The relaxation of a nonlinear oscillator interacting with acoustic vibrations in a solid

A. A. Ovchinnikov and N. S. Érikhman

L. Ya. Karpov Physico-chemical Research Institute
(Submitted January 15, 1975)
Zh. Eksp. Teor. Fiz. **69**, 266–273 (July 1975)

The problem of the relaxation of a high-frequency oscillator interacting nonlinearly with the acoustic vibrations in a solid is considered. The problem is solved in the framework of classical mechanics. It is shown that at zero temperature there is no damping of the vibrations. On the other hand, in the case $T>0$ the vibrations of the high-frequency oscillator are damped, the damping constant Γ being anomalously small ($\sim 10^2\ \mathrm{sec}^{-1}$) compared with the characteristic frequencies of the system ω and Ω_D ($\sim 10^{12}-10^{14}\ \mathrm{sec}^{-1}$), and also compared with the interaction parameter $(\mu\lambda)^{1/2}\sim 10^{10}-10^{12}\ \mathrm{sec}^{-1}$. It is shown that under the condition $\kappa=\omega/\Omega_D\gg 1$ the function $\Gamma(T)$ has the form $\Gamma\sim T^{\kappa}$ and coincides with exponential accuracy with the temperature dependence obtained in the quantum method of calculation.

PACS numbers: 62.90.

The problem of the relaxation of a linear oscillator coupled with a continuous medium has been the subject of numerous studies. We shall mention here the work of Bogolyubov[1], in which it is rigorously shown that in a harmonic solid an excited linear oscillator does not lose energy if its frequency ω is greater than the Debye frequency Ω_D of the crystal. It is interesting to determine the conditions for the absence of energy dissipation for an excited oscillator coupled with a medium by a weak nonlinear interaction. It then turns out that when the temperature of the crystal equals zero there is no energy dissipation (in the framework of classical mechanics) if the condition $\omega>\Omega_D$ is fulfilled (in this sense, the conclusion of the previous papers[2,3] is confirmed). But if $T>0$ (in classical mechanics, a nonzero temperature means that the energy of the crystal $E\sim TV$), this leads at once to energy relaxation, which is, however, fairly slow. If, e.g., $\kappa=\omega/\Omega_D\gg 1$ and $T/\epsilon\ll 1$ (ϵ is the energy of the oscillator when the vibration amplitude is of the order of the range of the interaction of the oscillator with the medium), the relaxation time τ behaves like $\tau\sim(T/\epsilon)^{-\kappa}$. It must be assumed, however, that $T/\hbar\Omega_D>1$, in order that the classical limit have meaning.

In the framework of quantum mechanics relaxation occurs even when $T=0$, since in quantum mechanics all processes in which energy is conserved are allowed. In the first two sections an equation for determining the asymptotic solutions is obtained and it is shown that the imaginary part of the oscillator frequency (or damping constant) equals zero at $T=0$. In the third section the lifetime is determined for $T>0$. In particular, this time is expressed in terms of the correlation two-time Green function of the solid. It is noted also that the temperature dependences of the lifetime of the high-frequency vibration in the quantum and classical cases coincide with exponential accuracy ($\tau\sim T^{-\kappa}$).

1. DERIVATION OF THE EQUATIONS DETERMINING THE BEHAVIOR OF THE SYSTEM AS $t\to\infty$

We shall consider a system consisting of a high-frequency oscillator interacting weakly with a set of low-frequency oscillators ($\omega_0\gg\Omega_D=\max\{\Omega_k\}$). In such a model the low-frequency oscillators describe the acoustic lattice vibrations and the high-frequency oscillator corresponds to an optical vibration excited in some cell in the crystal. The equations of motion of such a system have the form1)

$$\ddot{x}+\omega_0^2x=\mu\left(\frac{m}{m_0}\right)^{1/2}U\left(\frac{1}{N^{1/2}}\sum_{k=1}^{N}\gamma_kq_k\right)$$

$$\ddot{q}_k+\Omega_k^2q_k=\mu\left(\frac{m_0}{m}\right)^{1/2}x\frac{\partial}{\partial q_k}\left\{U\left(\frac{1}{N^{1/2}}\sum_{k=1}^{N}\gamma_kq_k\right)\right\}. \quad (1)$$

