Bose system with a condensate, near the terminus point of the homogeneous phase

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The behavior of a compressed Bose system with a condensate as the roton minimum $\Delta = \epsilon(p_0)$ of the system is lowered on increase of the pressure up to the point of absolute instability of the homogeneous phase is studied. The following differences between the exact solution of the problem and the solution in the Bogolyubov approximation are established: 1) at the point of onset of the roton instability the static susceptibility $\chi(p \sim p_0, 0)$ to short-wavelength perturbations remains finite; 2) the roton instability is preceded by a phonon instability to which it is related.

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1. INTRODUCTION

Solidification of the quantum liquids He^4 and He^3 leads to the formation of crystals with strongly pronounced anharmonicity of the zero-point vibrations of the particles at the sites. There are reasons to suppose that the effect of tunneling of particles between sites, leading to violation of the correspondence "one particle per site" and to a whole series of properties connecting the state with that of a quantum liquid[1,2], plays a significant role in these crystals. The possibility of the formation of crystals with nonlocalized particles and the properties of these crystals were investigated earlier[2] with the aid of a model of a "compressed" system (the interaction energy of a pair of particles is small compared with the kinetic energy, while, by virtue of the compression, the mean potential energy per unit volume is of the order of or greater than the kinetic energy). In the present paper the compressed model is used to analyze the behavior of a quantum liquid near the point of absolute instability of the homogeneous phase.

We draw attention to the fact that the compressed model is particularly simple and, at the same time, sufficiently realistic. Even in the framework of the self-consistent approximation (the Hartree-Fock approximation for a Fermi system and the Bogolyubov approximation for a Bose system), the compressed model permits an exact treatment of most of the problems; in the special cases where this approximation is insufficient, it is possible to construct an adequate and simple generalization of the self-consistent approximation (see below). At the same time, the compressed model is capable of reflecting qualitatively all the macroscopic quantum features of real systems: superfluidity, Cooper pairing, quantum crystallization[2], features of the spectrum[3], the effects arising on mixing of quantum liquids[4], etc.

In the framework of the self-consistent approximation, the terminus point of the homogeneous phase is characterized by the fact that the collective pole $\epsilon = \epsilon(p_0)$ of the retarded Green function intersects, while passing through zero, the boundary of the upper half-plane (forbidden by the Lehmann relations) of the frequency; the energy of the system then ceases to correspond to a local minimum, the homogeneous state is found to be unstable to arbitrarily small short-wavelength ($\lambda \sim 1/p_0$) perturbations of the density, and the static susceptibility $\chi(p = p_0, \epsilon = 0)$ becomes infinite. For a Bose system (to the study of which we confine ourselves in the present paper), the Bogolyubov roton minimum $\Delta_B = \epsilon^B(p_0)$ vanishes at the critical point (see Fig. 1; the spectrum with $\Delta_B = 0$ is depicted by the dashed line). This prediction of the self-consistent approximation is in sharp disagreement with the experimental situation, well-studied for He^4, in which the roton minimum decreases only very slightly with increase of the pressure before crystallization.

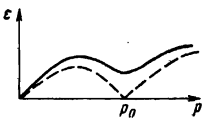

FIG. 1

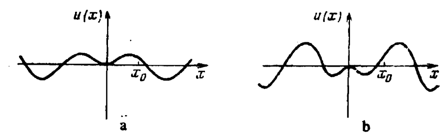

FIG. 2

The impossibility of making the roton minimum vanish can be explained using the example of a compressed system. A rigorous treatment of the compressed model shows that the self-consistent approximation turns out to be invalid near the critical point, and the exact picture of the appearance of the instability differs from the Bogolyubov picture in two respects: 1) a short-wavelength instability sets in at a nonzero value $\Delta_c \neq 0$ of the roton minimum (i.e., the value $\Delta = 0$ is found to be unattainable in principle), and as a result the short-wavelength static susceptibility $\chi(p_0, 0)$ remains finite at the critical point (contrary to what is stated by, e.g., Schneider and Enz[5]); 2) the approach of Δ to Δ_c leads to a long-wavelength instability, which thus sets in earlier than the short-wavelength one.

The reason for the inapplicability of the self-consistent approximation near the critical point is well-known: the small factor in the correlation corrections is compensated by the divergence associated with the unlimited growth of the amplitude of the fluctuations (i.e., of the zero-point or thermal oscillations of the collective modes)[1)]. We shall note the characteristics of our problem: 1) the fluctuations, by renormalizing the spectrum of the collective modes, can either hasten or delay the appearance of the instability (cf. the one-dimensional analogs in Fig. 2; x_0 characterizes the amplitude of the zero-point or thermal vibrations; case (a) corresponds to premature disruption of the stability and case (b) to delay in the onset of the instability); in our case the first possibility is realized; 2) the divergence of the short-wavelength fluctuations is removed in our problem when a certain extremely simple subsequence of diagrams is taken into account, such that, moreover, the diagrams not taken into account are finite and negligibly small; in other words, the fluctuations are self-consistent in the framework of a finite integral equation; the latter circumstance makes it possible to carry out a simple and rigorous treatment of the state of the system, right up to the point at which it is absolutely unstable.

