Fluctuations in a system of parametrically excited magnons

A. S. Mikhaĭlov

Moscow State University
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Starting from the Fokker-Planck equation we obtain in the self-consistent field approximation the distribution function of the complex magnon amplitude for the stationary state of a system of parametrically excited magnons when ferromagnets are subject to parallel pumping. Knowledge of the distribution function enables us to study the fluctuations in the stationary state. It turned out to be convenient to change to the amplitudes and phases of standing spin waves as variables for a discussion of the physical meaning of the state obtained.

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When an external uniform variable magnetic field, parallel to the static magnetization of the ferromagnetic, is applied to a ferrodielectric there arises the possibility of parametric resonance, caused by the decay of a photon of the pumping field into two magnons.[1,2] Nonlinear effects arising when spin waves (SW) are excited in this way in a ferromagnetic (it is called the parallel pumping method) has turned out to be the subject of a thorough theoretical analysis. Apart from the possibility to establish a self-oscillating regime in a system of parametrically excited magnons[3], the possibility was noted of the existence of a stationary regime, i.e., a regime in which the power absorbed by the system does not change with time. Gottlieb and Suhl[4] connected the realization of this possibility with the non-linear dependence of the magnon damping on the total number of parametrically excited magnons (PEM).[1)] Zakharov, L'vov, and Starobinets[6] indicated the existence of another mechanism which also can lead to the establishment of a stationary regime in a PEM system. This mechanism is connected with the non-linear renormalization of the magnon frequency and of the pumping due to four-magnon interaction processes. A further discussion of the different properties of such a stationary state of a PEM system was given in a large number of papers by the same authors (see the review[7]). The studies in those papers made it possible to determine the range of parameters in which the stationary state is stable and outside which the PEM system is in a self-oscillating regime; moreover, both the non-linear damping and the renormalization of the frequency and of the pumping were taken into account.

Stationary states arising for parallel pumping and described in a number of papers[4–7] refer to the class of so-called "flux equilibrium states"[8] for which the energy flux "through" the system is characteristic: the energy entering the system from some external source later dissipates into a thermal bath connected with the system. It is well known that the "dissipative structures" which appear can be of high order.

A non-trivial example, which has been extensively studied recently, of a system in a "flux equilibrium state" is a continuously working laser. It has turned out to be most convenient for the analysis of the processes which take place in a laser to use the Fokker-Planck equation formalism which enables us to find the photon density matrix and to study the fluctuations in the laser emission (see [9,10]). The possibility of realizing a flux-less thermodynamic equilibrium state of a PEM system—a state in which the PEM system does not "remove" energy from the external pumping and hence does not dissipate energy—was studied in [11].

In the present paper we do not touch upon the problems connected with the existence and stability of the state proposed in [11] and we restrict ourselves to a study of the stationary flux state of a PEM system which is described in [6,7,12]. We use in what follows the Fokker-Planck equation to evaluate the distribution function and the fluctuations.

1. HAMILTONIAN OF A PEM SYSTEM AND EQUATIONS OF MOTION

When there is no pumping the spin system is characterized by the Hamiltonian

$$\mathscr{H}^{(s)}=\sum_{\mathbf{k}}\hbar\omega_{\mathbf{k}}a_{\mathbf{k}}^{+}a_{\mathbf{k}}+\mathscr{H}_{int}, \tag{1}$$

where $\mathscr{H}_{int}$ describes the magnon-magnon interaction. We shall only consider the classical limit in which the average occupation number $n_{\mathbf{k}} = \langle a_{\mathbf{k}}^{+}a_{\mathbf{k}}\rangle$ is large ($n_{\mathbf{k}} \gg 1$).[2)] This enables us to neglect the fact that the magnon creation and annihilation operators ($a_{\mathbf{k}}^{+}$ and $a_{\mathbf{k}}$) do not commute and to consider them to be c-numbers.

If the magnetic pumping field varies as $h(t) = h\exp(-i\omega_p t)$ the Hamiltonian of the interaction between the (parallel) pumping and the SW system in the ferromagnetic can be written in the form

$$\mathscr{H}^{(v)}=\sum_{\mathbf{k}}\frac{\mu_0 h}{\hbar\omega_{\mathbf{k}}}\left\{2A_{\mathbf{k}}\exp(-i\omega_p t)a_{\mathbf{k}}^{+}a_{\mathbf{k}}+\frac{1}{2}B_{\mathbf{k}}^{*}\exp(-i\omega_p t)\cdot a_{\mathbf{k}}a_{-\mathbf{k}}+\text{c.c.}\right\},$$

where μ_0 is the Bohr magneton, $A_{\mathbf{k}}$ and $B_{\mathbf{k}}$ are the coefficients in the not-diagonalized spin Hamiltonian (see [13], p. 179). We emphasize that the $B_{\mathbf{k}}$ appear when the dipole-dipole interaction is taken into account.

The first term in $\mathscr{H}^{(v)}$, proportional to $a_{\mathbf{k}}^{+}a_{\mathbf{k}}$, conserves the magnon number. We shall take into account in what follows only the second term

$$\mathscr{H}^{(v)}=\frac{\hbar}{2}\sum_{\mathbf{k}}(V_{\mathbf{k}}^{*}\exp(-i\omega_p t)a_{\mathbf{k}}a_{-\mathbf{k}}+\text{c.c.}), \tag{2}$$

where $\hbar V_{\mathbf{k}} = h\mu_0 B_{\mathbf{k}}/\hbar\omega_{\mathbf{k}}$. One can show that if the terms dropped are taken into account they lead to a correction (see [5]) of order $h/M_0 \ll 1$ (M_0 is the static magnetic moment of the ferromagnetic).