Eliminating the coordinates of the medium with the aid of the second equation, we obtain

$$\ddot{x}+\omega_0^2x=\mu\left(\frac{m}{m_0}\right)^{1/2}U(q),$$

$$q=\mu\left(\frac{m_0}{m}\right)^{1/2}\int_0^t M(t-\tau)x(\tau)\frac{\partial U}{\partial q}(q(\tau))d\tau+Q(t), \quad (2)$$

where

$$q=\frac{1}{N^{1/2}}\sum_{k=1}^{N}\gamma_kq_k(t),$$

$$Q(t)=\frac{1}{N^{1/2}}\sum_{k=1}^{N}\gamma_kA_k\cos(\Omega_kt+\varphi_k), \quad (3)$$

$$M(\tau)=\frac{1}{N}\sum_{k=1}^{N}\frac{\gamma_k^2}{\Omega_k}\sin(\Omega_k\tau)\approx\int_0^{\Omega_D}\gamma_\Omega^2\frac{\sin(\Omega\tau)}{\Omega}\rho(\Omega)d\Omega=\int_0^{\Omega_D}M(\Omega)\sin(\Omega\tau)d\Omega.$$

Eq. (2) is a very complicated nonlinear integro-differential equation. We shall assume that the solution of the system (2) can be represented in the form of a sum of two functions:

$$x(t)=x_p(t)+y(t),\quad q(t)=q_p(t)+r(t), \quad (4)$$

where $x_p(t)$ and $q_p(t)$ are periodic functions with the same period, and $y(t)$ and $r(t)$ tend to zero in a power-law manner as $t\to\infty$. It is obvious that $x_p(t)$ and $q_p(t)$ determine the asymptotic behavior of the solutions as $t\to\infty$, which will be of main interest to us in the following.

In order to obtain equations for the functions $x_p(t)$ and $q_p(t)$, we substitute (4) into (2). As a result we obtain

$$\ddot{x}_p+\omega_0^2x_p+\ddot{y}+\omega_0^2y=\mu\left(\frac{m}{m_0}\right)^{1/2}U(q_p+r),$$

$$q_p+r=\mu\left(\frac{m_0}{m}\right)^{1/2}\int_0^t M(t-\tau)[x_p(\tau)+y(\tau)]\frac{\partial U}{\partial q}(q_p(\tau)+r(\tau))d\tau+Q(t). \quad (5)$$

In the following, for the sake of simplicity, we shall assume that $m=m_0$, which in no way limits the generality.

Taking into account that $x_p(t)$ and $q_p(t)$ are periodic functions and are, consequently, defined for all t, and also that $M(\tau)$ too is defined for all times, as follows from (3), we rewrite the system (5) in the form

$$\ddot{x}_p+\omega_0^2x_p=\mu U(q_p)+\mu\{U(q_p+r)-U(q_p)\}-\{\ddot{y}+\omega_0^2y\},$$

$$q_p=\mu\int_0^\infty M(\tau)x_p(t-\tau)\frac{\partial U}{\partial q}(q_p(t-\tau))d\tau+\Big\{Q(t)-r(t)-\mu\int_t^\infty M(\tau)x_p(t-\tau)$$

$$\times\frac{\partial U}{\partial q}(q_p(t-\tau))d\tau+\mu\int_0^t M(t-\tau)\Big[(x_p(\tau)+y(\tau))\Big(\frac{\partial U}{\partial q}(q_p(\tau)+r(\tau))$$

$$-\frac{\partial U}{\partial q}(q_p(\tau))\Big)+y(\tau)\frac{\partial U}{\partial q}(q_p(\tau))\Big]d\tau\Big\}. \quad (6)$$

As $t\to\infty$ the expressions in the curly brackets tend to zero, since, as $\tau\to\infty$, $M(\tau)$ and $Q(\tau)\to 0$ by definition[2)], and $y(t)$ and $r(t)\to 0$ by assumption. As a result of passing to the limit $t\to\infty$, for the periodic functions $x_p(t)$ and $q_p(t)$ at large t we obtain a system of the equations of the form

$$\ddot{x}_p+\omega_0^2x_p=\mu U(q_p),$$

$$q_p=\mu\int_0^\infty M(\tau)x_p(t-\tau)\frac{\partial U}{\partial q}(q_p(t-\tau))d\tau. \quad (7)$$