2. CHOICE OF DIAGRAMS. SELF-CONSISTENT EQUATION FOR THE ROTON MINIMUM

We shall consider a condensed Bose system with a condensate:

$$\alpha \equiv m|V_{p_0}|p_0 \ll 1, \quad p_0 \ll n^{1/3}, \quad m|V_{p_0}|np_0^{-2} \gg 1 \tag{1}$$

(V_p is the Fourier transform of the pair potential and p_0 is the characteristic momentum of the interaction). The maximum contribution to Σ_{ik} is made by diagrams that to not contain integrations over intermediate momenta (Fig. 3). They correspond to the Bogolyubov approximation:

$$G_B'(p) = \frac{\varepsilon + \varepsilon_p^0 + n_0 V_p}{\varepsilon^2 - \varepsilon_p^{B2} + i\delta}, \quad \hat{G}_B(p) = \frac{-n_0 V_p}{\varepsilon^2 - \varepsilon_p^{B2} + i\delta}$$
$$\varepsilon_p^B = [\varepsilon_p^0(\varepsilon_p^0 + 2n_0 V_p)]^{1/2}, \quad \varepsilon_p^0 = p^2/2m. \tag{2}$$

Under extremely general assumptions about the interaction V_p the spectrum ϵ_p^B has a nonmonotonic (phonon-roton) form. With increase of the density $n \approx n_0$ (or of the pressure) at a fixed temperature, the Bogolyubov roton minimum tends to zero:

$$\Delta_B(n) = \varepsilon_{p_0}^0 \left(\frac{n_c^B - n}{n_c^B}\right)^{1/2} \tag{3}$$
$$(\Delta_B = \min \varepsilon_p^B \equiv \varepsilon_{p_0}^B, \quad n_c^B = \varepsilon_{p_0}^0 / 2|V_{p_0}|).$$

In the diagrams G supplementing the Bogolyubov approximation (2) it is natural to use G_B (2) as the "unperturbed" Green functions. In their analytic expressions all these diagrams contain integrations over intermediate momenta, and each integration introduces the small parameter α (1) and a factor that diverges as $n \to n_c^B$. In particular the diagrams of lowest order in α contain a single integration over the momentum transfer and diverge like $\sim \ln \Delta_B$ and $\sim 1/\Delta_B$. These diagrams include the diagrams analogous to the Hartree-Fock diagrams for a Fermi system ($\sim \ln \Delta_B$) and also those which describe the decay (virtual or real) of the initial excitation into two others ($\sim 1/\Delta_B$).

Inasmuch as the reason for the divergence of the diagrams lies in the unrestricted decrease of Δ_B, in the case when the exact roton minimum Δ does not tend to zero the divergence should disappear when the Bogolyubov Green functions are simply replaced by the exact Green functions (i.e., when we go over to "skeleton" diagrams of a certain kind). The opposite possibility ($\Delta \to 0$) would imply the necessity of seeking other ways of effectively summing the diagrams to remove their divergence. We shall show that for our problem the first possibility is realized, it being sufficient, in the entire region of stability of the homogeneous phase, to take into account the simplest skeleton diagrams $\Delta\Sigma^{(1)}$ with a single integration; the self-consistent equation for the Green function with $\Delta\Sigma^{(1)}$ (Fig. 4)

$$G_{ik}(p) = G_{ik}^0(p) + G_{im}^0(p)\Sigma_{mn}(p)G_{nk}(p),$$
$$\Sigma_{mn} = \Sigma_{mn}^B + \Delta\Sigma_{mn}^{(1)}, \tag{4}$$
$$G_{ik}(p) = \begin{pmatrix} G'(p) & \hat{G}(p) \\ \check{G}(p) & G'(-p) \end{pmatrix}, \quad G_{ik}^0(p) = \begin{pmatrix} G^0(p) & 0 \\ 0 & G^0(-p) \end{pmatrix}$$

gives everywhere a value $\Delta \geq \Delta_c > 0$ such that all the disregarded skeleton diagrams $\Delta\Sigma - \Delta\Sigma^{(1)}$ are negligibly