The Hamiltonian (1), (2) corresponds to the following equations of motion for the c-numbers $a_{\mathbf{k}}$:

$$\dot{a}_{\mathbf{k}}=-i\omega_{\mathbf{k}}a_{\mathbf{k}}-iV_{\mathbf{k}}a_{-\mathbf{k}}^{+}\exp(i\omega_p t)-\frac{i}{\hbar}\frac{\delta\mathscr{H}_{int}}{\delta a_{\mathbf{k}}^{+}}.$$

We shall assume that the magnon system is "immersed" in a thermostat. We do not really consider in the present paper those systems which may serve as a thermostat for PEM. We merely note two possibilities. Firstly, by virtue of the fact that, as we shall show below, the SW excitation in parallel pumping proceeds in a narrow region of the wavevector space ($\omega_{\mathbf{k}} \approx \omega_p/2$) one could restrict the summation in the expression for the Hamiltonian $\mathscr{H}^{(s)}$ to solely that region of $\mathbf{k}$-space in which certainly all PEM are concentrated while one considers the set of SW with wavevectors outside that region as the thermostat. Secondly, one can consider the system of non-magnon excitations of the ferromagnetic (i.e., the phonon system, and so on) as thermostat. Both in the first and in the second case one assumes, of course, that the absorbed power is not so large that the parameters of the "thermostat" are changed.

Following Zakharov and L'vov[12] we shall take the thermostat into account purely phenomenologically by including additional "thermostat" terms in the equations of motion:

$$\dot{a}_{\mathbf{k}} = -(i\omega_{\mathbf{k}}+\gamma_{\mathbf{k}})a_{\mathbf{k}} - iV_{\mathbf{k}}a_{-\mathbf{k}}^{+}\exp(i\omega_p t) - \frac{i}{\hbar}\frac{\delta\mathscr{H}_{int}}{\delta a_{\mathbf{k}}^{+}} + f_{\mathbf{k}}. \qquad (3)$$

We have introduced in this equation a term $(-\gamma_{\mathbf{k}}a_{\mathbf{k}})$ describing the damping of the PEM due to the coupling with the thermostat, and a random Gaussian force $f_{\mathbf{k}}(t)$ with correlators

$$\langle f_{\mathbf{k}}(t)f_{\mathbf{k}'}^{*}(t')\rangle = \alpha_{\mathbf{k}}\delta_{\mathbf{k}\mathbf{k}'}\delta(t-t'), \qquad \langle f_{\mathbf{k}}(t)f_{\mathbf{k}'}(t')\rangle = 0,$$

which enables us to take into account the "noise" action of the thermostat on the PEM system (see [14]). The coefficients $\alpha_{\mathbf{k}}$ and $\gamma_{\mathbf{k}}$ are related (see below).

It is necessary for us for what follows to give a definite form of $\mathscr{H}_{int}$ which enters in the equations of motion. A study of Eqs. (3) with the exact form of the Hamiltonian $\mathscr{H}_{int}$ is an extraordinarily complex problem. An approximate "S-theory" was constructed in [6,7] with a model Hamiltonian $\mathscr{H}_{int}$:

$$\mathscr{H}_{int} = \sum_{\mathbf{k}\mathbf{k}'}\hbar\left(T_{\mathbf{k}\mathbf{k}'}a_{\mathbf{k}}^{+}a_{\mathbf{k}}a_{\mathbf{k}'}^{+}a_{\mathbf{k}'} + \frac{1}{2}S_{\mathbf{k}\mathbf{k}'}a_{\mathbf{k}}^{+}a_{-\mathbf{k}}^{+}a_{\mathbf{k}'}a_{-\mathbf{k}'}\right). \qquad (4)$$

The "S-theory" is a self-consistent field theory in which one assumes that the following equations hold:

$$\frac{1}{\hbar}\frac{\delta\mathscr{H}_{int}}{\delta a_{\mathbf{k}}^{+}} = \left(\sum_{\mathbf{k}'}2T_{\mathbf{k}\mathbf{k}'}a_{\mathbf{k}'}^{+}a_{\mathbf{k}'}\right)a_{\mathbf{k}} + \left(\sum_{\mathbf{k}'}S_{\mathbf{k}\mathbf{k}'}a_{\mathbf{k}'}a_{-\mathbf{k}'}\right)a_{-\mathbf{k}}^{+}$$
$$= \left(\sum_{\mathbf{k}'}2T_{\mathbf{k}\mathbf{k}'}n_{\mathbf{k}'}\right)a_{\mathbf{k}} + \left(\sum_{\mathbf{k}'}S_{\mathbf{k}\mathbf{k}'}\sigma_{\mathbf{k}'}\right)a_{-\mathbf{k}}^{+},$$

where $n_{\mathbf{k}} = \langle a_{\mathbf{k}}^{+}a_{\mathbf{k}}\rangle$ and $\sigma_{\mathbf{k}} = \langle a_{\mathbf{k}}a_{-\mathbf{k}}\rangle$.

It is thus assumed that the relative fluctuations of the integral quantities

$$\left(\sum_{\mathbf{k}'}S_{\mathbf{k}\mathbf{k}'}a_{\mathbf{k}'}a_{-\mathbf{k}'}\right), \quad \left(\sum_{\mathbf{k}'}2T_{\mathbf{k}\mathbf{k}'}a_{\mathbf{k}'}^{+}a_{\mathbf{k}'}\right)$$

are small and (neglecting in the estimates the wavevector dependence of T and S) that the inequalities

$$\left\langle\left[\sum_{\mathbf{k}}(a_{\mathbf{k}}^{+}a_{\mathbf{k}}-n_{\mathbf{k}})\right]^2\right\rangle \ll \left(\sum_{\mathbf{k}}n_{\mathbf{k}}\right)^2,$$
$$\left|\left\langle\left[\sum_{\mathbf{k}}(a_{\mathbf{k}}a_{-\mathbf{k}}-\sigma_{\mathbf{k}})\right]^2\right\rangle\right| \ll \left|\sum_{\mathbf{k}}\sigma_{\mathbf{k}}\right|^2, \qquad (5)$$

hold, or rewriting them in a more convenient form,

$$\sum_{\mathbf{k}\mathbf{k}'}(\langle a_{\mathbf{k}}^{+}a_{\mathbf{k}}a_{\mathbf{k}'}^{+}a_{\mathbf{k}'}\rangle - n_{\mathbf{k}}n_{\mathbf{k}'}) \ll \left(\sum_{\mathbf{k}}n_{\mathbf{k}}\right)^2,$$
$$\left|\sum_{\mathbf{k}\mathbf{k}'}(\langle a_{\mathbf{k}}a_{-\mathbf{k}}a_{\mathbf{k}'}a_{-\mathbf{k}'}\rangle - \sigma_{\mathbf{k}}\sigma_{\mathbf{k}'})\right| \ll \left|\sum_{\mathbf{k}}\sigma_{\mathbf{k}}\right|^2. \qquad (6)$$

We emphasize that we do not require that fluctuations in quantities referring to $\mathbf{k}$ are small: such fluctuations can, in principle, be large. As will be seen below, it is important that fluctuations referring to different $\mathbf{k}$ are uncorrelated: they have "random phases." This leads to the fact that after summation over all $\mathbf{k}$ we may get small corrections to the average values of the integral quantities even when the fluctuations in the "individual" quantities are large.

