$, which does not contradict the experimental data. The scatter in the values of the diffusion coefficient is due to inaccuracy of the available data on the electron temperature and the ambipolar electric potential. It must be remembered at the same time that our results cannot claim only order-of-magnitude accuracy, inasmuch as the calculations were simplified by replacing the axial symmetry by planar symmetry, and we used the model equation (5) to take the Coulomb collisions into account.

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[1)]Replacement of the lower integration limit by $-\infty$ is valid when we consider slow processes with a characteristic time scale $t_0 \gg \nu^{-1}$ ($\alpha_0 \gg 1$).

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