Influence of viscosity on the character of cosmological evolution

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The character of the cosmological solutions of a homogeneous model of Bianchi type I is investigated taking into account dissipative processes due to viscosity. It is shown that the influence of viscosity is not capable of removing the cosmological singularity, but leads to a qualitatively new behavior of the solutions near the singularity. The energy density of matter vanishes near the initial cosmological singularity and then increases during the subsequent expansion. Thus, the model under consideration exhibits an interesting property: at the instant of the "big bang" the gravitational field creates matter. It is also shown that the action of viscosity may lead to a sufficiently large accumulation of entropy, a fact that may be related to the anomalously large entropy per particle encountered in the present-day universe.

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1. INTRODUCTION

In investigating the character of the cosmological solutions in relativistic theories of gravitation one usually selects the energy-momentum tensor of matter in a form corresponding to an ideal fluid. In this case one can show [1] that near the cosmological singularity which inevitably appears, the influence of matter is negligibly small, the behavior of the gravitational field is determined by the Einstein equation in free space and is described by a complicated oscillatory regime. Formally, the character of the singularity does not depend on whether we consider it as a singularity in the past or a singularity in the future.

It is interesting to investigate the degree of universality of these conclusions with respect to generalizations of the form of the energy-momentum tensor, taking into account the behavior of matter more realistically. One of these generalizations is related to taking into account dissipative processes which are caused by viscosity. It is clear from general considerations that viscosity counteracts the cosmological collapse, and the final outcome of the evolution is not a priori obvious. Moreover, Murphy [2] has given an example of an exactly solvable flat cosmological model of the Friedmann type, taking into account the second viscosity coefficient, where in a certain sense the action of the latter removes the singularity.

We shall show that this effect is unstable and disappears when one goes over to more general, anisotropic, models. It turns out that the influence of viscosity is unable to remove the cosmological singularity, although it introduces qualitatively new elements into the character of the solution. In the presence of energy dissipation the process of cosmological evolution becomes time-irreversible, which leads to essentially different pictures of the contraction and expansion of the Universe. In the cases considered here the asymptotic behavior of the solution for the final stage of the cosmological collapse depends on the assumptions made on the form of the viscosity coefficients as functions of the energy density. A general feature, however, is the fact that the collapse occurs at some finite instant of proper time and is accompanied by the energy and curvature invariants becomining infinite. When one considers the expansion of the Universe, the singularity corresponding to the beginning of this process has a completely different character. For the physically most reasonable form of the viscosity coefficients the influence of matter near the initial singularity is negligibly small and the metric is determined by the free space Einstein equations. What is more important, at the initial instant the energy density vanishes. Then, in the course of the expansion, the energy density increases and subsequently behaves according to the Friedmann laws, which govern the model in the later stages of expansion. Thus, the model considered here exhibits a remarkable property: in the process of the appearance of the Universe the gravitational field creates the matter.

We carry out our analysis on the example of the anisotropic cosmological model of Bianchi type I, since all the basic features of the phenomena we are investigating appear already in this simple case. At the end we briefly dwell on the cosmological model of type IX.

2. INVESTIGATION OF THE EINSTEIN EQUATIONS

The metric of the homogeneous type I model has the form[1)]

$$-ds^2=-dt^2+R_1^2(t)\,dx^2+R_2^2(t)\,dy^2+R_3^2(t)\,dz^2. \tag{1}$$

The energy-momentum tensor of a viscous fluid is written in the form [3]:

$$\begin{gathered}T_{ik}=(\varepsilon+p')u_iu_k+p'g_{ik}-\eta(u_{i;k}+u_{k;i}+u^ku^lu_{i;l}+u_iu^lu^k_{;l}),\\ p'=p-\left(\zeta-\frac{2}{3}\eta\right)u^k_{;k},\quad u^ku_k=-1.\end{gathered} \tag{2}$$

Here ϵ is the energy density, η and ζ are the first and second viscosity coefficients. In the homogeneous models all these quantities depend only on the time t, and therefore we may consider them as functions only of the energy density ϵ. In order that the gravitational equations based on the metric (1) be compatible it is necessary to select an appropriate reference system in which $u^0 = 1$, $u^\alpha = 0$ (comoving frame). Then the components T_i^k are of the form

$$T_0^0=-\varepsilon,\quad T_\alpha^0=0,\quad T_\alpha^\beta=p'\delta_\alpha^\beta-\eta\varkappa_\alpha^\beta,\quad p'=p-(\zeta-{}^2/_3\eta)(\ln\sqrt{g})^\cdot. \tag{3}$$

Here g is the determinant of the metric tensor $g_{\alpha\beta}$ of Eq. (1) and $\kappa_\alpha^\beta = g^{\beta\gamma}\dot{g}_{\gamma\alpha}$, where the superior dot denotes differentiation with respect to the time t. For the sequel it will be convenient to introduce the quantities R, H, and ψ defined as follows:

$$\sqrt{g}=R_1R_2R_3=R^3,\quad (\ln R)^\cdot=H,\quad \eta=-{}^1/_2\psi. \tag{4}$$