It is convenient to change in Eqs. (3) to a "rotating" system of coordinates through the transformation $a_{\mathbf{k}} \to a_{\mathbf{k}}\exp(-i\omega_p t/2)$ after which the explicit time dependence disappears in them. Finally we get in the "S-theory" approximation the following equations of motion:

$$\dot{a}_{\mathbf{k}} = -(i\tilde{\omega}_{\mathbf{k}}+\gamma_{\mathbf{k}})a_{\mathbf{k}} - iP_{\mathbf{k}}a_{-\mathbf{k}}^{+} + f_{\mathbf{k}}(t), \qquad (7)$$

where

$$\tilde{\omega}_{\mathbf{k}} = \omega_{\mathbf{k}} - {}^1/_2\omega_p + 2\sum_{\mathbf{k}'}T_{\mathbf{k}\mathbf{k}'}n_{\mathbf{k}'}, \qquad (8)$$

$$P_{\mathbf{k}} = V_{\mathbf{k}} + \sum_{\mathbf{k}'}S_{\mathbf{k}\mathbf{k}'}\sigma_{\mathbf{k}'}.$$

2. PEM DISTRIBUTION FUNCTION

Any differential equation in variables x_i in which there occurs linearly a Gaussian random force $f_i(t)$,

$$\dot{x}_i = Q_i(x) + f_i(t),$$

corresponds to a Fokker-Planck equation for the distribution function of the quantities x_i:

$$\frac{\partial\Phi}{\partial t} = -\frac{\partial}{\partial x_i}[Q_i(x)\Phi] + D_{ij}\frac{\partial^2\Phi}{\partial x_i\partial x_j},$$

in which D_{ij} can be found from the condition

$$\langle f_i(t)f_j(t')\rangle = 2D_{ij}\delta(t-t').$$

The distribution function $\Phi(x)$ has the meaning of a probability density for finding the random quantities x_i in the range $(x_i, x_i + dx_i)$ such that

$$\int\Phi\prod_i dx_i = 1.$$

The scheme for obtaining the Fokker-Planck equation is well known (see, e.g., [9]). This scheme can without changes be applied to the equation of motion (7) with a Gaussian force. As a result we can obtain for the distribution function $\Phi_{\mathbf{k}}$ the following Fokker-Planck equation

$$\frac{\partial\Phi_{\mathbf{k}}}{dt} = \frac{\partial}{\partial a_{\mathbf{k}}}\{[(i\tilde{\omega}_{\mathbf{k}}+\gamma_{\mathbf{k}})a_{\mathbf{k}} + iP_{\mathbf{k}}a_{-\mathbf{k}}^{+}]\Phi_{\mathbf{k}}\} \qquad (9)$$
$$+ \frac{\partial}{\partial a_{-\mathbf{k}}}\{[(i\tilde{\omega}_{\mathbf{k}}+\gamma_{\mathbf{k}})a_{-\mathbf{k}} + iP_{\mathbf{k}}a_{\mathbf{k}}^{+}]\Phi_{\mathbf{k}}\} + \frac{\alpha_{\mathbf{k}}}{2}\frac{\partial^2\Phi_{\mathbf{k}}}{\partial a_{\mathbf{k}}^{+}\partial a_{\mathbf{k}}}$$
$$+ \frac{\alpha_{\mathbf{k}}}{2}\frac{\partial^2\Phi_{\mathbf{k}}}{\partial a_{-\mathbf{k}}^{+}\partial a_{-\mathbf{k}}} + \text{c.c.}$$

in which the symmetry of the coefficients of the coefficients of $\omega_{\mathbf{k}}$, $\gamma_{\mathbf{k}}$, $\alpha_{\mathbf{k}}$, $V_{\mathbf{k}}$, $S_{\mathbf{k}\mathbf{k}'}$, and $T_{\mathbf{k}\mathbf{k}'}$ under inversion, $\mathbf{k} \to -\mathbf{k}$, is taken into account.

The distribution function $\Phi_{\mathbf{k}}(a_{\mathbf{k}}^{+}, a_{\mathbf{k}}, a_{-\mathbf{k}}^{+}, a_{-\mathbf{k}})$ is a "pair" distribution function. Its normalization condition is:

$$\int\Phi_{\mathbf{k}}da_{\mathbf{k}}^{+}\,da_{\mathbf{k}}\,da_{-\mathbf{k}}^{+}\,da_{-\mathbf{k}} = 1.$$

By the integral $\int(\ldots)da_{\mathbf{k}}^{+}da_{\mathbf{k}}$ we mean the integral

$$\int_0^\infty\int_0^{2\pi}(\ldots)d\rho_{\mathbf{k}}\,d\phi_{\mathbf{k}}, \qquad a_{\mathbf{k}} = \rho_{\mathbf{k}}^{1/2}\exp(i\phi_{\mathbf{k}}).$$

As the equations for the distribution functions $\Phi_{\mathbf{k}}$ corresponding to different values of $\mathbf{k}$ do not interconnect with one another, the complete distribution function Φ decomposes into a product of "pair" functions[3)]

$$\Phi(\{a_k^+, a_k\}) = \left(\prod_k \Phi_k\right)^{1/2}.$$

The square root is necessary as each "pair" distribution function Φ_k is twice taken into account in the product.

We restrict ourselves to finding a stationary solution of the equation for Φ_k $\partial\Phi_k/\partial t = 0$. We shall look for its solution in the form

$$-\frac{\alpha_k}{2}\ln\Phi_k = x_k(a_k^+a_k + a_{-k}^+a_{-k}) + 2\,\mathrm{Re}(y_k^* a_k a_{-k}), \qquad (10)$$

where x_k and y_k are coefficients to be determined.