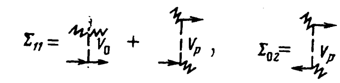

FIG. 3

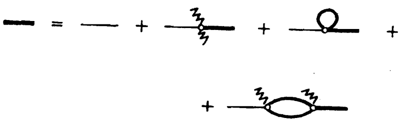

FIG. 4

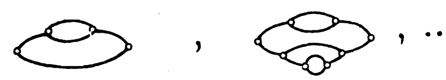

FIG. 5

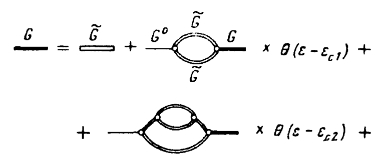

FIG. 6

small. In other words, the perturbation theory series, rearranged as a result of going over in the diagrams from the Bogolyubov Green functions to the "exact" (self-consistent in the sense of Eq. (4)) Green functions, is characterized by a new small parameter ($\alpha^* = \alpha(\epsilon_p^0)^2/\Delta_c^2$) accompanying each integration over the intermediate momenta, and, since the new series does not have divergences, it is natural that an adequate approximation is given by taking into account the contribution of lowest order in α^* to $\Delta\Sigma$.

The skeleton diagrams with a single integration effectively sum an innumerable set of diagrams of the old perturbation-theory series with "insertions" in the intermediate particle lines (such diagrams arise in iterations of the right-hand side of Eq. (4); see Fig. 5). Diagrams with insertions are, naturally, omitted when we go over to skeleton diagrams. It is possible to retain them, however, if we agree to associate the intermediate Green functions with the pole contribution only ($G \to \tilde{G}$, Fig. 6; ϵ_{cn} is the threshold for decay into (n + 1) stable particles). The diagrams of Fig. 6 give a clear picture of the characteristics of all the non-pole singularities of the Green functions and give an expression for their contribution in terms of the exact values of the real pole frequencies. This approach corresponds to representing Im $\Delta\Sigma$ in the form of the sum of the probabilities of all real decays (a well-known consequence of the unitarity of the S-matrix).

We note that diagrams with insertions are distinguished from all other diagrams with multiple integration by the fact that their order in α^* is not less than that of diagrams with a single integration, but the same (this is why they are taken into account in (4)); thus, in integrating the product of two Green functions in $\Delta\Sigma^{(1)}$ (4), besides the pole contribution it is also necessary to take into account the integral along the cut; for more detail about this, see below.

The essential point is that the corrections to Σ_{mn}^B despite their fundamental role in the region $n \sim n_c^B$, remain small everywhere ($\Sigma_{mn}^B \sim \epsilon_{p_0}^0 \gg \Delta\Sigma^{(1)} \sim \alpha^*\Sigma_{mn}^B$; see below). Therefore, in the exact equation for the spectral curve[6, 7]

$$[\varepsilon-A(p)]^2-[\varepsilon_p{}^0+S(p)-\mu+\Sigma_{02}(p)][\varepsilon_p{}^0+S(p)-\mu-\Sigma_{02}(p)]=0$$

they must be taken into account only in the expression $[\epsilon_p^0 + S(p) - \mu - \Sigma_{02}(p)]$, where they appear in a sum with the smallest of the quantities used $(\epsilon_p^0 + 2n_0V_p \sim \Delta^2/\epsilon_p^0)$ (which determines the order of magnitude of the corrections); in the term with $\epsilon \sim \Delta$ the correction $A(p) \sim \Delta^2/\epsilon_p^0$ must also be neglected.

Corrections to the Bogolyubov spectrum are important only in the region $p \approx p_0$, where

$$\varepsilon_p^{B2}\approx\Delta_B^2+{}^1/_2c_B(p-p_0)^2,\quad \varepsilon_p^2\approx\Delta^2+{}^1/_2c(p-p_0)^2;$$

inasmuch as $|c_B - c| \ll c_B$, we can neglect the correction in the coefficient c. Thus,

$$\varepsilon_p^2\approx\Delta^2+{}^1/_2c_B(p-p_0)^2\quad (p\approx p_0), \tag{5}$$

$$\Delta^2=\varepsilon_{p_0}{}^0[\varepsilon_{p_0}{}^0+2n_0V_{p_0}+{}^1/_2(\Delta\Sigma_{11}^{(1)}(p_0,\Delta)+\Delta\Sigma_{11}^{(1)}(p_0,-\Delta))-\Delta\mu^{(1)}+\Delta\Sigma_{02}(p_0,\Delta)].$$