Setting up the Einstein equations we note that the $\alpha\beta$ components of these equations admits two integrals

$$(\ln R_1)^{\cdot}-(\ln R_2)^{\cdot}=\text{const}\cdot R^{-3}e^{\psi},\quad (\ln R_1)^{\cdot}-(\ln R_3)^{\cdot}=\text{const}\cdot R^{-3}e^{\psi},$$

which can be written in the following symmetric form:

$$(\ln R_\alpha)^{\cdot}=H+s_\alpha R^{-3}e^{\psi},\quad \sum_\alpha s_\alpha=0, \tag{5}$$

where s_α are integration constants. After this only one equation remains for the $\alpha\beta$ components:

$$\dot H=\varepsilon-3H^2-{}^1/{}_2w+{}^3/{}_2\zeta H, \tag{6}$$

and the 00 component yields with the use of (6)

$$\varepsilon=3H^2-q^2R^{-6}e^{2\psi}, \tag{7}$$

where $q^2 = {}^1/{}_2(s_1^2 + s_2^2 + s_3^2)$ and w is the enthalpy: $w = \epsilon + p$. The relations (4)—(7) represent the complete set of gravitational equations for the model under consideration. In the sequel we shall need the hydrodynamic equation $T^k_{i;k} = 0$ which is contained in this system and which can be written with the help of (7) in the form

$$\dot\varepsilon=9\zeta H^2+4\eta(3H^2-\varepsilon)-3Hw. \tag{8}$$

This equation is related to the law of increase of entropy for the given energy dissipation. It is easy to show that the entropy density σ in the model under consideration equals

$$\sigma=\sigma_c\exp\left(\int\frac{d\varepsilon}{w(\varepsilon)}\right),$$

where σ_c = const., and (8) can be represented in the form

$$R^{-3}(R^3\sigma)^{\cdot}=\frac{\sigma}{w}[9\zeta H^2+4\eta(3H^2-\varepsilon)].$$

Since the quantities σ, w, η, and ζ are positive and $3H^2 \geq \epsilon$, as follows from (7), the right-hand side of this equation is positive. Thus, the law of increase of entropy admits only of evolutions that correspond to an increase of the proper time t.

To investigate the character of the solutions in the case under consideration it is convenient to turn to the qualitative theory of dynamical systems, noting that for a given dependence of the quantities w, η, and ζ on the energy density ϵ, the equations (6) and (8) describe such a system on the phase space (H, ϵ). Let us note some properties of this system. We are first of all interested in the integral curves only in the physical region of the phase plane, which is singled out by the obvious condition $0 \leq \epsilon \leq 3H^2$. This region lies between the axis $\epsilon = 0$ and the parabola $\epsilon = 3H^2$ (cf. Fig. 1—3). These boundaries themselves are also integral curves of the equations (6)—(8). We shall assume that the enthalpy $w(\epsilon)$ and the viscosity coefficients $\eta(\epsilon)$ and $\zeta(\epsilon)$ vanish as $\epsilon \to 0$. Thus, the axis $\epsilon = 0$ represents the well known vacuum solution obtained by Kasner[1]. The solution corresponding to the parabola $\epsilon = 3H^2$ is nothing else but the Friedmann solution. Indeed, from Eq. (7) it can be seen that such a solution exists only for q = 0, hence for $s_\alpha = 0$. In this case the first viscosity coefficient does not manifest itself at all, and we obtain the solution involving the second viscosity coefficient investigated by Murphy[2].

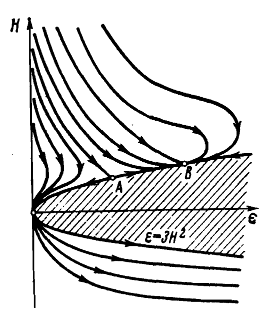

FIG. 1

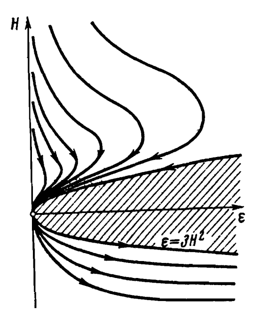

FIG. 2

FIG. 1. In the upper half-plane (expansion) all trajectories come out of the node (H, ϵ) = (+∞, 0), where matter is produced and the asymptotic behavior of the solutions is given by Eqs. (16) and (17). The evolution ends in the node (0, 0) (a purely Friedmann stage with the asymptotic behavior (15), or in the node B (with the asymptotic behavior (12) as $t \to +\infty$) with uncurtailed accumulation of entropy. In the lower half-plane (contraction) the trajectories come out of the Friedmann node (0, 0) and in general approach asymptotically the parabola $\epsilon = 3H^3$, ending the evolution in a singularity of the Friedmann type (asymptotic behavior (19)).

FIG. 2. The case when there are no singularities (H_0, ϵ_0). The expansion starts at the node (+∞, 0) with matter production and ends in a Friedmann solution at the node (0, 0). The contraction looks qualitatively like in the picture described in Fig. 1.