We shall demand that $x_p(t)$ and $q_p(t)$ satisfy Eqs. (7) for all times. Then, finding x_p and q_p from (7) and substituting them into (6), we obtain an equation for $y(t)$ and $r(t)$ with the boundary conditions $y(t)\to 0$ and $r(t)\to 0$ as $t\to\infty$. It can be shown that, to within terms of order μ^3, such an approach is consistent: for certain restrictions on $M(t)$ (see below), Eq. (7) has periodic solutions, and the system (6) has solutions $y(t)$ and $r(t)$ that fall off in a power-law manner as $t\to\infty$.

In the case when $U(q)$ is a linear function, Eq. (6) can be solved excatly and the solution can be represented in the form of a sum of a periodic function and a function decreasing as $t\to\infty$. It is of importance here that the periodic part can be obtained by solving the system (7). Moreover, if the linear system $(U(q)=\lambda q)$ does not have a purely periodic component but has an oscillating solution damping exponentially with a small damping constant $\sim\mu^2$, then this solution can be found, again by solving the system (7), and the damping constant turns out to be equal to the imaginary contribution to the frequency (in this case, the system (7) also has no purely periodic solutions).

The above properties of linear systems, and also the fact that these properties also hold for nonlinear systems, at least to accuracy $\sim\mu^3$, make it possible to find the asymptotic form of the solutions from Eqs. (7). The frequency of the periodic solution of the system (7) can turn out to be complex, generally speaking. In this case, its imaginary part will determine the damping constant (growth constant) of the solutions of the original system (2) as $t\to\infty$. Taking the above considerations into account, in studying the behavior at large times of a high-frequency oscillator interacting nonlinearly with a set of low-frequency oscillators, we shall start from Eq. (7).

The only difference in the following will be the fact that for $T>0$ the function $Q(t)$ is assumed to be random and therefore does not tend to zero as $t\to\infty$. This means that in the second equation of the systems (6) and (7) the term with $Q(t)$ must be retained on passing to the limit $t\to\infty$ (see below).

2. DETERMINATION OF THE FREQUENCY OF THE VIBRATIONS AT T = 0

We shall find the periodic solutions of the system (7) and elucidate the restriction on the kernel $M(t-\tau)$ that guarantees the existence of such solutions. In solving (7) we shall use a method described in the work of Bogolyubov on nonlinear mechanics[4]. We expand the solution in a Fourier series[3)]:

$$x=\sum_{n=-\infty}^{+\infty}x_ne^{in\omega t},\quad q=\sum_{n=-\infty}^{+\infty}q_ne^{in\omega t},\quad U(q)=\sum_{n=-\infty}^{+\infty}U_ne^{in\omega t}, \quad (8)$$

where

$$U(q)=\sum_{\nu=0}^{\infty}a_\nu q^\nu,\quad U_n=\sum_{\nu=0}^{\infty}a_\nu\sum_{n_1,\ldots,n_\nu=-\infty}^{+\infty}\delta\Big(n-\sum_{\lambda=1}^{\nu}n_\lambda\Big)q_{n_1}q_{n_2}\ldots q_{n_\nu}.$$

Substituting (8) into (7), we arrive at the following nonlinear system for the Fourier coefficient x_n and q_n:

$$(\omega_0^2-n^2\omega^2)x_n=\mu\Big\{a_0\delta(n)+a_1q_n+a_2\sum_{n_1,n_2=-\infty}^{+\infty}\delta(n-n_1-n_2)q_{n_1}q_{n_2}+\ldots\Big\},$$

$$q_n=\mu M_n(\omega)\Big\{a_1x_n+2a_2\sum_{m,n_1=-\infty}^{+\infty}\delta(n-m-n_1)x_mq_{n_1}$$

$$+3a_3\sum_{m,n_1,n_2=-\infty}^{+\infty}\delta(n-m-n_1-n_2)x_mq_{n_1}q_{n_2}+\ldots\Big\}, \quad (9)$$

where

$$M_n(\omega)=\int_0^\infty M(\tau)e^{-in\omega\tau}d\tau. \quad (10)$$

We seek the solution in the form of a series in powers of μ:

$$x_n=x_n^{(0)}+x_n^{(1)}+\ldots,\quad q_n=q_n^{(0)}+q_n^{(1)}+\ldots,\quad \omega=\omega_0+\omega^{(1)}+\omega^{(2)}+\ldots \quad (11)$$