In the calculation of $\Delta\Sigma_{mn}^{(1)}$, $\Delta\mu = \Delta\Sigma_{11}^{(1)}(0) - \Delta\Sigma_{02}^{(1)}(0)$, as the integrable Green functions in the diagrams we must take functions in which the numerator coincides with the Bogolyubov expression (2) while the corrections are taken into account in the denominator:

$$G'(p,\varepsilon)=\frac{\varepsilon+\varepsilon_p{}^0+n_0V_p}{\varepsilon^2-\varepsilon_p{}^2-i\,\mathrm{Im}\,\varepsilon_p{}^0\delta\Sigma^{(1)}(p,\varepsilon)},$$
$$\hat{G}(p,\varepsilon)=\frac{-n_0V_p}{\varepsilon^2-\varepsilon_p{}^2-i\,\mathrm{Im}\,\varepsilon_p{}^0\delta\Sigma^{(1)}(p,\varepsilon)}, \tag{6}$$
$$\varepsilon_p=\{\varepsilon_p{}^0[\varepsilon_p{}^0+2n_0V_p+\mathrm{Re}\,\delta\Sigma(p,\varepsilon_p)]\}^{1/2},$$
$$\delta\Sigma^{(1)}(p,\varepsilon)={}^1/_2[\Delta\Sigma_{11}^{(1)}(p,\varepsilon)+\Delta\Sigma_{11}^{(1)}(p,-\varepsilon)]-\Delta\mu^{(1)}+\Delta\Sigma_{02}^{(1)}(p,\varepsilon).$$

Substituting (6) into the diagrams $\Delta\Sigma_{mn}^{(1)}$ gives a closed equation for the function $\delta\Sigma^{(1)}(p,\epsilon)$; the solution of this equation is substantially simplified if we take into account that, in the integration of G' and $\hat{G}$, inclusion of $\delta\Sigma^{(1)}(p,\epsilon)$ is, in fact, important only in the region $p \approx p_0$, $\epsilon \sim \Delta$, so that the matter reduces to the self-consistent calculation of the constants[2)] Δ (or $\delta\Sigma^{(1)}(p_0,\Delta)$) and $\mathrm{Im}\,\delta\Sigma^{(1)}(p_0,2\Delta)$ only. If we use the representation of Fig. 6, only Δ, as the sole constant requiring self-consistent calculation, will appear in the right-hand side of Eqs. (4).

3. ESTIMATE OF THE INCLUDED AND DISCARDED DIAGRAMS. BEHAVIOR OF THE ROTON MINIMUM

The principal contribution fo $\delta\Sigma^{(1)}$ (6) is made by the diagrams $\Delta\Sigma_{mn}^{(1)}$ with two intermediate Green functions (i.e., by those which, in the old perturbation theory, diverge fastest: $\sim 1/\Delta_B$, $\Delta_B \to 0$). The correction $\Delta\mu^{(1)}$ also contains the diagrams with two Green functions, their divergence as $\Delta_B \to 0$ being higher than in $\Delta\Sigma_{mn}^{(1)}(p \neq 0, \epsilon \neq 0)$ $(\sim 1/\Delta_B^2)$; however, when these diagrams are summed there appears in the numerator of the integrand an extra small factor $\sim\Delta^2$, which lowers the divergence to a logarithmic one: the numerator

$$\{G(q)[G'(q)-\check{G}(q)]+g(q)[g(-q)-g(q)]\}$$
$$=2[\omega^2-\varepsilon_q{}^0(\varepsilon_q{}^0+2n_0V_q)],\quad q=(\mathbf{q},\omega).$$

Here we have used the notation:

$$g(p)=G'(p)+\check{G}(p)\quad(\text{or } G'(p)+\hat{G}(p)),$$
$$G(p)=G'(p)+G'(-p)+\hat{G}(p)+\check{G}(p).$$

In the integration over the 3-momenta in the diagrams $\Delta\Sigma_{mn}^{(1)}$ with two Green functions the divergent contribution is made by the region in which both intermediate 3-momenta are close to p_0. Thus, $\delta\Sigma^{(1)}$ includes a sum of terms of the form

$$n_0V_{p_0}{}^2i\int G_1(\pm p+q)G_2(q)\frac{d\omega}{2\pi}\quad (p\sim p_0), \tag{7}$$

integrated over d^3q in the region $|\pm \mathbf{p} + \mathbf{q}| \sim p_0$, $q \sim p_0$ (here G_1 and G_2 are any functions from the set G', $\hat{G}$, G, g). The Green functions G_i (6) can be represented in the form of a pole term (differing from the Bogolyubov function only by a shift in the pole frequency $\epsilon_p^B \to \epsilon_p$) and an integral along the cut:

$$G_i(q,\omega)=\frac{A_{iq}}{\omega-\varepsilon_q+i\delta}-\frac{B_{iq}}{\omega+\varepsilon_q-i\delta}+\int_{2\Delta}^{\infty}\left[\frac{A_i(q,E)}{\omega-E+i\delta}-\frac{B_i(q,E)}{\omega+E-i\delta}\right]dE \tag{8}$$

(we assume that decay into rotons appears before other decays, i.e., $\mathrm{Im}\,\delta\Sigma^{(1)} = \mathrm{Im}\,\delta\Sigma^{(1)}\theta(\epsilon - 2\Delta)$).