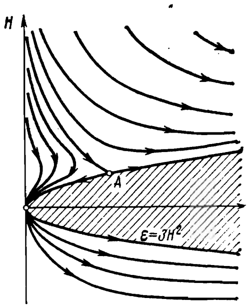

FIG. 3. The special case when the first viscosity coefficient η behaves like $\sqrt{\epsilon}$ as $\epsilon \to \infty$. The expansion starts, as before, with matter production at the node (+∞, 0) but may end in a singularity (H, ϵ) $\to$ (+∞, −∞) with the asymptotic form of the solution (22).

We now determine the singular points of the dynamical system (6), (8) corresponding to finite values of the variables H and ϵ and situated in the physical part of the phase plane. Equating to zero both right-hand sides of (6) and (8) simultaneously, we easily establish that in the physical region the singularities can be situated only on the parabola $\epsilon = 3H^2$, and moreover only in the upper half-plane $H \geq 0$ (in the lower half-plane the singularities end up in the region $\epsilon > 3H^2$). The equations that determine the positions of the singularities are $H = (\epsilon/3)^{1/2}$, $w(\epsilon) = (3\epsilon)^{1/2}\zeta(\epsilon)$. We denote the solutions of these equations by H_0 and ϵ_0, and all quantities taken at the point (H_0, ϵ_0) will carry the subscript 0. Thus, at the singularities (if they exist) we have the relations

$$\zeta_0(\varepsilon_0)=w_0(\varepsilon_0)\sqrt{3\varepsilon_0},\quad H_0=\sqrt{\varepsilon_0/3}. \tag{9}$$

It is easy to establish that the characteristic numbers λ_1 and λ_2 of the singularities (9) are real and take the form

$$\lambda_1=3\varepsilon_0\left[\frac{d}{d\varepsilon}\left(\zeta-\frac{w}{\sqrt{3\varepsilon}}\right)\right]_0,\quad \lambda_2=-6H_0-4\eta_0, \tag{10}$$

and near the singularities the solutions of the (6) and (8) are of the form

$$H=H_0+\frac{C_1}{6H_0}e^{\lambda_1 t}-\frac{C_2[2\lambda_1+12H_0-3\zeta_0]}{6H_0(8\eta_0+3\zeta_0)}e^{\lambda_2 t},$$
$$\varepsilon=\varepsilon_0+C_1e^{\lambda_1 t}+C_2e^{\lambda_2 t}, \tag{11}$$

where C_1 and C_2 are arbitrary constants. Since $H_0 > 0$ it follows that λ_2 is always negative, and the character of the singularities depends only on the sign of λ_1. For $\lambda_1 > 0$ we have a saddle point and for $\lambda_1 < 0$ we have an attractor node. Let us consider both cases in more detail.

1. The saddle point (represented by the point A in the

figures), for $\lambda_1 > 0$. Setting $C_2 = 0$ in (11) we obtain the equations of the outgoing whiskers ($t \to -\infty$) of the first separatrix. This separatrix is the Friedmann parabola $\epsilon = 3H^2$. For $C_1 = 0$ we have the equations of the incoming ($t \to +\infty$) whiskers of the second separatrix. However, the physical region contains only one whisker: the one which comes down into the saddle from the region of large positive values of H. Its slope at the saddle point is given by $(dH/d\epsilon)_0 = -(2\lambda_1 + 12H_0 - 3\zeta_0)/6H_0(8\eta_0 + 3\zeta_0)$ and one can see that $(dH/d\epsilon)_0 < 0$ for physically reasonable conditions ($d\zeta/d\epsilon > 0$, $dp/d\epsilon < 1$). The first non-vanishing terms of the asymptotic form of the solutions corresponding to the separatrices of the saddle point will be

$$H=H_0, \quad \varepsilon=\varepsilon_0, \quad R=e^{H_0t}, \quad R_\alpha=e^{H_0t} \quad (t\to\pm\infty). \qquad (12)$$

(Here and in the sequel we omit for simplicity in the asymptotic expressions the dimensionless arbitrary constants that can be removed by the scale transformation $(x, y, z) \to (c_1x, c_2y, c_3z)$.) As $t \to -\infty$ we go out of the saddle point in which all the metric coefficients vanish. However, the energy density (and also the curvature invariants) are finite here. Exactly this case, where in a certain sense the cosmological singularity is removed, has been described by Murphy[2]. We see, however, that such a solution is possible only on a single integral curve: the separatrix of the saddle point corresponding to the isotropic Friedmann case. The inclusion of an arbitrarily small anisotropy ($q \neq 0$, $s_\alpha \neq 0$) makes the existence of solutions with "singularities" of this character impossible. The second separatrix ($t \to +\infty$) describes the late stages of the infinite isotropic expansion of the Universe, but the energy density here tends not to zero, as in the usual Friedmann model, but to some finite value. At the nodal point, which we shall consider later, this effect occurs for all integral curves entering the node.