Substituting expression (10) into Eq. (9) in the stationary case we find the values of the coefficients x_k and y_k and find the stationary distribution function:

$$\Phi_k = Z_k^{-1}\exp\left\{-\frac{2\gamma_k}{\alpha_k}\left[(a_k^+a_k + a_{-k}^+a_{-k}) + 2\,\mathrm{Im}\left(\frac{P_k^*}{-i\tilde\omega_k+\gamma_k}a_k a_{-k}\right)\right]\right\}$$

Z_k is here a normalization constant, to be found below.

When there is no pumping (when $V_k = 0$, and therefore $P_k = 0$) the distribution function Φ_k must go over into the equilibrium Gibbs function:

$$\Phi_k^{(0)} = Z_k^{(0)^{-1}}\exp\left\{-\frac{\hbar\omega_k}{\Theta}(a_k^+a_k + a_{-k}^+a_{-k})\right\},$$

where Θ is the thermostat temperature. When there is no pumping the relation $\alpha_k = 2\Theta\gamma_k/\hbar\omega_k$ must thus be satisfied. We shall assume that this relation remains valid for all levels of pumping. We neglect thus in the given model the non-linear damping effects and the effects of the "natural" noise of the PEM system, a consistent taking into account of which would require the application of a diagram technique (see the review [7]). In many cases for not too strong pumping such effects are weak; the appropriate estimates are given in [12].

We can write the distribution function Φ_k in the form

$$\Phi_k = Z_k^{-1}\exp(-E_k/\Theta),$$

$$E_k = \hbar\omega_k\left[(a_k^+a_k + a_{-k}^+a_{-k}) + 2\,\mathrm{Im}\left(\frac{P_k^*}{-i\tilde\omega_k+\gamma_k}a_k a_{-k}\right)\right]. \qquad (11)$$

Notwithstanding the external similarity of distribution (11) and the Gibbs one, they are in an essential way different. The "energy" E_k in Eq. (11) is not a true energy of the magnon excitations (cf. the Hamiltonian (1), (2), (4)); the parameters of the integral of the collisions of the PEM and the particles of the thermostat—the relaxation frequencies γ_k—occur in E_k. The distribution (11) describes a system in a "flux equilibrium" state; this manifests itself in the fact that one can show that a PEM system in the state with the distribution function (11) absorbs the pumping power and dissipates it into the thermostat.

The quantities P_k and $\tilde\omega_k$ in expression (11) must be determined from the self-consistency conditions (8).

Using the normalization of the function Φ_k we find

$$Z_k = \frac{1}{\eta_k}\left(\frac{2\pi\Theta}{\hbar\omega_k}\right)^2, \quad \eta_k = 1 - \frac{|P_k|^2}{\tilde\omega_k^2+\gamma_k^2}.$$

It is convenient to find also the "single-particle" distribution function defined by the equation

$$\Phi_k^{(1)} = \int \Phi_k\, da_{-k}^+\, da_{-k}.$$

We get the following expression for $\Phi_k^{(1)}$:

$$\Phi_k^{(1)} = \eta_k\frac{\hbar\omega_k}{2\pi\Theta}\exp\left\{-\frac{\hbar\omega_k\eta_k}{\Theta}a_k^+a_k\right\}.$$

We shall show below that in the region of $\mathbf{k}$-space, in which PEM are excited, the relation $1 \gg \eta_k > 0$ is satisfied when $|V_k| \gg \gamma_k$.

3. AVERAGE CHARACTERISTICS OF A PEM SYSTEM

Using the fact that

$$a_k^+a_k\Phi_k^{(1)} = -(\hbar\omega_k\eta_k)^{-1}Z_k^{(1)^{-1}}\frac{\partial}{\partial(1/\Theta)}(Z_k^{(1)}\Phi_k^{(1)}),$$

where $Z_k^{(1)} = 2\pi\Theta/\hbar\omega_k\eta_k$ is the normalization constant of the "single-particle" distribution function $\Phi_k^{(1)}$, we get

$$n_k = \Theta/\hbar\omega_k\eta_k. \qquad (12)$$

Similarly, using

$$a_k a_{-k}\Phi_k = -i\frac{\Theta}{\hbar\omega_k}(-i\tilde\omega_k+\gamma_k)Z_k^{-1}\frac{\partial}{\partial P_k^*}(Z_k\Phi_k),$$

we get

$$\sigma_k = -\frac{iP_k}{i\tilde\omega_k+\gamma_k}n_k. \qquad (13)$$

As the coefficients P_k and $\tilde\omega_k$ depend on n_k and σ_k (see the definition (8) of these quantities) Eqs. (12) and (13) together with (8) are a set of integral equations to find P_k, $\tilde\omega_k$, n_k, and σ_k. This set was obtained (and solved) before by Zakharov and L'vov in [12] where they started directly from the equations for the correlators n_k and σ_k (i.e., from the moment equations). They gave in the same paper an analysis of the solutions obtained. They showed, in particular, that the symmetry properties and the actual form of the coefficients $S_{kk'}$, $T_{kk'}$, and V_k primarily determine the geometric character of the PEM distribution in $\mathbf{k}$-space.[4)]

We give the scheme of the solution and the results of the calculations for a simple model with constant coefficients:

$$V_k = V,\ S_{kk'} = S,\ T_{kk'} = T,\ \omega_k = \omega(|k|),\ \gamma_k = \gamma,$$

where S and T are real. In this model $P_k = P = V + S\sigma$, $\tilde\omega_k = \omega_k - 1/2\,\omega_p + 2TN$ and

$$\sigma = \sum_k \sigma_k, \quad N = \sum_k n_k,$$

so that if follows from Eq. (13) that

$$\sigma = -\frac{4\pi i\Omega}{(2\pi)^3}P\int\frac{-i\tilde\omega_k+\gamma}{\tilde\omega_k^2+\nu^2}n_k^0 k^2\,dk, \qquad (14)$$

where $\nu^2 = \gamma^2 - |P|^2$, $n_k^0 = \Theta/\hbar\omega_k$, Ω is the volume of the ferromagnetic, $\Omega \to \infty$, $\nu^2 > 0$.[5)] Below we shall show that there exists a value $k = k_0 \neq 0$ such that $\tilde\omega_{k_0} = 0$. Bearing in mind that the main contribution to the integral (14) comes from a narrow region $|\tilde\omega_k| \lesssim \nu$, $\nu \le \gamma$, we find then that

$$\sigma = -iP(\gamma/\nu)\xi_0, \qquad (15)$$

where

$$\xi_0 = k_0^2 n_0\Omega/2\pi v_0,\quad v_0 = d\omega/dk|_{k=k_0},\quad n_0 = n^0(k_0).$$