The positivity of the coefficients A,B has been proved for the function G' in its general form (cf., e.g.,[6]); in the present case the same property is possessed by $\hat{G}$ and, consequently, by G and g. In fact, in our approximation (6),

$$\hat{A}_q=\hat{B}_q=-n_0V_q/2\varepsilon_q>0\quad (V_{q\approx p_0}<0),$$
$$\hat{A}(q,E)=\hat{B}(q,E)=-\pi^{-1}\,\mathrm{Im}\,\hat{G}(q,\pm E)$$
$$=\frac{n_0V_q\varepsilon_q{}^0\,\mathrm{Im}\,\delta\Sigma^{(1)}}{\pi[E^2-\varepsilon_q{}^0(\varepsilon_q{}^0+2n_0V_q+\mathrm{Re}\,\delta\Sigma^{(1)})]^2+(\varepsilon_q{}^0\,\mathrm{Im}\,\delta\Sigma^{(1)})^2}>0 \tag{9}$$
$$(\mathrm{Im}\,\delta\Sigma^{(1)}<0).$$

Taking into account what has been said, it is not difficult to determine the sign and order of magnitude of $\delta\Sigma^{(1)}$ (both characteristics are the same for all terms of $\delta\Sigma^{(1)}$ (8)). Substituting (8) into (7), we find

$$n_0V_{p_0}{}^2i\int G_1(\pm p+q)G_2(q)\frac{d\omega}{2\pi}=-n_0V_{p_0}{}^2\left\{\frac{A_{1(\pm p+q)}B_{2q}}{\varepsilon_{\pm p+q}+\varepsilon_q\mp\varepsilon-i\delta}+\frac{B_{1(\pm p+q)}A_{2q}}{\varepsilon_{\pm p+q}+\varepsilon_q\pm\varepsilon-i\delta}\right.$$
$$\left.+\iint_{(2\Delta)}^{(\infty)}\left[\frac{A_1(\pm p+q,E)B_2(q,E')}{E+E'\mp\varepsilon-i\delta}+\frac{B_1(\pm p+q,E)A_2(q,E')}{E+E'\pm\varepsilon-i\delta}\right]dE\,dE'\right\}. \tag{10}$$

Near the roton minimum ($p \approx p_0$, $\epsilon \approx \Delta$) there are no real decays, so that the sign of the denominators of all the fractions in (10) is positive. Hence we find that $\delta\Sigma^{(1)}(p_0,\Delta) < 0$.

The relations (9) give the following estimate for the coefficients near the minima of the denominators of the fractions in (10):

$$A_q,B_q\sim\frac{n_0|V_{p_0}|}{\Delta},\quad A(q,E),B(q,E)\sim\frac{n_0|V_{p_0}|}{\Delta^2}. \tag{11}$$

Here it has been taken into account that

$$\varepsilon_{p_0}{}^0\,\mathrm{Im}\,\delta\Sigma^{(1)}(p_0,2\Delta)\sim\varepsilon_{p_0}{}^0\delta\Sigma^{(1)}(p_0,\Delta)\sim\Delta^2.$$

It can be seen that both terms in the curly brackets in (10)—the pole contribution and the integral along the cut—are of the same order; qualitatively, this is explained by the fact that for small Δ the branch point (2Δ) turns out to be close to the pole.

The main contribution to $\delta\Sigma^{(1)}$ when (10) is integrated over d^3q is made by the region near the minima of the denominators, where ϵ_q, $\epsilon_{\pm p+q} \sim \Delta$:

$$\frac{p_0\Delta^2}{c_B}\qquad c_B=\frac{d^2}{dp^2}[\varepsilon_p{}^0(\varepsilon_p{}^0+2n_0V_p)]_{p=p_0}\sim\frac{p_0{}^2}{m^2}. \tag{12}$$

Using (10), (11) and (12), we find

$$-\delta\Sigma^{(1)}\sim n_0V_{p_0}{}^2\frac{n_0{}^2V_{p_0}{}^2}{\Delta^3}\frac{p_0\Delta^2}{c_B}\sim\alpha\frac{(\varepsilon_{p_0}{}^0)^2}{\Delta}$$
$$(n_0|V_{p_0}|\approx\varepsilon_{p_0}{}^0/2\sim\varepsilon_{p_0}{}^0). \tag{13}$$

Taking (5), (6) and (13) into account, we obtain an equa-

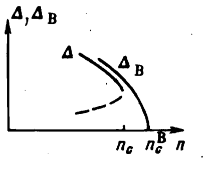

FIG. 7

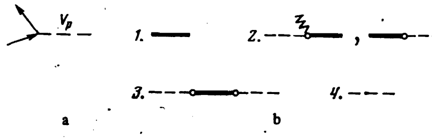

FIG. 8

tion for the corrected value of the roton minimum Δ as a function of the density:

$$\Delta^2+\alpha R(\varepsilon_{p_0}^0)^3/\Delta=\Delta_B^2(n), \quad R\sim 1,$$
$$\Delta_B=[\varepsilon_{p_0}^0(\varepsilon_{p_0}^0+2nV_{p_0})]^{1/2}=\varepsilon_{p_0}^0\left(\frac{n_c^B-n}{n_c^B}\right)^{1/2} \tag{14}$$