2. The nodal point (denoted by B in the figure), where $\lambda_1 < 0$. Assuming that $d\zeta/d\epsilon > 0$ and $dp/d\epsilon < 1$, it is easy to show that in this case $|\lambda_2| > |\lambda_1|$. Consequently, as $t \to +\infty$ the equations of the integral curves near the node are given by the expressions (11), in which one can neglect the terms containing the constant C_2. All these curves have a common tangent, the slope of which is determined by the derivative $(dH/d\epsilon)_0 = 1/6H_0$. This shows that the integral curves at the nodal point are tangent to the Friedmann parabola $\epsilon = 3H^2$, since the latter has the same value of the derivative $(dH/d\epsilon)_0$. An exception is only one integral curve: the one corresponding to $C_1 = 0$ in (11). The slope of this curve is given by the same expression as the slope of the second separatrix at the saddle point A, consequently $(dH/d\epsilon)_0 < 0$. If one draws the vertical and horizontal axes with the origin in the node, then this integral curve descends into the node through the second quadrant from the direction of large positive values of H. The other trajectories enter the node from the first and third quadrants.

The asymptotic form of the solution near the node is given by the same formulas (12), but with $t \to +\infty$. All trajectories which enter the node describe infinite isotropic expansion of the Universe, in the same manner as in the Friedmann model. However, owing to viscosity, the energy of motion of the expanding matter is dissipated and converted into internal energy in such a manner that in spite of the increase of the volume the density ϵ is maintained constant. At the same time the "production" of entropy density tends to a stationary regime in the limit $t \to +\infty$. The entropy density is $\sigma \sim \exp(\int d\epsilon/w) \to$ const, and the energy momentum tensor takes on the asymptotic form $T_0^0 = -\epsilon_0$, $T_\alpha^\beta = -\epsilon_0\delta_\alpha^\beta$. Formally matter in this region looks like an ideal fluid with an effective "pressure" $\bar{p}$ and the "equation of state" $\epsilon + \bar{p} = 0$, the peculiar form of which is related to the existence of a dissipative process on account of the second viscosity coefficient (the quantities H_0 and ϵ_0 do not depend on the first viscosity coefficient).

Above we have talked about the behavior of the scalar entropy density σ near the nodal point under consideration. The entropy per particle in a distinguished volume of the comoving space is proportional to the quantity $\int \sigma\sqrt{g}\, d^3x$ and, consequently, behaves as a function of time as $R^3\sigma$. As t tends to $+\infty$ this quantity increases as $R^3 = e^{3H_0t}$ (since $\sigma \to$ const). Thus, the influence of viscosity leads in this case to an accumulation of entropy even during the latest stages of isotropic expansion, which could be related to the observed anomalously large entropy "per baryon" in the present-day Universe.

We also note the case when $\lambda_1 = 0$. It can be seen from (9) and (10) that the value of ϵ_0 at the singularities corresponds to an intersection of the curves $\zeta(\epsilon)$ and $w(\epsilon)/\sqrt{3\epsilon}$ and the sign of λ_1 is determined by the difference of the derivatives of these functions at the intersection point. The case $\lambda_1 = 0$ corresponds to tangency of the indicated curves. It is easy to understand the character of these singularities, if one considers them as limits obtained through the confluence of two intersection points. One of these points must be a saddle point, and the other a node. The confluence yields a complicated equilibrium state of the saddle-node type. The separatrix of the saddle point and the singular integral curve of the node (both descend into the singularity from the region of large positive H) are fused into one curve, the separatrix of the saddle-node. The singularity behaves like a saddle point for the integral curves on one side of this separatrix, and like an attractor node on the other side. This reasoning does not apply to the special case of tangency, when the curve $\zeta(\epsilon)$ lies on different sides of the curve $w(\epsilon)/\sqrt{3\epsilon}$ near the singularity. To clarify the character of the singularity in this case, additional analysis would be required. However, we shall not consider these possibilities in this paper, since this would require some special conditions on the functions $\zeta(\epsilon)$ and $w(\epsilon)$, for which there are no physical foundations.

It may turn out that for finite values of H and ϵ (with the exception of the coordinate origin of the phase plane) there are no singular points at all, i.e., the equation $\zeta(\epsilon) = w(\epsilon)/\sqrt{3\epsilon}$ has no solutions. One could consider this case in general, but in order not to make the exposition too cumbersome by analyzing a situation which is not quite satisfactory from a physical point of view, we return to this possibility later, when the equation of state is made somewhat more concrete, as well as the dependence of the viscosity coefficients on the energy density.

We define the equation of state in the following simplest form:

$$w=\gamma\varepsilon, \quad 1\leq\gamma=\text{const}\leq2. \qquad (13)$$

As far as the viscosity coefficients are concerned, we require to know only their asymptotic form for small and large values of the energy density. An analysis of the relativistic kinetic equation for some simple cases[2)] shows that in these asymptotic regions the viscosity coefficients can be approximated by power functions of the energy density with definite requirements on the exponents of these functions. For small values of ϵ it is reasonable to consider both exponents large or, in the extreme case, equal to one. For large ϵ the power of the second viscosity coefficient ζ should be considered smaller (or equal) to 1/2, and the difference between the power of the first viscosity coefficient η and the power of the second viscosity coefficient ζ is reasonably considered to be larger (or equal) to 1/2. Such requirements have been obtained as a result of approximate estimates for some simplified gas models and are, of course, not widely applicable. However, the character of the results obtained is in its main traits sufficiently independent of these conditions.