Using Eq. (15) and the definition of P ($P = V + S\sigma$) we get the following equation for ν^2:

$$\nu^4 + \nu^2[|V|^2 + \gamma^2(\xi_0 S)^2 - \gamma^2] - \gamma^4(\xi_0 S)^2 = 0. \qquad (16)$$

According to the estimates in [12] $\xi_0 S \sim \Theta(ak_0)^3/\hbar\omega_p \ll 1$, where a is the lattice constant.[6)]

As a result of solving Eq. (16) we find the amplitude $|P|$ of the renormalized pumping as function of the amplitude of the external pumping $|V|$ ($|P|^2 = \gamma^2 - \nu^2$) as shown in Fig. 1. In the limiting cases the following relations are valid:

a) $\nu^2 \approx \gamma^2 - |V|^2, \quad P \approx V$ when $|V| \ll \gamma$,

b) $(\nu/\gamma) \approx (\xi_0 S)(\gamma/|V|), \quad P \approx -i\gamma(V/|V|)$ when $|V| \gg \gamma$.

The total number of PEM turns out to equal (see Fig. 2)

$$N - N_0 = \xi_0(\gamma^2 - \nu^2)/\nu, \quad N_0 = \sum_k n_k^0,$$

where N_0 is the total number of thermal magnons. In the limiting cases we get:

a) $N - N_0 \approx (\gamma\xi_0)(|V|/\gamma)^2$ when $|V| \ll \gamma$,

b) $N - N_0 \approx |V/S|$, when $|V| \gg \gamma$;

k_0 is determined from the equation $\tilde{\omega}_{k_0} = 1/2\,\omega_p - 2TN$. It is necessary to note that this equation has a solution, when $T > 0$, only provided the inequality $1/2\omega_p - \omega_0 > 2T\,|V|/|S|$ ($\omega_0 = \omega(k = 0)$) is satisfied, i.e., when half the pumping frequency does not lie too close to the "bottom" of the magnon spectrum. If the above-mentioned inequality is violated, the existence of a stationary regime will nevertheless be possible, but the PEM distribution in **k**-space is then concentrated near **k** = 0 and we must take into account the discreteness of the **k**-values (Walker modes). We do not study this case in the present paper.

When $|V| < \gamma$ the SW distribution will thus differ little from thermal equilibrium:

$$n_k - n_k^0 \approx |V|^2[(\omega_k - {}^1/_2\omega_p)^2 + \gamma^2]^{-1} n_k^0.$$

We note that also when $|V| < \gamma$ there is a non-vanishing anomalous correlator:

$$\sigma_k \approx -iV[i(\omega_k - {}^1/_2\omega_p) + \gamma]^{-1} n_k^0.$$

After passing through the threshold $|V| = \gamma$ the number of SW increases steeply in a narrow region of width $\nu \ll \gamma$ in the vicinity of the sphere of radius k_0 in **k**-space

$$n_k - n_k^0 \approx \frac{\gamma^2}{\tilde{\omega}_k^2 + \nu^2} n_k^0,$$

where $\nu^2 = \gamma^2 - |P|^2$ and $|P| \to \gamma$ as $|v| \to \infty$.

These results remain qualitatively valid also in the general case for any wavevector dependence of the coefficients V_k, $S_{kk'}$, and $T_{kk'}$ and of the quantities ω_k and γ_k. In particular, for those **k** where the PEM distribution is concentrated, $|P_k| \to \gamma_k$ as $h \to \infty$, where h is the amplitude of the variable external pumping magnetic field.

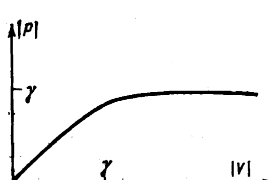

FIG. 1

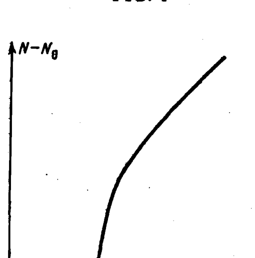

FIG. 2

4. FLUCTUATIONS IN THE PEM SYSTEM

We determine the quaternary correlators $\langle(a_k^+ a_k)^2\rangle$, $\langle|a_k a_{-k}|^2\rangle$, and $\langle(a_k a_{-k})^2\rangle$ in the stationary state with the distribution function (11). To do this we use the identities

$$(a_k^+ a_k)^2 \Phi_k^{(1)} = (\hbar\omega_k\eta_k)^{-2} Z_k^{(1)^{-1}} \frac{\partial^2}{\partial(1/\Theta)^2}(Z_k^{(1)}\Phi_k^{(1)}),$$

$$|a_k a_{-k}|^2 \Phi_k = \left(\frac{\Theta}{\hbar\omega_k}\right)^2 (\tilde{\omega}_k^2 + \gamma_k^2) Z_k^{-1} \frac{\partial^2}{\partial P_k \partial P_k^*}(Z_k\Phi_k),$$

$$(a_k a_{-k})^2 \Phi_k = -\left(\frac{\Theta}{\hbar\omega_k}\right)^2 (i\tilde{\omega}_k - \gamma_k)^2 Z_k^{-1} \frac{\partial^2}{(\partial P_k^*)^2}(Z_k\Phi_k).$$

After some simple calculations we get the following exact expressions for the relative fluctuations:

$$[\langle(a_k^+ a_k)^2\rangle - n_k^2]^{1/2} n_k^{-1} = 1, \quad [\langle(a_k a_{-k})^2\rangle - \sigma_k^2]^{1/2} \sigma_k^{-1} = 1,$$
$$[\langle|a_k a_{-k}|^2\rangle - |\sigma_k|^2]^{1/2} n_k^{-1} = 1.$$

The relative fluctuations in the quantities $a_k^+ a_k$ and $a_k a_{-k}$ are thus large in the "S-theory."

