The theory of quantum processes in the field of a strong electromagnetic wave

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We apply an operator diagram technique, previously developed for considering phenomena in a uniform external field, to processes in the field of a plane electromagnetic wave. The calculations are based upon a specific technique of disentangling operator expressions. We find the mass operator of scalar and spinor particles in the field of an elliptically polarized wave of a general shape described by a double integral. Its imaginary part gives a new representation of the total probability for the emission of a particle in the field of a wave. We analyze polarization effects for spinor particles.

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1. INTRODUCTION

The fast development of laser techniques which make it possible to obtain waves with a very high electromagnetic field strength (up to 10^9 V/cm) has stimulated a broad study of quantum processes in the field of a strong electromagnetic wave. Recently a large number of papers has appeared devoted to effects which occur when an electron and a photon move in the field of a strong plane wave.[1–5] Solutions of the Dirac equation in the field of a plane wave have been used in these papers. The interaction of charged particles with the field of an electromagnetic wave is thereby taken exactly into account. However, the interaction with the radiation field is considered in the lowest order of perturbation theory. The probabilities which are found are the imaginary part of the mass and polarization operators. Of considerable interest are also the real parts of these operators (corrections to the mass, and so on). Unfortunately, it is difficult to find these quantities in the framework of the above-mentioned technique and this has up to the present not been done.

Recently three of the present authors have developed an operator approach to problems of electromagnetic interactions in an external electromagnetic field.[6] We have found in the framework of that approach the mass operator in a spatially and temporally constant electromagnetic field. We use this operator approach in the present paper to find the mass operator in the field of a plane wave of arbitrary shape for spin-zero and spin-1/2 particles.[1)] Notwithstanding the generality of the approach and of the initial expressions, a rational method of calculation for the case of motion in the field of a plane wave differs appreciably from the calculation for the case of motion in a uniform field. The basis of this method is a specific technique for disentangling operator expressions. Using that we are able to obtain an expression for the mass operator of a particle in an elliptically polarized wave of arbitrary shape. We need to give the actual shape of the wave only when evaluating the average over the mass shell in the last stage of the calculation. This average is given by a double integral, and in the particular case of a circularly polarized wave by a single integral.

We shall describe a plane wave of arbitrary shape by the potential

$$A_\mu(\varphi)=a_{1\mu}\psi_1(\varphi)+a_{2\mu}\psi_2(\varphi), \tag{1.1}$$

where $\varphi \equiv \kappa x = \kappa^0 x^0 - \boldsymbol{\kappa}\cdot\mathbf{x}$, while

$$\varkappa^2=0, \qquad \varkappa a_1=\varkappa a_2=a_1 a_2=0. \tag{1.2}$$

In the case of linear polarization $a_{2\mu} = 0$. The field strength of the wave is

$$F_{\mu\nu}=(\varkappa_\mu a_{1\nu}-\varkappa_\nu a_{1\mu})\psi_1'(\varphi)+(\varkappa_\mu a_{2\nu}-\varkappa_\nu a_{2\mu})\psi_2'(\varphi), \tag{1.3}$$

where $\psi_{1,2}'(\varphi) \equiv d\psi_{1,2}/d\varphi$. It sometimes turns out to be convenient to introduce a "special" coordinate system in which $A_0 = 0$, the vector $\boldsymbol{\kappa}$ lying along the 3-axis, i.e., $\kappa^0 = \kappa^3$, while the field strength vectors of the wave lie in the 1,2-plane.

Section 2 is devoted to the case of scalar particles. We find the mass operator in the field of the wave (1.1) and analyze its imaginary part. In Sec. 3 we find the mass operator of a spinor particle and analyze its properties, including spin correlations. The Appendix is devoted to an explanation of technical details of the proposed approach.