Let us consider the region of small values of ϵ. According to what was said, the asymptotic form of the viscosity coefficients is here the following:

$$\eta=\eta_1(\varepsilon/\varepsilon_1)^{a_1},\quad \zeta=\zeta_2(\varepsilon/\varepsilon_2)^{a_2}\quad (a_1\geq 1,\quad a_2\geq 1,\quad \varepsilon\ll\varepsilon_1,\ \varepsilon_2). \tag{14}$$

A simple analysis show that there are three singularities in this region: the origin (H, ϵ) = (0, 0) and the two infinite points (H, ϵ) = ($\pm\infty$, 0). The singularity at (H, ϵ) = ($-\infty$, 0) has a saddle-point character, with only one trajectory ending up in it, namely Kasner's vacuum solution ($\epsilon\equiv 0$), which describes the collapse of empty space-time. The origin is a node, where all trajectories enter from the side of positive H, and from which trajectories exit in the direction of negative H (either of these corresponds to a growth of the proper time t). In Eqs. (6) and (8) the terms containing the viscosity coefficients are negligibly small near these points, and we obtain the Friedmann asymptotic behavior, describing the isotropic stages of the infinitely expanded state of the Universe:

$$H=2/3\gamma t,\quad \varepsilon=4/3\gamma^2t^2,\quad R=(t^2/t_c^2)^{1/3\gamma},$$
$$R_\alpha=(t^2/t_c^2)^{1/3\gamma}\quad (t\to\pm\infty). \tag{15}$$

(Here and in the sequel the quantities with the subscript c denote arbitrary constants.) The integral curves which enter into the node (H, ϵ) = (0, 0) for $t\to+\infty$ correspond to late stages of the expansion, and the outgoing curves ($t\to-\infty$) describe the initial stages of compression (time increases, starting from $-\infty$).

The most interesting singularity is the one at (H, ϵ) = ($+\infty$, 0), which corresponds to a node from which all trajectories exit. This node corresponds to a finite instant of time, which one may take to be equal to zero. The metric coefficients have here a Kasner singularity, but the energy density vanishes, in distinction from the usual cosmological singularity, where ϵ would become infinite. The asymptotic form of the solution here is indifferent to which of the two viscosity coefficients dominates. In all cases in the right-hand side of Eq. (6) for (H, ϵ) = ($+\infty$, 0) one may neglect all terms compared to the one which is quadratic in H. The right-hand side of Eq. (8), as well as the asymptotic form (14), is homogeneous in ζ and η. Let, for instance, η be larger than ζ (i.e., $a_1 < a_2$); then the limiting forms of Eqs. (6) and (8) will be

$$\dot H=-3H^2,\quad \dot\varepsilon=12\eta H^2-3\gamma H\varepsilon.$$

The solution of these equations yields

$$H=1/3t,\quad \varepsilon=\text{const}\cdot(3\varepsilon_1t/4\eta_1)^{1/(a_1-1)},$$
$$R=(\sqrt{3}\,qt)^{1/3},\quad R_\alpha=(\sqrt{3}\,q\,t)^{p_\alpha},\quad p_\alpha=1/3+s_\alpha/\sqrt{3}\,q\quad (t>0,\quad t\to 0) \tag{16}$$

(the constant in the expression for ϵ is expressed in a definite manner in terms of the constants γ, a_1, ϵ_1) where the exponents p_α automatically satisfy the Kasner relations $\Sigma p_\alpha=\Sigma p_\alpha^2=1$. The expression for ϵ in (16) is not applicable for $a_1 = 1$. In this case $\eta=\eta_1\epsilon/\epsilon_1$ and we have the following asymptotic behavior:

$$\varepsilon=\varepsilon_c(3\varepsilon_1t/4\eta_1)^{-\gamma}\exp(-4\eta_1/3\varepsilon_1t)\quad (a_1=1). \tag{17}$$

The quantities H, R, and R_α remain the same as in (16).

Thus, the node under consideration corresponds to the start of the cosmological expansion from a singularity of the Kasner type. At the same time there appears a phenomenon which one might call the production of matter by a singularity of the gravitational field.

Let us now consider the behavior in the region $\epsilon\to\infty$. As was already stated, here the asymptotic behavior of the viscosity coefficients must be taken in the form

$$\eta=\eta_1(\varepsilon/\varepsilon_1)^{b_1},\quad \zeta=\zeta_2(\varepsilon/\varepsilon_2)^{b_2}\quad (0\leq b_2\leq 1/2,\quad b_1\geq b_2+1/2,\quad \varepsilon\gg\varepsilon_1,\ \varepsilon_2) \tag{18}$$

(the use of the same constants ϵ_2, ζ_2 and ϵ_1, η_1 in this region does not restrict the generality, since the exponents a and b are different). Depending on the relations between the exponents b_1 and b_2 three qualitatively different cases may occur here, corresponding to the following possibilities:

I) $1/2>b_2\geq 0,\quad b_1>1/2$; II) $b_2=1/2,\quad b_1\geq 1$; III) $b_2=0,\quad b_1=1/2$.