Substituting (11) into (9), we can obtain the solution of the equations to any order in μ. We shall write out the expressions for the frequency ω, accurate to terms $\sim\mu^2$:

$$\omega=\omega_0-{}^1/_2\mu^2a_1^2M_1(\omega_0). \quad (12)$$

It can be seen from formula (12) that if the quantity $M_1(\omega_0)$ is complex there are no periodic solutions. It is not difficult to convince oneself that, in any order in μ, imaginary frequencies will appear only in the case when the $M_n(\omega)$ are complex. From this it follows that the question of the existence of periodic solutions reduces to studying the reality of the functions $M_n(\omega)$.

Substituting (3) into (10), we find that $\operatorname{Im}M_n(\omega)$ is given by the following expression:

$$\operatorname{Im}M_n(\omega)=\begin{cases}0, & n\omega-\Omega_D>0\\ -{}^1/_2\pi M(n\omega), & -\Omega_D<n\omega-\Omega_D<0.\end{cases} \quad (13)$$

It follows from (13) that in the case when

$$\omega=\omega_0+\omega^{(1)}+\omega^{(2)}+\ldots>\Omega_D, \quad (14)$$

the quantities $M_n(\omega)$ are purely real, and, consequently, a periodic solution exists. On the other hand, if

$$\omega_0<\Omega_D, \quad (15)$$

then, for sufficiently small μ, certain of the quantities $M_n(\omega)$ will turn out to be complex and there will be no periodic solutions. In this case, the damping constant Γ, to second order in μ, is equal to

$$\Gamma=\operatorname{Im}\omega=-{}^1/_2\mu^2a_1^2\operatorname{Im}M_1(\omega_0). \quad (16)$$

From the treatment given it can be seen that a high-frequency oscillator interacting nonlinearly with a set of low-frequency weakly-excited oscillators will perform undamped oscillations (the large frequency difference guarantees the fulfilment of (14), while the small degree of excitation of the low frequencies ensures that $Q(t) = 0$ in Eq. (2)).

3. DETERMINATION OF THE RATE OF DAMPING OF THE OSCILLATIONS IN THE CASE $T > 0$

We shall consider the case $Q(t) \neq 0$. This corresponds to the situation in which the low-frequency oscillators are excited, i.e., the temperature T is nonzero. This implies that the function Q(t) is random. We shall take the law by which Q(t) is distributed to be the following: the amplitudes A_k and phases $\varphi_{k'}$ determining Q(t) (cf. (3)) are independent for all k and k', and the phase φ_k is uniformly distributed. From this definition it follows that the correlator $\langle Q(t-\tau)Q(t)\rangle$ has the form

$$\langle Q(t-\tau)Q(t)\rangle = \frac{1}{2N}\sum_{k=1}^{N}\gamma_k^2\langle A_k^2\rangle\cos(\Omega_k\tau) \underset{N\to\infty}{\longrightarrow} \frac{1}{2}\int_0^{\Omega_D} Q^2(\alpha)\cos(\alpha\tau)\rho(\alpha)\,d\alpha,$$
$$Q^2(\Omega_k) = \gamma_k^2\langle A_k^2\rangle. \tag{17}$$

Since, with this definition of the function Q(t), it does not tend to zero as $t\to\infty$, Eqs. (7) in this case have the form

$$\ddot{x}+x=\mu U(q),$$
$$q=Q(t)+\mu\int_0^\infty M(\tau)x(t-\tau)\frac{\partial U}{\partial q}(q(t-\tau))\,d\tau, \tag{18}$$

where the frequency $\omega_0 = 1$. We shall seek x(t) to terms of order μ^2. To this end, we find the function q(t) to order μ from the second equation of the system (18) and substitute it into the first equation, as a result of which we have for x(t) the equation

$$\ddot{x}+x=\mu U\left\{Q(t)+\mu\int_0^\infty M(\tau)x(t-\tau)\frac{\partial U}{\partial q}(Q(t-\tau))\,d\tau\right\}. \tag{19}$$