2. MASS OPERATOR OF A SCALAR PARTICLE

The mass operator of a spin-zero particle can be written in the form (see [6], Eq. (1.10))

$$M^{(0)}=-\frac{ie^2}{(2\pi)^4}\int d^4k\,(2P-k)^\mu\frac{1}{(P-k)^2-m^2+i\varepsilon}(2P-k)_\mu\frac{1}{k^2+i\varepsilon}, \tag{2.1}$$

where $P_\mu \equiv P_\mu(\varphi) = i\partial_\mu - eA_\mu$. It is convenient to use the exponential operator representation. To do this we carry out a parametrization of the kind

$$\frac{1}{k^2+i\varepsilon}\,\frac{1}{(P-k)^2-m^2+i\varepsilon}$$

$$=-\int_0^\infty ds\int_0^s du\,\exp\{-ium^2\}\exp\{i[u(P^2-2Pk)+sk^2]\}. \tag{2.2}$$

Using (2.2) we can write the mass operator (2.1) in the form

$$M^{(0)}=\frac{ie^2}{(2\pi)^4}\int d^4k\int_0^\infty ds\Big\{\int_0^s du[4P^\mu\exp\{iu(P-k)^2\}P_\mu$$
$$-\{(P^2-m^2)\exp\{iu(P-k)^2\}\}]\exp\{i(s-u)k^2\}\exp\{-ium^2\}$$
$$-i(\exp\{is((P-k)^2-m^2)\}-2\exp\{isk^2\})\Big\}, \tag{2.3}$$

where $\{,\}$ indicates an anticommutator. We have in Eq. (2.3) integrated by parts the term containing Pk in the coefficient of the exponential; this enabled us to put $M^{(0)}$ in a form where only the index of the exponential depends on k. Integrating over k thus reduces to evaluating

$$Q^{(0)}=\int d^4k\,e^{iu(P-k)^2}e^{i(s-u)k^2}=\int d^4k\,e^{-ikX}e^{iuP^2}e^{ikX}e^{i(s-u)k^2}, \tag{2.4}$$

where we used the shift operator in momentum space: for any function f(P) one has

$$e^{-ikX}f(P)e^{ikX}=f(P-k), \quad [P_\mu, X_\nu]=ig_{\mu\nu}.$$

The exponential operator expression $\exp(iuP^2)$ in (2.4) contains non-commuting operators. One of the central points of the discussion here is the disentangling of this expression (see the Appendix) which for the case of a linearly polarized wave $A_\mu(\varphi) = a_\mu\psi(\varphi)$ gives (see (A.19))

$$\exp(iuP^2)=\exp\left\{i\int_0^u [aP-ea^2\Delta(\tau)]^2\frac{d\tau}{a^2}\right\}\exp(iuP_\perp^2), \quad (2.5)$$

where $\Delta(\tau) = \psi(\varphi - 2(\kappa P)\tau) - \psi(\varphi)$ and $P_\perp = P - a(aP)/a^2$. As a result of the disentangling we obtained a product of two operators which are such that in the index of each of them commuting operators occur. After this we can complete the integration over k. Substituting (2.5) into (2.4) we find after simple transformations:

$$Q^{(0)}=\int d^4k\exp\left\{i\int_0^u (aP(\varphi,k))^2\frac{d\tau}{a^2}-2iak\int_0^u (aP(\varphi,k))\frac{d\tau}{a^2}\right\}$$
$$\times\exp[iu(P_\perp^2-2P_\perp k_\perp)]\exp(isk^2), \quad (2.6)$$

where $k_\perp = k - a(ak)/a^2$,

$$P(\varphi,k)=P-ea\Delta(\tau,k), \quad \Delta(\tau,k)=\psi(\varphi-2((P-k)\varkappa)\tau)-\psi(\varphi). \quad (2.7)$$

We expand all the vectors in (2.6) in terms of the basis vectors

$$\mathbf{e}_1=\varkappa a/\varkappa^0\sqrt{-a^2}, \quad \mathbf{e}_3=\varkappa/\varkappa^0, \quad \mathbf{e}_2=[\mathbf{e}_3\mathbf{e}_1], \quad (2.8)$$

which we choose as the axes of a Cartesian system of coordinates. In this system we introduce the variables

$$y=k^0+k^3, \quad z=k^0-k^3, \quad v=k^2, \quad w=k^1; \quad (2.9)$$

and then

$$k^2=yz-v^2-w^2, \quad ak=a^0z+v\sqrt{-a^2}, \quad \varkappa k=\varkappa^0 z,$$
$$P_\perp k_\perp=b_1z+b_2y-P_\perp^2v-P_\perp^1w, \quad (2.10)$$