We discuss these cases in order.

I) It is easy to show that for $b_2 < 1/2$ and $b_1 > 1/2$ the integral curves cannot go off to infinity with ϵ in the upper half-plane $H > 0$. This is possible in the lower half-plane. As (H, ϵ) $\to$ ($-\infty$, $+\infty$) the solutions describe the approach to a cosmological singularity of the Friedmann type, occurring at some finite instant of time t_1. The asymptotic behavior of the solutions is the following:

$$H=\frac{2}{3\gamma\tau},\quad \varepsilon=\frac{4}{3\gamma^2\tau^2},\quad R=(\tau^2/\tau_c^2)^{1/3\gamma},\quad R_\alpha=(\tau^2/\tau_c^2)^{1/3\gamma},$$
$$\tau=t-t_1<0,\quad \tau\to 0. \tag{19}$$

For $t\to t_1$ all trajectories approach asymptotically the Friedmann parabola $\epsilon = 3H^2$. The law of approach is given by the expansion terms following after (19) and have the form

$$1-\varepsilon/3H^2\sim\exp[\rho^2\tau(-\tau)^{-2b_1}], \tag{20}$$

where ρ is a constant which can be expressed in terms of the constants η_1, ϵ_1, γ and b_1. Thus we see that the integral curves come out of the node (H, ϵ) = (0, 0) and enter the node (H, ϵ) = ($-\infty$, $+\infty$), describing the contraction of the Universe from a state of "infinite volume" to the singularity. During this the time increases from $-\infty$ to t_1. It is interesting to note that both end points correspond to the isotropic Friedmann solution. The anisotropy appears, of course, during some time interval, but is extinguished according to a very fast exponential law (20) as the singularity is approached, on account of the action of the first viscosity coefficient (the second viscosity coefficient is unimportant in this case and in Eqs. (6) and (8) one may neglect the terms with the coefficient ζ).

II) For $b_2 = 1/2$ and $b_1\geq 1$ the terms involving the second viscosity coefficient in (6) and (8) can no longer be neglected, but the integral curves behave in the lower half-plane $H < 0$ qualitatively in the same manner as in the preceding case. As $t\to t_1$ they again approach the Friedmann curve $\epsilon = 3H^2$ according to the law (20). The asymptotic behavior of the solution differs inessentially from (19) and is obtained from it by the formal substitution $\gamma \to \gamma(1+\beta)$, where

$$\beta=\sqrt{3}\,\zeta_2/\gamma\sqrt{\varepsilon_2}. \tag{21}$$

However, the case under consideration also admits integral curves that go off to infinity with ϵ, situated in the upper half-plane $H > 0$. This is possible only with the condition $\beta > 1$. All trajectories quickly approach the parabola $\epsilon = 3H^2$ according to the law (20) (where the constant ρ will now depend also on the constant β) as $\epsilon \to +\infty$ (which again corresponds to t tending to a finite value t_1). The asymptotic behavior of the solution is the following:

$$H=2/3\gamma(1-\beta)\tau, \quad \varepsilon=4/3\gamma^2(1-\beta)^2\tau^2,$$
$$R=(\tau^2/\tau_c^2)^{1/3\gamma(1-\beta)}, \quad R_\alpha=(\tau^2/\tau_c^2)^{1/3\gamma(1-\beta)}. \tag{22}$$

Since $1-\beta < 0$, the quantity R and the metric coefficients tend to infinity as $\tau = t - t_1 \to 0$. The energy density, the entropy density, and the "Hubble constant" H also tend to infinity. It is hard to say something definite about the physical meaning of such solutions. At any rate, it is clear that such solutions are not applicable sufficiently close to the instant t_1. It is quite conceivable that a singularity of this type is unstable and starting with some time $t_{cr} < t_1$ we must switch to a different expansion regime. However, as the time t is reached the entropy density $\sigma \sim \epsilon^{1/\gamma}$ may increase sufficiently so that at the later real stages of expansion, at $t > t_1$, it would correspond to the observed anomalously large entropy density "per baryon" in the present-day Universe. We also note that the only possible origin of the integral curves under consideration can be the point $(H, \epsilon) = (+\infty, 0)$, where matter is created.