Expanding U(q) in the right-hand side of (19) in a power series and retaining terms $\sim\mu^2$, we transform Eq. (19) to the form

$$\ddot{x}+x=\mu\sum_{\nu=0}^\infty a_\nu Q^\nu(t)+\mu^2\sum_{\substack{\nu=0\\ s=0}}^\infty \nu s a_\nu a_s\int_0^\infty M(\tau)x(t-\tau)Q^{\nu-1}(t)Q^{s-1}(t-\tau)\,d\tau. \tag{20}$$

It can be seen from (20) that the solution x(t) is a random function, and the random correction is $\sim\mu$. This circumstance enables us, in the statistical averaging of Eq. (20), to make use of the following relation:

$$\langle x(t-\tau)Q^\nu(t-\tau)Q^s(t)\rangle = \langle x(t-\tau)\rangle\langle Q^\nu(t-\tau)Q^s(t)\rangle. \tag{21}$$

Replacement of the averages in accordance with formula (21) leads to an error $\sim\mu^3$ and does not take us outside the limits of the assigned accuracy $\sim\mu^2$. Furthermore, from the definition of the random function Q(t) (cf. (3) and (17)) it follows that

$$\langle Q^\nu(t)\rangle=\text{const},\quad \langle Q^\nu(t)Q^s(t-\tau)\rangle=\langle Q^\nu(\tau)Q^s(0)\rangle. \tag{22}$$

Carrying out the statistical averaging of Eq. (20) and taking the relations (21) and (22) into account, for the averaged quantity $\langle x(t)\rangle$ we obtain the following equation:

$$\langle\ddot{x}\rangle+\langle x\rangle=\mu^2\sum_{\substack{\nu=0\\ s=0}}^\infty \nu s a_\nu a_s\int_0^\infty M(\tau)\langle x(t-\tau)\rangle\langle Q^{\nu-1}(\tau)Q^{s-1}(0)\rangle\,d\tau. \tag{23}$$

In the right-hand side of Eq. (23) we have omitted the term that does not depend on the time, since it leads only to a negligible ($\sim\mu$) shift of the averaged amplitude $\langle x(t)\rangle$. We shall solve (23) by perturbation theory. Putting $\langle x^{(0)}(t)\rangle = Ae^{it} + A^*e^{-it}$, we obtain for the correction $\omega^{(2)}\sim\mu^2$ to the frequency the expression

$$\omega^{(2)}=-\frac{\mu^2}{2}\sum_{\substack{\nu=0\\ s=0}}^\infty(\nu+1)(s+1)a_{\nu+1}a_{s+1}\int_0^\infty M(\tau)e^{-i\tau}\langle Q^\nu(\tau)Q^s(0)\rangle\,d\tau. \tag{24}$$

The terms of the series in (24) will be complex for sufficiently large ν and s. In fact, from the definition of the functions $Q(\tau)$ and $\langle Q^\nu(\tau)Q^s(0)\rangle$ (cf. (3) and (17)) it follows that the most rapidly oscillating part of the correlator in the integrand of (24) has the form

$$\langle Q^\nu(\tau)Q^s(0)\rangle=\frac{N!\cos^n(\Omega_D\tau)\langle Q^{|\nu-s|}(0)\rangle Q^{2n}(\Omega_D)}{2^n(N-n)!}, \tag{25}$$

where

$$n=\min(\nu,s),\quad N=\max(\nu,s).$$

Taking (13) and (25) into account, we arrive at the following relation:

$$\operatorname{Im}\int_0^\infty M(\tau)e^{-i\tau}\langle Q^\nu(\tau)Q^s(0)\rangle\,d\tau\neq0, \tag{26}$$

if $0 < 1 - n\Omega_D < \Omega_D$.