(we have neglected terms $\sim\alpha\epsilon_{p_0}^0$ and $\sim\alpha\epsilon_{p_0}^0\ln(\Delta/\epsilon_{p_0}^0)$ in $\delta\Sigma^{(1)}$, inasmuch as they are small compared with the leading correction $\sim\alpha(\epsilon_{p_0}^0)^2/\Delta\sim\alpha^{1/3}\epsilon_{p_0}^0$; correspondingly, as before, we need not distinguish n and n_0 in the expression for Δ_B).

It can be seen from (14) that the critical density value n_c, above which the real solution of the equation for Δ vanishes, is reduced in comparison with that in the Bogolyubov approximation (cf. (3)):

$$\Delta_B^2(n_c)=\min\left(\Delta^2+\alpha R(\varepsilon_{p_0}^0)^3/\Delta\right)=3(\alpha R/2)^{2/3}(\varepsilon_{p_0}^0)^2, \tag{15}$$

i.e.,

$$n_c=n_c^B[1-3(\alpha R/2)^{2/3}],$$

and the lowest value of Δ is found to be nonzero:

$$\min\Delta(n)=\Delta(n_c)\equiv\Delta_c=(\alpha R/2)^{1/3}\varepsilon_{p_0}^0. \tag{16}$$

Near the critical point,

$$\Delta-\Delta_c\approx\varepsilon_{p_0}^0\left(\frac{n_c-n}{3n_c}\right)^{1/2}. \tag{17}$$

For $n < n_c$, Eq. (14) has a further branch of solutions (below Δ_c; see the dashed line in Fig. 7); however, the corresponding state of the system cannot be realized, since, as will be shown, even the point $\Delta=\Delta_c$ at which the short-wavelength instability arises is inaccessible because of a long-wavelength instability that arises beforehand (following the onset of the long-wavelength instability, our treatment, which does not take into account the phonon region of the spectrum, ceases to be valid).

We shall now prove that the discarded diagrams are small. An arbitrary skeleton diagram can be regarded as an assembly of 2m three-point vertices (Fig. 8a) linked by lines of four types (Fig. 8b); in total there are 3m-1 internal lines; the number of integrations over independent intermediate 4-momenta is $(3m-1)-(2m-1)=m$ (the 2m vertices introduce $2m-1$ independent δ-functions). The maximum divergence of a diagram is attained, obviously, in the case when all the internal lines contain the Green function for particles above the condensate, i.e., when they belong to types 1–3 in Fig. 8b. All the $3m-1$ intermediate 3-momenta q can be made close in modulus to the momentum p_0 of the roton minimum $(c_B(q-p_0)^2\sim\Delta^2)$, since the m independent 3-momenta contain 3m parameters; thus, the effective region of integration spans a volume $p_0(\Delta/c_B^{1/2})^{3m-1}$. After integration over the m frequencies of m internal lines, a factor $(n_0|V_{p_0}|/\Delta)^m$ enters; the remaining $(2m-1)$ internal lines produce a factor $(n_0|V_{p_0}|/\Delta^2)^{2m-1}$. Taking into account that, in the absence of lines of type 4 in the diagram, each three-point vertex introduces a factor $n_0^{1/2}|V_{p_0}|$, we find the following estimate for the contribution of the most divergent skeleton diagram with 2m vertices:

$$\delta\Sigma^{(m)}\sim p_0\left(\frac{\Delta}{c_B^{1/2}}\right)^{3m-1}\left(\frac{n_0|V_{p_0}|}{\Delta}\right)^m\times$$
$$\left(\frac{n_0|V_{p_0}|}{\Delta^2}\right)^{2m-1}(n_0^{1/2}|V_{p_0}|)^{2m}\sim\Delta(\alpha n_0^2|V_{p_0}|^2/\Delta^2)^m,$$

i.e., taking into account that $\Delta\sim\Delta_c\sim\alpha^{1/3}\epsilon_{p_0}^0$,

$$\delta\Sigma^{(m)}\sim\alpha^{(m+1)/3}\varepsilon_{p_0}^0.$$

Thus, the corrections $\delta\Sigma^{(m>1)}$ can be neglected in comparison with $\delta\Sigma^{(1)}$.