III) If $b_2 = 0$, $b_1 = 1/2$ the terms involving the second viscosity coefficient again become unimportant in Eqs. (6) and (8). Then the trajectories cannot go off to infinity in the upper half-plane $H > 0$ with ϵ. For trajectories in the lower half-plane we have two possibilities, depending on the magnitude of the constant ω:

$$\omega=4\eta_1/\sqrt{3}\,(2-\gamma)\sqrt{\varepsilon_1}. \tag{23}$$

For $\omega > 1$ we obtain the already known approach of the trajectories to the parabola $\epsilon = 3H^2$, corresponding to $t \to t_1$. The asymptotic behavior of the solution coincides exactly with (19), but the law according to which the integral curves approach the Friedmann parabola is now a weaker one:

$$1-\varepsilon/3H^2\sim(\tau^2)^{(2-\gamma)(\omega-1)/\gamma}, \tag{24}$$

which corresponds to a weak dependence of the viscosity coefficient η on the energy density. Compared to the preceding cases, the isotropization of the solution as the singularity is approached occurs with great difficulty. If $\omega < 1$ the integral curves do not approach the Friedmann trajectory at all, and the solution does not become isotropic near the singularity. In this case all integral curves tend asymptotically to the parabola $\epsilon = 3\omega^2H^2$. The point $(H, \epsilon) = (-\infty, +\infty)$ corresponds to the final instant of time $t = t_2$ and the asymptotic behavior of the solution is

$$H=\frac{2}{3(2+\gamma\omega^2-2\omega^2)\tau}, \quad \varepsilon=3\omega^2H^2,$$
$$R=(\tau^2/\tau_c^2)^{1/3(2+\gamma\omega^2-2\omega^2)}, \quad R_\alpha=(\tau^2/\tau_c^2)^{\pi_\alpha/2},$$
$$\tau=t-t_2<0, \quad \tau\to 0. \tag{25}$$

Here the exponents π_α are the following

$$\pi_\alpha=\frac{2(1-s_\alpha\sqrt{3(1-\omega^2)}/q)}{3(2+\gamma\omega^2-2\omega^2)}, \tag{26}$$

but they do not satisfy the Kasner conditions. The law according to which the trajectories approach the parabola $\epsilon = 3\omega^2H^2$ is described by the expression

$$1-\varepsilon/3\omega^2H^2=\text{const}\cdot R^{3(1-\omega^2)(2-\gamma)\omega^2/2} \tag{27}$$

with the function R from (25). The constant in (27) can have both signs.

We can now construct a complete picture of the behavior of the integral curves of (6) and (8) represented in Figs. 1, 2, and 3. With the selected equation of state (13) and with the conventions (14) and (18) regarding the asymptotic character of the viscosity coefficients, the different possibilities depend on the magnitudes of the exponent b_2 and of the constant β (21), which determine the behavior of the coefficient $\zeta(\epsilon)$ and together with it also of the solution of the equations (9) for the singularities. Eliminating from consideration, as already remarked, the unlikely cases when the curves $\zeta(\epsilon)$ and $w(\epsilon)/\sqrt{3\epsilon}$ have a common tangency point, we obtain the following possibilities.

1) $b_2 < 1/2$ and the equations (9) have solutions. In this case the number of singularities $(H_0, \epsilon_0) \neq (0, 0)$, $(+\infty, +\infty)$ can only be even, the first of them (counting from small values of ϵ to large ones) being a saddle point and the last one being a node. Without loss of generality for the qualitative picture we may consider that there are only two singularities in this case. Otherwise all the singularities would alternate according to the rule saddle—node—saddle—node ... and the separatrices of the saddle-points would separate the phase plane into regions that duplicate one another. There would not appear qualitatively new solutions compared to the case of two singularities. If one considers the constant β (21) sufficiently large: $\beta \gg 1$, one can even find the solutions of Eq. (9) explicitly. In this case one of the solutions is situated in the region of applicability of the asymptotic behavior (14) for small values of ϵ/ϵ_2 and the second is in the region of large ϵ/ϵ_2, where the approximation (18) is applicable. These solutions (H_{01}, ϵ_{01}), (H_{02}, ϵ_{02}) are

$$H_{01}=(\varepsilon_2/3)^{1/2}\beta^{1/(1-2a_2)}, \quad \varepsilon_{01}=\varepsilon_2\beta^{2/(1-2a_2)} \quad (\varepsilon_{01}\ll\varepsilon_2),$$
$$H_{02}=(\varepsilon_2/3)^{1/2}\beta^{1/(1-2b_2)}, \quad \varepsilon_{02}=\varepsilon_2\beta^{2/(1-2b_2)} \quad (\varepsilon_{02}\gg\varepsilon_2). \tag{28}$$

From (10) we now obtain the characteristic numbers $\lambda_1^{(1)}$, $\lambda_2^{(2)}$ and $\lambda_1^{(2)}$, $\lambda_2^{(2)}$ of both singularities

$$\lambda_1^{(1)}=3\gamma H_{01}(a_2-1/2)>0, \quad \lambda_2^{(1)}=-6H_{01}-4\eta_{01}<0,$$
$$\lambda_1^{(2)}=3\gamma H_{02}(b_2-1/2)<0, \quad \lambda_2^{(2)}=-6H_{02}-4\eta_{02}<0. \tag{29}$$

Thus, the first singularity is a saddle point and the second one is a node. The picture of the integral curve for this case is shown in Fig. 1.

2) $b_2 < 1/2$ and Eqs. (9) do not have solutions. In this case the inequality $\zeta < w/\sqrt{3\epsilon}$ will hold for all values of ϵ. In the upper half-plane all trajectories come out of the node $(H, \epsilon) = (+\infty, 0)$ and enter into the Friedmann node $(H, \epsilon) = (0, 0)$. The picture of the integral curves in this case is shown in Fig. 2.