where $b_1 = (P_\perp^0 + P_\perp^3)/2$ and $b_2 = (P_\perp^0 - P_\perp^3)/2$. Changing in the integral (2.6) to the variables (2.9) and substituting (2.10) in it, we find that the integral over y is of the form

$$\int_{-\infty}^{+\infty} dy\exp[iy(sz-2b_2u)]=2\pi\delta(sz-2b_2u), \quad (2.11)$$

so that the integration over z reduces to integrating a δ-function, i.e., to making in all expressions the substitution

$$z\to 2b_2u/s=u(\varkappa P)/s\varkappa^0$$

(the quantity $P(\varphi, k)$ depends only on z). The remaining integrals over v and w can be evaluated directly (they reduce to Fresnel integrals). As a result we get after some manipulations

$$Q_l^{(0)}=-\frac{i\pi^2}{s^2}\exp\left\{i\eta\int_0^1 [aP-ea^2\Delta(\eta y)]^2\frac{dy}{a^2}\right\}e^{i\beta}\exp\{i\eta P_\perp^2\}, \quad (2.12)$$

where we have made the substitution $u \to us$; $\eta = u(1-u)s$,

$$\Delta(\eta y)=\psi(\varphi-2(\varkappa P)\eta y)-\psi(\varphi), \quad (2.13)$$
$$\beta=-\xi^2sm^2u^2\left[\int_0^1\Delta^2(\eta y)dy-\left(\int_0^1\Delta(\eta y)dy\right)^2\right];$$

here

$$\xi^2=-e^2a^2/m^2 \quad (2.14)$$

is the parameter characterizing the strength of the plane wave[2] (see, e.g., (98.1) in [5]). Using (2.5) we can write (2.12) in the form

$$Q_l^{(0)}=-\frac{i\pi^2}{s^2}e^{i\beta}e^{i\eta P^2}. \quad (2.15)$$

To complete the calculations it is necessary to transform the combination occurring in the first term of (2.3) after completing the integration:

$$P^\mu e^{i\beta}e^{i\eta P^2}P_\mu. \quad (2.16)$$

The corresponding consideration is given in the Appendix. For a linearly polarized wave $A_\mu = a_\mu\psi(\varphi)$ we have (see (A.28) and (A.35))

$$P^\mu e^{i\beta}e^{i\eta P^2}P_\mu=\left(P^2+\frac{\xi^2m^2\Delta^2(\eta)}{2}+(\varkappa P)\frac{d\beta}{d\varphi}\right)e^{i\beta}e^{i\eta P^2}, \quad (2.17)$$

where $\Delta(\eta)$ is given by Eq. (2.13).

The integral over k of the last term on the right-hand side of (2.3), which is independent of u, can be done directly. One can also use Eq. (2.12), putting u = 1 (u = s in Eq. (2.4)) in it. As a result we get

$$\int d^4k\, e^{is(P-k)^2}=\int d^4k\, e^{isk^2}=-i\pi^2/s^2. \quad (2.18)$$

Substituting (2.12), (2.17), and (2.18) into (2.3) we have

$$M^{(0)}=\frac{\alpha}{2\pi}m^2\int_0^\infty\frac{ds}{s}\left\{\int_0^1 du\left[1+\frac{P^2}{m^2}+\xi^2\Delta^2(\eta)+\frac{\varkappa P}{m^2}\frac{d\beta}{d\varphi}\right]e^{-isum^2}e^{i\beta}e^{i\eta P^2}\right.$$
$$\left.+\frac{i}{m^2s}\left(1-\frac{1}{2}e^{-ism^2}\right)\right\}. \quad (2.19)$$

The mass operator we have found diverges in the same way as the mass operator of a scalar particle when there is no external field. Its renormalization is therefore standard:

$$M_R^{(0)}=M^{(0)}-M^{(0)}(P^2=m^2, A_\mu=0)-(P^2-m^2)\frac{dM^{(0)}}{dP^2}(P^2=m^2, A_\mu=0). \quad (2.20)$$

As a result we find the renormalized mass operator of a scalar particle in the field of a linearly polarized plane wave:

$$M_R^{(0)}=\frac{\alpha}{2\pi}m^2\int_0^\infty\frac{ds}{s}\int_0^1 du\left\{\left[1+\frac{P^2}{m^2}+\xi^2\Delta^2(\eta)+\frac{\varkappa P}{m^2}\frac{d\beta}{d\varphi}\right]e^{-isum^2}e^{i\beta}e^{i\eta P^2}\right.$$
$$\left.-e^{-isu^2m^2}\left[2+\left(\frac{P^2}{m^2}-1\right)(1+2isu(1-u))\right]\right\}; \quad (2.21)$$

here β and $\Delta(\eta)$ are given by Eqs. (2.13). We note that apart from the diagrams which we have considered there is also the so-called contact diagram of the self-energy in which a photon is emitted and absorbed in a single point. The contribution of this diagram drops out in the regularization.