We now determine the fluctuations in the integral quantities (5):

$$\sum_{kk'} (\langle a_k^+ a_k a_{k'}^+ a_{k'}\rangle - n_k n_{k'}) = \sum_k (n_k^2 + |\sigma_k|^2),$$

$$\sum_{kk'} (\langle a_k a_{-k} a_{k'} a_{-k'}\rangle - \sigma_k \sigma_{k'}) = \sum_k \sigma_k^2.$$

(We used the fact that fluctuations with different **k** and **k**′ (**k** ≠ −**k**′) are uncorrelated.)

In order that the condition for the applicability of the "S-theory" is fulfilled it is thus necessary that

$$\left(\sum_k n_k^2\right) \Big/ \left(\sum_k n_k\right)^2 \ll 1, \quad \left|\sum_k \sigma_k^2\right| \Big/ \left|\sum_k \sigma_k\right|^2 \ll 1,$$

where n_k and σ_k are given by Eqs. (12), (13) and the self-consistency condition (11). As $\sum n_k^2$ is proportional to the volume Ω, and $(\sum n_k)^2 \propto \Omega^2$, we have

$$\left(\sum n_k^2\right)^{1/2} \Big/ \left(\sum n_k\right) \propto \Omega^{-1/2}$$

and it tends to zero as $\Omega \to \infty$. The same considerations are valid for $(\sum \sigma_k^2)^{1/2}/\sum \sigma_k$. We give the exact expression for the model with constant coefficients considered in Sec. 3 (the notation is explained there also):

$$\frac{\sum n_k^2}{\left(\sum n_k\right)^2} = \frac{1}{\Omega}\frac{2\pi v_0}{\nu k_0^2}.$$

The relative fluctuations of the integral quantities decreases with increasing volume of the ferromagnetic as $\Omega^{-1/2}$ and, hence, the assumption (5) that these quantities are small, which is the basis of the "S-theory," is valid in the case of an infinite ferromagnet ($\Omega \to \infty$) considered by us.

5. DISTRIBUTION FUNCTION IN TERMS OF AMPLITUDE AND PHASE VARIABLES

We change to new variables $\mathscr{P}_k$, p_k, Ψ_k, and δ_k:

$$\mathscr{P}_k = {}^1/_2(\rho_k + \rho_{-k}), \quad p_k = \rho_k - \rho_{-k},$$
$$\Psi_k = \phi_k + \phi_{-k}, \quad \delta_k = \phi_k - \phi_{-k},$$

where ρ_k and ϕ_k are determined by the relations

$$a_k = \rho_k^{1/2} \exp(i\phi_k), \quad a_{-k} = \rho_{-k}^{1/2} \exp(i\phi_{-k}).$$

We elucidate the physical meaning of the variables introduced here.

Let a pair of SW with wavevectors $\mathbf{k}$ and $-\mathbf{k}$ be excited. In that case we have for the component of the magnetization vector which is perpendicular to the static magnetization vector of the ferromagnetic (see [12], p. 185)

$$m_x(\mathbf{r},t)=(2\mu_0 M_0)^{1/2}\,\mathrm{Re}\,[a_\mathbf{k}\exp(-i(\omega_\mathbf{k}t-\mathbf{kr}))+a_{-\mathbf{k}}^+\exp(i(\omega_\mathbf{k}t+\mathbf{kr}))].$$

After some transformations we get in terms of the variables ρ and ϕ

$$m_x(\mathbf{r},t)=(2\mu_0 M_0)^{1/2}(\rho_\mathbf{k}^{1/2}+\rho_{-\mathbf{k}}^{1/2})\cos[\mathbf{kr}+{}^1/_2(\phi_\mathbf{k}-\phi_{-\mathbf{k}})]\times\cos[\omega_\mathbf{k}t-{}^1/_2(\phi_\mathbf{k}+\phi_{-\mathbf{k}})]+(2\mu_0M_0)^{1/2}(\rho_\mathbf{k}^{1/2}-\rho_{-\mathbf{k}}^{1/2})\times\sin[\mathbf{kr}+{}^1/_2(\phi_\mathbf{k}-\phi_{-\mathbf{k}})]\sin[\omega_\mathbf{k}t-{}^1/_2(\phi_\mathbf{k}+\phi_{-\mathbf{k}})].$$

Thus, $\Psi_\mathbf{k}=\phi_\mathbf{k}+\phi_{-\mathbf{k}}$ is the temporal phase of the magnetic moment oscillations while $\delta_\mathbf{k}=\phi_\mathbf{k}-\phi_{-\mathbf{k}}$ is the spatial phase of the standing spin wave.

In the new variables one can write the distribution function in the form

$$\Phi_\mathbf{k}=Z_\mathbf{k}^{-1}\exp(-E_\mathbf{k}/\Theta),\quad E_\mathbf{k}=2\hbar\omega_\mathbf{k}\mathscr{P}_\mathbf{k}\{1-(1-p_\mathbf{k}^2/4\mathscr{P}_\mathbf{k}^2)^{1/2}\varkappa_\mathbf{k}\cos(\Psi_\mathbf{k}-\Psi_\mathbf{k}^0)\},\tag{17}$$

where

$$\varkappa_\mathbf{k}=(1-\eta_\mathbf{k})^{1/2},\quad \Psi_\mathbf{k}^0=-\pi/2+\arg(P_\mathbf{k}/(i\tilde{\omega}_\mathbf{k}+\gamma_\mathbf{k})).$$

The normalization constant $Z_\mathbf{k}$ stays as before as the Jacobian for the transformation from $a_\mathbf{k}^+$, $a_\mathbf{k}$, $a_{-\mathbf{k}}^+$, $a_{-\mathbf{k}}$ to $\mathscr{P}_\mathbf{k}$, $p_\mathbf{k}$, $\Psi_\mathbf{k}$, $\delta_\mathbf{k}$ equals unity. It is clear from Eqs. (17) that the distribution function is independent of the magnitude of the spatial phase $\delta_\mathbf{k}$ so that all its values have the same probability.