3) $b_2 = 1/2$, $\beta > 1$. In this case (9) must have as a minimum one solution which is a saddle point. There can only be an odd number of singularities and they will alternate according to the rule (in the direction of increasing ϵ): saddle—node—...—saddle. We shall consider that there exist only one saddle-point singularity. A larger number of solutions of (9) would require a physically unjustified complication of the function $\zeta(\epsilon)$ and in addition does not yield qualitatively new integral

curves, compared to those already observed. In the upper half-plane all trajectories come out of the node $(H, \epsilon) = (+\infty, 0)$, some end up in the Friedmann node $(H, \epsilon) = (0, 0)$, and some go off to infinity, where the asymptotic form of the solutions is given by the expressions (22). This case is illustrated in Fig. 3.

4) $b_2 = 1/2$, $\beta > 1$. This case does not require special consideration. There may be no singularities (H_0, ϵ_0) here and the picture of the trajectories corresponds to Fig. 2. If there are singularities, the situation is the same as in the already considered case 1. The picture of the integral curves corresponds to Fig. 1.

3. CONCLUSION

We have considered the cosmological solutions of the Einstein equations for a homogeneous model of type I taking into account dissipative processes due to viscosity. The analysis showed that the viscosity has an essential influence on the character of the solutions. From a quantitative point of view the results depend on the form of the viscosity coefficients as functions of the energy density, but qualitatively the picture is to a sufficient degree universal. We recall its characteristic traits.

1. The presence of a cosmological singularity remains, as before, an inevitable factor of the evolution both for the expansion of the Universe, as well as for its contraction. The singularity occurs at a finite instant of proper time and is accompanied by the curvature invariants becoming infinite.

2. The solutions corresponding to the expansion of the Universe and to its contraction are essentially different insofar as the behavior of the energy density of matter near the cosmological singularity is concerned. A contraction of the Universe starts with isotropic Friedmann stages (the node $(H, \epsilon) = (0, 0)$) and in general ends also with an isotropic Friedmann singularity (the node $(H, \epsilon) = (-\infty, +\infty)$), where the energy density becomes infinite. Thus the viscosity shows an essential isotropizing action, which is quite natural from a physical point of view.

3. In the case of expansion of the Universe there exists a richer selection of possibilities, but near the cosmological singularity (the node $(H, \epsilon) = (+\infty, 0)$) the picture is unique: the singularity has a Kasner character with infinite curvature invariants, but the matter energy density at the singularity vanishes. This phenomenon can be interpreted as production of matter by the gravitational field at the instant of the "big bang." It is not excluded that such a classical description of matter production has some relation to deeper quantum processes of particle production, as investigated, e.g., in [4].

4. The process of further expansion of the Universe will in general lead to an isotropic Friedmann expansion (as $t \to \infty$). Depending on the initial conditions we arrive either at the node $(H, \epsilon) = (0, 0)$ corresponding to a usual Friedmann picture with a negligible role of the viscosity, or we arrive at the node $(H, \epsilon) = (H_0, \epsilon_0)$, where the viscosity continues to exert a substantial influence up to $t = \infty$. The latter case is interesting in that in these final isotropic stages of expansion the monotonic accumulation of entropy per particle is not curtailed. The entropy can become arbitrarily large and this fact has possibly a relation to the actually observed anomalously high entropy "per baryon" in the present-day Universe.

It is not difficult to consider the behavior of the cosmological solutions near the "big bang" also in the more general Bianchi type IX model. One can show that near the initial singularity the influence of matter will be negligibly small, since the energy density, and together with it the energy-momentum tensor, vanish here. Consequently, the only distinction will be the fact that in place of a singularity of the Kasner type, the initial singularity in the type IX model will be described by an oscillatory regime. As was shown[1], the latter is stable and is present also in the general solution of the Einstein equations near the cosmological singularity. Thus, we arrive at the conclusion that it is apparently possible to construct a general solution of the gravitational equations in which there is matter production on account of the influence of viscosity.

It should be kept in mind that taking into account the dissipation by means of the two viscosity coefficients is a valid approach only in the case when the terms with the highest derivatives with respect to the velocity are small. One can hope, however, that the kinetic coefficients of higher order will be proportional to the energy taken to a power higher than the one characterizing the coefficients η and ζ. In this case such terms will turn out to be negligibly small near the initial singularity (in particular, if the energy density is exponentially small). One way or another, there are reasons to hope that taking into account the dissipative terms the picture obtained by us will not change substantially from a qualitative point of view. It is not excluded that the model under consideration gives us an idea of what could happen in the appearance and evolution of the Universe.

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1) We use a system of units in which the velocity of light and the Einstein gravitational constant are equal to one. The metric is written in the form $-ds^2 = g_{ik}dx^i dx^k$, where g_{ik} has the signature $(-,+,+,+)$. Latin indices run from 0 to 3 and Greek indices take on the values (1, 2, 3).

2) The appropriate estimates have been obtained by A. A. Starobinskiĭ.

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