The average value of $M_R^{(0)}$ on the mass shell $P^2\Phi = m^2\Phi$ is of considerable interest for physical applications. As $M_R^{(0)}$ depends only on P^2 and κP, it follows directly from (2.21) that (cf. Eq. (1.8) or [6])

$$\langle M_R^{(0)}\rangle\equiv\int d^4x\,\Phi^+(x)M_R^{(0)}\Phi(x)=\frac{\alpha}{\pi}m^2\frac{1}{V}\int d^4x\int_0^\infty\frac{ds}{s}\int_0^1 du\,e^{-isu^2m^2}$$
$$\times\left\{\left(1+\frac{\xi^2}{2}\Delta^2(\eta)+\frac{\lambda}{4}\frac{d\beta}{d\varphi}\right)e^{i\beta}-1\right\}, \quad (2.22)$$

where Φ is a solution of the Klein-Gordon equation in the field of the wave. V is the normalization 4-volume (we have used only the properties $P^2\Phi = m^2\Phi$, $2\kappa P\Phi/m^2 = \lambda\Phi$).

Above we found the mass operator in the field of a linearly polarized wave. In the general case of an elliptically polarized wave we must substitute in the integral (2.4) the disentangled expression (A.20). The further calculation is similar to the one given above, and as a result terms containing the amplitudes of the waves a_1 and a_2 occur additively. As a result it turns out that we must in (2.15) make the substitution

$$\beta\to\beta_1+\beta_2,\quad \beta_k=-\xi_k^2 su^2m^2\left[\int_0^1\Delta_k^2(\eta y)dy-\left(\int_0^1\Delta_k(\eta y)dy\right)^2\right], \quad (2.23)$$

where

$$\Delta_k(\eta y)=\psi_k(\varphi-2(\varkappa P)\eta y)-\psi_k(\varphi),$$
$$\xi_k^2=-e^2a_k^2/m^2 \quad (k=1,2), \quad (2.24)$$

and in (2.17) we must apart from this make the substitution

$$\xi^2\Delta^2(\eta)\to\xi_1^2\Delta_1^2(\eta)+\xi_2^2\Delta_2^2(\eta),$$
$$d\beta/d\varphi\to d(\beta_1+\beta_2)/d\varphi. \quad (2.25)$$

This can be understood as follows: in the answer (2.21) and (2.22) there occurs

$$[A_\mu(\varphi)-A_\mu(\varphi)]^2.$$

In the case of linear polarization we put $A_\mu(\varphi) = a_\mu\psi(\varphi)$; in the general case we must put $A_\mu = a_{1\mu}\psi_1(\varphi) + a_{2\mu}\psi_2(\varphi)$ which leads to the substitutions (2.23) and (2.25). As a result the explicit form of the mass operator of a scalar particle in an elliptically polarized wave follows straight from (2.21) and (2.22) where we must perform the substitutions (2.23) and (2.25).

To complete the integration over the variable x we bear in mind that the integrand depends only on the variable φ (in the special system $\varphi = \kappa^0(x^0 - x^3)$). For a monochromatic plane wave

$$\psi_1(\varphi)=\cos\varphi,\quad \psi_2(\varphi)=\sin\varphi \quad (2.26)$$

we have a periodic function φ so that we can restrict the integration over a single period.[3)] Using this we have

$$\frac{1}{V}\int d^4x[\quad]=\frac{1}{2\varkappa^0L}\int_{-\varkappa^0L}^{\varkappa^0L}d\varphi[\quad]=\frac{1}{2\pi}\int_{-\pi}^{\pi}d\varphi[\quad]. \quad (2.27)$$