It is clear that the condition (26) will be fulfilled for sufficiently large ν and s, and the corresponding terms of the series (24) will be complex, indicating the absence of periodic solutions. We shall assume that the coefficients a_ν fall off sufficiently rapidly with the index ν. In this case, the imaginary part of the frequency $\omega^{(2)}$ is determined by that term in the series for which the sum $\nu + s$ is a minimum and, at the same time, the condition (26) is fulfilled. It is obvious that this will be the term in the series with $\nu = s = n$. Taking account of what has been said, for the damping constant Γ we obtain the expression

$$\Gamma=\operatorname{Im}\omega^{(2)}=-{}^1/_2\mu^2(n+1)^2a_{n+1}^2\left(\operatorname{Im}\int_0^\infty M(\tau)e^{-i\tau}\right.$$
$$\left.\times Q^n(\tau)Q^n(0)\rangle\,d\tau\right)=\frac{\pi\mu^2(n+1)^2n!a_{n+1}^2}{2^{2n+1}}\int_0^{\Omega_D}\rho(\alpha_1)\,d\alpha_1\ldots$$
$$\ldots\int_0^{\Omega_D}\rho(\alpha_n)\,d\alpha_n\,Q^2(\alpha_1)\ldots Q^2(\alpha_n)M\left(1-\sum_1^n\alpha_k\right)\theta\left(1-\sum_1^n\alpha_k\right), \tag{27}$$

where $\theta(\alpha)$ is the usual theta-function and $M(\alpha)$ and $Q^2(\alpha)$ are defined by formulas (3) and (17) respectively. In the integration over the variable τ in formula (27) the representation (25) of the correlator has been used.

We put

$$\rho(\alpha)=3\alpha^2/\Omega_D^3,\quad Q^2(\alpha)=2T/m\alpha^2,\quad M(\alpha)=\rho(\alpha)/\alpha. \tag{28}$$

This corresponds to $\gamma_k = 1$ in (3) and (17), and to ordinary classical statistics, in which the mean energy of an oscillator with frequency α is equal to $m\alpha^2Q^2(\alpha)/2 = T$. Substituting (28) into (27) and performing the integration, we obtain

$$\Gamma={}^1/_2\pi\mu^2(n+1)a_{n+1}^2\,(3T/2m\Omega_D^2)^n\{(n+1)\Omega_D-1\}^n(3/\Omega_D^3). \tag{29}$$

We put $U(q) = e^{\lambda q}$; then $a_n = \lambda^n/n!$. Substituting this expression for a_n into (28) and taking into account that, according to (26), $n = [1/\Omega_D] \approx 1/\Omega_D$, and also $1/\Omega_D \gg 1$, we obtain for the damping constant Γ the formula

$$\Gamma=\frac{3}{4}\frac{\mu^2\lambda^2}{\Omega_D}\left(\frac{3e^2\lambda^2T}{2m}\right)^{1/\Omega_D}. \tag{30}$$

In formula (30) the frequency of the high-frequency oscillator is $\omega = 1$, and therefore the quantity Γ is dimensionless.

In order to write Γ in $\sec^{-1}$ it is convenient to introduce the following notation:

$$\beta=\mu\lambda/\omega(\omega\Omega_D)^{1/2},\quad \Delta=(3e^2\lambda^2T/2m\omega^2)^{1/2}. \quad (31)$$

In formula (31) all parameters have cgse dimensions, including μ–[cm · sec $^{-2}$] (these dimensions are determined from Eq. (1)). Taking (31) into account, we obtain for the damping constant Γ the following expression:

$$\Gamma={}^3/_4\omega\beta^2(\Delta^2)^{\varkappa}. \quad (32)$$

It is interesting to note that the quantum calculation by perturbation theory leads, for large values of the ratio $\omega/\Omega_D \gg 1$, to exactly the same temperature dependence $\Gamma \sim T^{\kappa}$.