In conclusion, we shall convince ourselves that for diagrams with "insertions" (Figs. 5, 6) the additional integrations do not introduce a small factor: the contribution of two Green functions with an insertion $(n_0|V_{p_0}|/\Delta^2)^2\delta\Sigma^{(1)}\sim(\alpha^{1/3}\epsilon_{p_0}^0)^{-1}$ coincides with that of a single Green function $(n_0|V_{p_0}|/\Delta^2)\sim(\alpha^{1/3}\epsilon_{p_0}^0)^{-1}$. This circumstance corresponds to the necessity of taking self-consistent Green functions into account in the decay diagram for $\Delta\Sigma$ (which has been done in Eq. (14)).

4. LONG-WAVELENGTH INSTABILITY

We shall show that the critical point $\Delta=\Delta_c$ due to the "self-consistency of the rotons" (i.e., of the roton fluctuations) is, like $\Delta_B=0$, unattainable in practice. To be precise, in the immediate vicinity of the point $\Delta=\Delta_c$ the correction (associated with the diagrams for virtual decay into two rotons) in the concentration derivative of the chemical potential is negative and diverges, leading to instability of the phonon part of the spectrum: for a certain $\Delta=\Delta_c'>\Delta_c$ the sound velocity

$$u=\left(\frac{n}{m}\frac{d\mu}{dn}\right)^{1/2}$$

vanishes (the compressibility is infinite)[3]. An important point is that the unattainability of the point $\Delta=\Delta_c$ does not remove the necessity of taking into account the "self-consistency of the rotons": the phonon instability arises significantly later than the point at which this self-consistency becomes important. We find

$$\mu=\Sigma_{11}(0)-\Sigma_{02}(0)=nV_0+i\int\frac{d^4q}{(2\pi)^4}V_q\frac{\omega+\varepsilon_q^0+2n_0V_q}{\omega^2-\varepsilon_q^2-i\varepsilon_q^0\,\mathrm{Im}\,\delta\Sigma^{(1)}}$$
$$+i\int\frac{d^4q}{(2\pi)^4}n_0V_q^2\frac{-2(\omega^2-\varepsilon_q^{B2})}{(\omega^2-\varepsilon_q^2-i\varepsilon_q^0\,\mathrm{Im}\,\delta\Sigma^{(1)})^2}$$
$$=nV_0-\int\frac{d^3q}{(2\pi)^3}V_q\left[\frac{1}{2}\left(1-\frac{\varepsilon_q^{B2}}{\varepsilon_q\varepsilon_q^0}\right)\right. \tag{18}$$
$$\left.+\int_{-\infty}^{-2\Delta}\frac{d\omega}{2\pi}\frac{-(\omega+\varepsilon_q^0+2n_0V_q)\varepsilon_q^0\,\mathrm{Im}\,\delta\Sigma^{(1)}}{(\omega^2-\varepsilon_q^2)^2+(\varepsilon_q^0)^2\,\mathrm{Im}\,\delta\Sigma^{(1)2}}\right]-\int\frac{d^3q}{(2\pi)^3}n_0V_q^2\left[\frac{\varepsilon_q^{B2}+\varepsilon_q^2}{2\varepsilon_q^3}\right.$$
$$\left.-\int_{-\infty}^{-2\Delta}\frac{d\omega}{2\pi}\frac{4(\omega^2-\varepsilon_q^{B2})\varepsilon_q^0(\omega^2-\varepsilon_q^2)\,\mathrm{Im}\,\delta\Sigma^{(1)}}{[(\omega^2-\varepsilon_q^2)^2+(\varepsilon_q^0)^2\,\mathrm{Im}\,\delta\Sigma^{(1)2}]^2}\right].$$

Near $\Delta=\Delta_c$ the term $(d\mu/d\Delta)(d\Delta/dn)$ makes a divergent contribution to $d\mu/dn$ (in fact, $d\Delta/dn\approx-\epsilon_{p_0}^0[3(n_c-n)n_c]^{-1/2}\to-\infty$; cf. (17)). The principal contribution to the coefficient $d\mu/d\Delta$ is associated with the last term of (18); this contribution is certainly positive ($\mathrm{Im}\,\delta\Sigma^{(1)}<0$), and corresponds to the estimate

$$p_0^2\frac{\Delta}{c_B^{1/2}}\frac{n_0V_0^2}{\Delta^2}\sim\alpha^{2/3}\qquad(V_0\sim|V_{p_0}|)$$

(the principal role in the integral over d^3q is played by the region $q\sim p_0$, where $\epsilon_q=[\Delta^2+1/2c_B(q-p_0)^2]^{1/2}\sim\Delta$). Thus, the sound velocity vanishes when $n=n'$, where

$$K\frac{\alpha^{2/3}\varepsilon_{p_0}{}^0}{[n_c(n_c-n')]^{1/2}}=V_0(k\sim 1),\quad \text{i.e.,}\ n_c-n'/n_c\sim\alpha^{4/3}.$$

It can be seen that n′ actually lies inside the region in which the self-consistency of the rotons is important, i.e., in which $\Delta - \Delta_c \sim \Delta_c$, $(n_c - n)/n_c \sim \alpha^{2/3}$ (cf. (16), (17)).