An important advantage of this approach is the possibility to obtain the result for the general case of a plane wave of an arbitrary configuration (see (2.21) and (2.22)) and only in the last stage of the calculation do we need to give the actual form of the wave (see (2.26)). After simple transformations we easily check that terms depending on φ are met with only in the combination

$$(\xi_1^2-\xi_2^2)\cos[2(\varphi-\eta\lambda/2)].$$

After splitting off this combination the integration over φ is performed easily. For a monochromatic plane wave with elliptical polarization we have for the mass operator of a scalar particle

$$\langle M_R^{(0)}\rangle_{el}=\frac{\alpha}{\pi}m^2\int_0^\infty\frac{dt}{t}\int_0^\infty\frac{dv}{(1+v)^2}\{e^{-iZ_1}[[1+(\xi_1^2+\xi_2^2)\sin^2t]J_0(Z_0)$$
$$-i(\xi_1^2-\xi_2^2)\sin^2tJ_1(Z_0)]-e^{-2ivt/\lambda}\}, \quad (2.28)$$

where $J_0(Z_0)$ and $J_1(Z_0)$ are Bessel functions, $\lambda = 2(\kappa p)/m^2$,

$$Z_0=\frac{1}{\lambda}vt(\xi_1^2-\xi_2^2)\left[\frac{\sin^2t}{t^2}-\frac{\sin 2t}{2t}\right],$$
$$Z_1=\frac{2vt}{\lambda}\left[1+\frac{\xi_1^2+\xi_2^2}{2}\left(1-\frac{\sin^2t}{t^2}\right)\right] \quad (2.29)$$

in Eq. (2.28) we changed to the new variables

$$v=\frac{u}{1-u},\quad t=\frac{\lambda}{2}m^2\frac{sv}{(1+v)^2}.$$

The mass operator (2.28) depends on the invariant characteristics of the wave intensity ξ_1^2 and ξ_2^2, and on the invariant integral of motion of the particle in the field of the wave, $\lambda = 2(\kappa p)/m^2$. If we put $\xi_2^2 = 0$ in (2.28), we get the mass operator for the case of linear polarization. In the particular case of a circularly polarized wave, $\xi_1^2 = \xi_2^2 = \xi^2$, the expression for the mass operator can be simplified considerably:

$$\langle M_R^{(0)}\rangle_{cr}=\frac{\alpha}{\pi}m^2\int_0^\infty\frac{dt}{t}\{(1+2\xi^2\sin^2t)[1+i\eta e^{i\eta}\,\mathrm{Ei}(-i\eta)]$$
$$-[1+i\eta_0e^{i\eta_0}\,\mathrm{Ei}(-i\eta_0)]\}, \quad (2.30)$$

where Ei(x) is the exponential integral,[7]

$$\eta=\frac{2t}{\lambda}\left[1+\xi^2\left(1-\frac{\sin^2t}{t^2}\right)\right],\quad \eta_0=\frac{2t}{\lambda}.$$

The imaginary part of $\langle M_R^{(0)}\rangle$ is in the well-known way connected with the total probability $W^{(0)}$ for the emission of a particle in the field of the wave (cf.[6], Eq. (2.43)):

$$W=-\frac{1}{\varepsilon}\mathrm{Im}\langle M_R^{(0)}\rangle, \quad (2.31)$$

where ϵ is the zero component of the average kinetic momentum (quasi-momentum) of the particle in the field of the wave. Equations (2.31) and (2.28) and its particular case (2.30) are a new representation for the total probability for the emission which is convenient, in particular, for an analysis of the situation when $\xi^2 \sim 1$. Performing a number of transformations we can obtain the total probability in the form which is analogous to the well-known representation for spinor particles

$$W_{cr}^{(0)}=\frac{\alpha m^2}{\varepsilon}\sum_{n=1}^\infty\int_0^{v_n}\frac{dv}{(1+v)^2}$$
$$\times\left\{-J_n^2(Z)+\xi^2\left[\frac{1}{2}(J_{n+1}^2(Z)+J_{n-1}^2(Z))-J_n^2(Z)\right]\right\}, \quad (2.32)$$

where

$$Z=\frac{2n\xi}{[1+\xi^2]^{1/2}}\left[\frac{v}{v_n}\left(1-\frac{v}{v_n}\right)\right]^{1/2},\quad v_n=\frac{n\lambda}{1+\xi^2}. \quad (2.33)$$