We shall estimate the magnitude of Γ. For real molecular crystals the parameters appearing in (31) have the following orders of magnitude[5]:

$$\frac{\omega}{\Omega_D}\sim 5,\quad \lambda\sim 10^8\ \text{cm}^{-1},\quad \beta\sim\frac{\mu\lambda}{\omega(\omega\Omega_D)^{1/2}}\sim\frac{\Omega_D{}^2\lambda^{-1}\lambda}{\omega(\omega\Omega_D)^{1/2}}\sim\left(\frac{\Omega_D}{\omega}\right)^{3/2}\sim 0.1,$$

$$\Delta\sim\left(\frac{3e^2\lambda^2T}{2m\omega^2}\right)^{1/2}\sim\left(\frac{10T}{m\omega^2\lambda^{-2}}\right)^{1/2}\sim 0{,}1,\quad \omega\sim 10^{14}\ \sec^{-1} \quad (33)$$

Substituting (33) into (31), and then into (32), we obtain $\Gamma \sim 10^2\ \sec^{-1}$. Inasmuch as we have obtained an anomalously small value of Γ, it is necessary to estimate the contribution to Γ of higher orders in the parameter μ. To do this, we note that the expression (32) for Γ is a product of powers of two small quantities β^2 and Δ^2, the total power of which is equal to $\omega/\Omega_D + 1 = n + 1$. This structure of the expression for Γ is a consequence of the fact that damping of the high-frequency vibrations occurs because of their resonance with the (n + 1)-st harmonic of the low-frequency vibrations, the coupling with these being provided by the part of the nonlinear interaction with the form $\mu x(\lambda q)^{n+1}$. It is obvious that a coupling between the indicated resonances also arises in the (n + 1)-st iteration of the linear part of the interaction, of the form $\mu x\lambda q$. As a result of taking this part of the interaction into account, we obtain for the damping constant Γ an expression of the type (32), in which the powers of the quantities β^2 and Δ^2 have exchanged places, i.e.,

$$\Gamma\sim\omega(\beta^2)^{\varkappa}\Delta^2. \quad (34)$$

Taking (33) into account, we see that (34) gives the same order of magnitude ($10^2\ \sec^{-1}$) for the damping constant as (32).

The investigation carried out shows that the rate of dissipation of energy of a high-frequency oscillator interacting nonlinearly with acoustic vibrations is anomalously small compared with the characteristic frequencies ω and Ω_D of the system, and also compared with the interaction parameter $(\mu\lambda)^{1/2}$, and tends to zero as $T \to 0$.

---

1)In writing out Eq. (1) we have omitted all terms that are anharmonic in the coordinate x, since their contribution to the damping will be $\sim(m_{el}/m_{nuc})^{1/2}\Gamma$, where Γ is the damping constant, calculated below, and m_{el} and m_{nuc} are the electron and nuclear masses. For the same reason we can disregard the effect of the other electronic terms. The arguments given are the consequence of the Born-Oppenheimer adiabatic theory.

2)The function $Q(t) \to 0$ as $t \to \infty$ only in the case when the coefficients A_k and φ_k are smooth functions of the index k (cf. (3)). Below we shall assume them to be random quantities, and in this case Q(t) does not tend to zero as $t \to \infty$ (see below).

3)Everywhere below we shall write simply x(t) and q(t), meaning the periodic functions $x_p(t)$ and $q_p(t)$.

1N. N. Bogolyubov, p. 77 in: Izbrannye trudy (Selected Works), Vol. 2, Naukova dumka, Kiev, 1970.

2A. A. Ovchinnikov and N. S. Erikhman, Zh. Eksp. Teor. Fiz. 61, 2391 (1971) [Sov. Phys.-JETP 34, 1281 (1972)].

3A. A. Ovchinnikov, Teor. Mat. Fiz. 11, 366 (1972) [Theor. Math. Phys. (USSR) 11, (1972)].

4N. N. Bogolyubov and Yu. A. Mitropol'skiĭ, Asimptoticheskie metody v teorii nelineĭnykh kolebaniĭ (Asymptotic Methods in the Theory of Nonlinear Oscillations), Fizmatgiz, M., 1963.

5Handbook "Termodinamicheskie svoĭstva individual'nykh veshchestv" (Thermodynamic Properties of Individual Substances), ed. L. V. Gurvich, Izd. AN SSSR, M., 1962.

Translated by P. J. Shepherd.
28