5. CONCLUSION

We shall compare the exact picture of the behavior of the spectral curve near the point of onset of instability of the homogeneous phase with the Bogolyubov picture. In the Bogolyubov approximation the instability is associated with the vanishing of the roton minimum, following which the pole of the Green function is found to be imaginary and forbidden by the Lehmann relations:

$$\varepsilon^B(p_0,\ n<n_c^B)=\pm\Delta_B(n),\quad \Delta_B(n_c^B)=0,\quad \varepsilon^B(p_0,\ n>n_c^B)=\pm i|\varepsilon^B|.$$

In the exact approach the roton minimum satisfies the self-consistent equation $\Delta^2 + \alpha R(\epsilon^0_{p_0})^3/\Delta = \Delta^2_B(n)$, the real positive solutions of which vanish for $\Delta = \Delta_c \neq 0$ and $n = n_c < n_c^B$ $(n_c^B - n_c/n_c^B \sim \alpha^{2/3}$, $\Delta_c/\epsilon p_0^0 \sim \alpha^{1/3})$; at the point $n = n_c$ the pair of real roots of the equation (one of them is physical) merge together and are transformed into the pair of complex roots $\epsilon\,(p_0; n > n_c = \Delta_c \pm i\gamma$ corresponding to the instability. By virtue of the inequality $\Delta_c \neq 0$, the static susceptibility $\chi(p_0, 0)$ does not diverge at the critical point (unlike in the Bogolyubov approximation).

A still deeper difference between the exact picture of the instability and the Bogolyubov picture lies in the fact that the roton instability is preceded by a related phonon instability: as $n \to n_c$ $(\Delta \to \Delta_c)$ the roton region gives a negative divergent contribution to the elasticity, so that at a certain point $n = n' < n_c$ (in the immediate vicinity of n_c, $n_c - n'/n_c \sim \alpha^{4/3}$) the sound velocity vanishes. Thus, at the true terminus of the homogeneous phase the phonon pole frequencies pass through zero and become imaginary: $\epsilon = up$,

$$u^2=\frac{n}{m}\frac{d\mu}{dn}\bigg|_{n>n'}<0.$$

In contrast to the case $V_0 \leq \int d^3p V_p^2/2\epsilon_p^0$, the phonon instability here does not lead to the appearance of a new stable homogeneous (liquid) phase; the only local stable state in the region $n > n'$ for a model with $\alpha \ll 1$ is the coherent-crystal state[2], which becomes energetically favorable long before the appearance of absolute instability of the homogeneous phase (at an appreciable distance from $n = n_c$).

The analysis carried out above of the behavior of the spectrum near the terminus of the homogeneous phase is an exact solution of the problem in the case of a compressed Bose system with the weak interaction (1). However, the results obtained–the impossibility of vanishing of the roton minimum and the connection between the roton and phonon instabilities–have a general character (they are valid for any Bose system, including superfluid He^4). In fact, these effects are due to processes of virtual decay into two rotons and can be obtained qualitatively without model limitations (by analyzing the decay self-energy diagrams that diverge as $\Delta \to 0$). The proximity of the short-wavelength instability to the critical point makes it possible to postulate that, in the general case, the stable phase in the supercritical region of densities (pressures) will be the crystalline state; the extent to which quantum anharmonicity is manifested in such crystals will be determined by the character of the interaction between the particles.

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1)In special cases the role of the fluctuations can be suppressed by the small size of their phase volume.

2)We note that formal application of the method described in the region $p, \epsilon \to 0$ would lead to difficulties associated with the divergence (integrals containing two phonon Green functions $G(p/2 + q)G(p/2 - q)$ and the approximate value of the three-point vertex $\gamma(p \to 0, q \to 0)$ diverge). These difficulties arise in any model of a Bose system with a condensate: the divergences in the long-wavelength region neutralize the small parameter. Concerning the surmounting of these difficulties, see [8]; the treatment carried out there justifies the use of convergent diagrams with the lowest power of the small parameter, so that, in lowest order in the small parameter, the Bogolyubov approximation is valid for the Green functions and spectrum in our problem for $p, \epsilon \to 0$.

3)Unlike the case of the roton minimum, the vanishing of the sound velocity does not lead to the appearance of a new divergence (this is obvious from the form of the Green functions in the long-wavelength limit [6,8]: $G' = -\hat{G} = n_0 m u^2/n(\omega^2 - u^2q^2 + i\delta)$). Thus, the point at which the sound velocity goes to zero is attainable in practice.

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