We consider now the way the "energy" $E_\mathbf{k}$ depends on the variables $\mathscr{P}$, p, and Ψ. First of all, it follows from the consideration in Sec. 3 that $0\le\kappa_\mathbf{k}<1$. Moreover, as $\mathscr{P}_\mathbf{k}=1/2(\rho_\mathbf{k}+\rho_{-\mathbf{k}})$, $p_\mathbf{k}=\rho_\mathbf{k}-\rho_{-\mathbf{k}}$, $\rho_\mathbf{k}$, $\rho_{-\mathbf{k}}\ge 0$, we have $|p_\mathbf{k}|\le 2\mathscr{P}_\mathbf{k}$. Therefore $E_\mathbf{k}\ge 0$ for any values of the variables $\mathscr{P}_\mathbf{k}$, $p_\mathbf{k}$, and $\Psi_\mathbf{k}$. For fixed $\mathscr{P}_\mathbf{k}$ and $p_\mathbf{k}$ the minimum of the function $E_\mathbf{k}$ is reached for the values $\Psi_\mathbf{k}=\Psi_\mathbf{k}^0$.

When there is no pumping all values of the phase $\Psi_\mathbf{k}$ are equally probable and the "energy" $E_\mathbf{k}$ is independent of $\Psi_\mathbf{k}$ so that $E_\mathbf{k}$ as function $\mathscr{P}_\mathbf{k}$ and $\Psi_\mathbf{k}$ describes the surface of a circular cone. When the amplitude $|V_\mathbf{k}|$ of the pumping increases the surface $E_\mathbf{k}(\mathscr{P}_\mathbf{k},\Psi_\mathbf{k})$ loses its symmetry: there appears a "dip" (or a "beak") in the direction $\Psi_\mathbf{k}=\Psi_\mathbf{k}^0$. If we would neglect non-linear effects, when we go through threshold the depth of the "dip" would become such that in it we would have $E_\mathbf{k}<0$ and $E_\mathbf{k}\to-\infty$ as $\mathscr{P}_\mathbf{k}\to\infty$. This would mean that there would not be a stationary distribution as the corresponding distribution function would be unnormalizable. Taking non-linear effects into account leads to a limitation of the depth of the "dip" such that for all $\mathscr{P}$ and Ψ, $E_\mathbf{k}\ge 0$ (see Fig. 3).

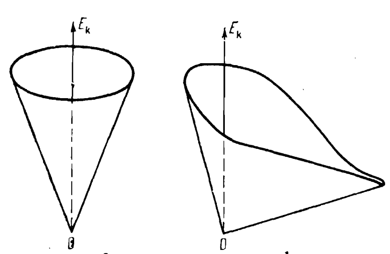

FIG. 3. The surface $E_\mathbf{k}=E_\mathbf{k}(\mathscr{P}_\mathbf{k},\Psi_\mathbf{k})$, $p_\mathbf{k}=0$ drawn in cylindrical coordinates $\rho=\mathscr{P}_\mathbf{k}$, $\varphi=\Psi_\mathbf{k}$; a) pumping amplitude well below threshold, b) pumping amplitude above the threshold.

Integrating the "pair" distribution function $\Phi_\mathbf{k}$ over the variables $\delta_\mathbf{k}$, $p_\mathbf{k}$, and $\mathscr{P}_\mathbf{k}$ we get the distribution function $\Phi_\mathbf{k}(\Psi_\mathbf{k})$

$$\Phi_\mathbf{k}(\Psi_\mathbf{k})=\frac{\eta_\mathbf{k}}{2\pi}\left[1+\left(\frac{\pi}{2}+\beta_\mathbf{k}\right)\left(\frac{1-\eta_\mathbf{k}}{\eta_\mathbf{k}+\mathrm{tg}^2\Delta_\mathbf{k}}\right)^{1/2}\right][\eta_\mathbf{k}+(1-\eta_\mathbf{k})\sin^2\Delta_\mathbf{k}]^{-1},$$

where

$$\beta_\mathbf{k}=\arcsin(\sqrt{1-\eta_\mathbf{k}}\cos\Delta_\mathbf{k}),\quad \Delta_\mathbf{k}=\Psi_\mathbf{k}-\Psi_\mathbf{k}^0.$$

As $|V_\mathbf{k}|\to 0$, $\eta_\mathbf{k}\to 1$, and therefore $\Phi_\mathbf{k}(\Psi_\mathbf{k})\to 1/2\pi$, i.e., all values of the phase $\Psi_\mathbf{k}$ are equally probable. Above threshold the function $\Phi_\mathbf{k}(\Psi_\mathbf{k})$ has a sharp maximum when $\Psi_\mathbf{k}=\Psi_\mathbf{k}^0$:

$$\Phi_\mathbf{k}(\Psi_\mathbf{k})\approx(\eta_\mathbf{k}/2)(\eta_\mathbf{k}+\Delta_\mathbf{k}^2)^{-3/2}.\tag{18}$$

We bear in mind that for PEM $\eta_\mathbf{k}\to 0$ as $|V_\mathbf{k}|\to\infty$.

Using Eq. (18) we determine the fluctuations in the phase $\Psi_\mathbf{k}$ above threshold:

$$\langle|\Psi_\mathbf{k}-\Psi_\mathbf{k}^0|\rangle\approx\sqrt{\eta_\mathbf{k}}.$$

Using the expressions for the quaternary correlators which we obtained earlier we easily determine also the fluctuations in the quantities $\mathscr{P}_\mathbf{k}$ and $p_\mathbf{k}$:

$$(\langle\mathscr{P}_\mathbf{k}^2\rangle-\langle\mathscr{P}_\mathbf{k}\rangle^2)^{1/2}=\langle\mathscr{P}_\mathbf{k}\rangle,\quad(\langle p_\mathbf{k}^2\rangle)^{1/2}=(2\eta_\mathbf{k})^{1/2}\langle\mathscr{P}_\mathbf{k}\rangle,\quad\langle\mathscr{P}_\mathbf{k}\rangle=n_\mathbf{k}=\Theta/\hbar\omega_\mathbf{k}\eta_\mathbf{k},\quad\langle p_\mathbf{k}\rangle=0.$$

Above threshold the fluctuations in the phase $\Psi_\mathbf{k}$ are thus small, but the fluctuations in the amplitude $\mathscr{P}_\mathbf{k}$ are large.

CONCLUSIONS

In the stationary regime above threshold external pumping stabilizes the temporal phase Ψ of the excited pair of magnons. Non-linear effects lead to the appearance of an "additional" pumping which in phase and magnitude correlates in such a way with the external pumping that it nearly completely compensates it. The power entering the PEM system due to the incomplete compensation of the external pumping is exactly equal to the power which can be dissipated by the PEM system due to existing relaxation mechanisms. The temporal phase of the PEM is "coupled" to the pumping phase.

We can represent the PEM distribution as a set of standing waves[7)] while the spatial phase δ determines the position of the nodes of these standing waves. If (like Ψ) the phase δ were stabilized the position of the standing wave nodes would remain constant with time so that the PEM system would be a set of coherent standing waves. However, we have shown above that the phase δ remains random (this is physically connected with the spatial inhomogeneity of the external pumping) for parallel pumping. The position of the nodes of the standing waves thus changes randomly in time and independently for waves with different periods (i.e., with different $\mathbf{k}$). This manifests itself in the noise character of the PEM distribution (for details see [15]).

The presence of fluctuations in $\mathscr{P}_\mathbf{k}$, $\Psi_\mathbf{k}$, and $p_\mathbf{k}$ and the randomness of $\delta_\mathbf{k}$ must lead to the appearance of a finite spectral width of the steady SW generation process for parallel pumping. We note that for a study of the spectral characteristics one must evaluate the many-time correlators.

In the above we have shown the internal consistency of the "S-theory." The large fluctuations in the amplitude $\mathscr{P}$ above threshold must be blamed on the shortcomings of the "S-theory." This is clearly connected with the fact that, as in other self-consistent field schemes (see, e.g., the molecular field approximation in magnetic phase transitions) the "S-theory" does not enable us to take into account the short-range order which is established beyond the "exact phase transition"—above threshold. The PEM distribution obtained is a "gas" of standing SW which do not directly interact with one another but only through the self-consistent field which is determined collectively by all PEM. Therefore, however close the wavevectors **k** and **k**′ are to one another, magnons with those wavevectors are not correlated with one another, i.e., there is no short-range order. At the same time the temporal phase Ψ of all PEM pairs follows the phase of the external pumping. This leads to the appearance of the anomalous correlators $\sigma_{\mathbf{k}}$.

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[1)] A detailed survey of studies in that direction was given in [5].

[2)] Here and henceforth $\langle \ldots \rangle$ denotes averaging over the PEM ensemble.

[3)] Vibrations with wavevectors k, k′ ($k \neq -k'$) are thus uncorrelated. Magnon pairs (k, −k) interact with one another only through the self-consistent field.

[4)] PEM may be concentrated near surfaces, lines, or isolated points in k-space.

[5)] Negative ν^2 lead to the occurrence of k-values for which $n_k < 0$.

[6)] The coefficients S and T are defined such that $S \propto 1/\Omega$, $T \propto 1/\Omega$ so that $\xi_0 S$ is independent of the volume of the ferromagnetic.

[7)] To avoid confusion we emphasize that we considered the limit of an infinite ferromagnetic (volume $\Omega \to \infty$) so that the appearance of standing waves in the description is connected with the PEM creation process and not with boundary conditions.

[1] M. I. Kaganov and V. M. Tsukernik, Zh. Eksp. Teor. Fiz. **37**, 823 (1959) [Sov. Phys.-JETP **10**, 587 (1960)].
[2] E. Schlöman, J. Appl. Phys. **33**, 527 (1962).
[3] Ya. A. Monosov, Nelineĭnyĭ ferromagnetnyĭ rezonans (Non-linear Ferromagnetic Resonance) Mir, 1971.
[4] P. Gottlieb and H. Suhl, J. Appl. Phys. **33**, 1508 (1962).
[5] W. Haubenreisser, Th. Klupsch, and F. Voigt, Phys. stat. sol. (B) **66**, 9 (1974).
[6] V. E. Zakharov, V. S. L'vov, and S. S. Starobinets, Zh. Eksp. Teor. Fiz. **59**, 1200 (1970) [Sov. Phys.-JETP **32**, 656 (1971)].
[7] V. E. Zakharov, V. S. L'vov, and S. S. Starobinets, Usp. Fiz. Nauk **114**, 609 (1974) [Sov. Phys.-Uspekhi **17**, 896 (1975)].
[8] P. Glansdorff and I. Prigogine, Thermodynamic Theory of Structure, Stability, and Fluctuations, Wiley, New York, 1971.
[9] M. Lax, transl. in: Fluktuatsiĭ i kogerenthye yavleniya (Fluctuations and Coherent Effects) Mir, 1974.
[10] H. Haken, Laser Theory, in Handb. Phys. **25**/2c, 1970.
[11] V. M. Tsukernik and R. P. Yankelevich, Zh. Eksp. Teor. Fiz. **68**, 2116 (1975) [Sov. Phys.-JETP **41**, 1059 (1975)].
[12] V. E. Zakharov and V. S. L'vov, Zh. Eksp. Teor. Fiz. **60**, 2066 (1971) [Sov. Phys.-JETP **33**, 1113 (1971)].
[13] A. I. Akhiezer, V. G. Bar'yakhtar, and S. V. Peletminskiĭ, Spinovye volny (Spin Waves) Nauka, Moscow, 1967 [English translation published by North Holland Publishing Cy, Amsterdam, 1968].
[14] L. D. Landau and E. M. Lifshitz, Élektrodinamika sploshnykh sred (Electrodynamics of continuous media) Fizmatgiz, 1959, Sec. 87 [English translation published by Pergamon Press, Oxford, 1960].
[15] A. S. Mikhaĭlov, ZhETF Pis. Red. **20**, 353 (1974) [JETP Lett. **20**, 158 (1974)].

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