3. MASS OPERATOR OF A SPINOR PARTICLE

The mass operator for a spin-1/2 particle in an external field can be written in the form (see [6], Eq. (3.1))

$$M^{(1/2)}=-\frac{ie^2}{(2\pi)^4}\int\frac{d^4k}{k^2+i\varepsilon}\gamma^\mu\frac{\hat P-\hat k+m}{(\hat P-\hat k)^2-m^2}\gamma_\mu. \quad (3.1)$$

Using the parametrization (2.2) we rewrite it in the form

$$M^{(1/2)}=\frac{ie^2}{(2\pi)^4}\int_0^\infty s\,ds\int_0^1du\,\mathfrak{M}^{(1/2)}, \quad (3.2)$$

where

$$\mathfrak{M}^{(1/2)}=\int d^4k\,\gamma^\mu(\hat P-\hat k+m)\exp\{isu[(\hat P-\hat k)^2-m^2]\}\gamma_\mu\exp[is(1-u)k^2]. \quad (3.3)$$

As in the case of scalar particles we use the shift procedure in momentum space:

$$\exp[i(\hat P-\hat k)^2u]=e^{-ikx}e^{iu\hat P^2}e^{ikx}. \quad (3.4)$$

The disentangling of the exponential operator expression, which is one of the main points of our approach, proceeds in the same way as in the case of scalar particles with a few complications caused by the spin terms (see the Appendix). As a result we find for the case of a linearly polarized wave $A_\mu = a_\mu\psi(\varphi)$:

$$\exp[iu\hat{P}^2]=\exp\left[\frac{e\hat{a}\hat{\varkappa}\Delta(u)}{2(\varkappa P)}\right]\exp[iuP^2]$$
$$=\exp\left[\frac{ea\varkappa\Delta(u)}{2\varkappa P}\right]\exp\left[i\int_0(aP-ea^2\Delta(\tau))^2\frac{d\tau}{a^2}\right]\exp[iuP_\perp^2],\quad(3.5)$$
where we have used the same notation as in (2.5).

The integral (3.3) splits into two, one of which does not contain k_μ in the factor of the exponential
$$Q^{(1/2)}=\int d^4k\exp[ius(\hat{P}-\hat{k})^2]\exp[is(1-u)k^2].\quad(3.6)$$
and can be evaluated in the same way as the integral $Q^{(0)}$ in (2.4) in Sec. 2. In the factor which is extra as compared to $Q^{(0)}$ we have then according to (2.10) and (2.11)
$$(P-k)\varkappa\to\varkappa P(1-u),\quad\Delta(su)\to\Delta(\eta),$$
so that we have for a linearly polarized wave
$$Q_l^{(1/2)}=\exp\left[\frac{e\hat{a}\hat{\varkappa}\Delta(\eta)}{2(1-u)\varkappa P}\right]Q_l^{(0)}=-\frac{i\pi^2}{s^2}\left(1+\frac{e\hat{a}\hat{\varkappa}\Delta(\eta)}{2(1-u)\varkappa P}\right)e^{i\beta}e^{i\eta P^2},\quad(3.7)$$
where $Q^{(0)}$ is given by Eq. (2.15) and where we have used the notation (2.13) and in the last equality used the fact that all powers of $\hat{a}\hat{\kappa}$ higher than the first one vanish. To evaluate the second integral in (3.3) which contains k_μ in the factor of the exponential we shall start from the equation (cf. (2.11) in [6])
$$\int d^4k\frac{\partial}{\partial k^\mu}\{\exp[-ikX]\exp[ius\hat{P}^2]\exp[ikX]\exp[is(1-u)k^2]\}=0.\quad(3.8)$$
Differentiating in that expression we get
$$\int d^4k\,\hat{k}\exp[ius(\hat{P}-\hat{k})^2]\exp[is(1-u)k^2]=\frac{1}{2s(1-u)}\gamma^\mu[X_\mu,Q^{(1/2)}],\quad(3.9)$$
where we have used (3.4) which reduces the evaluation of the required integral to calculating the commutator of X_μ with $Q^{(1/2)}$ which we have already found in (3.7) and which is the product of three operators:
$$A=1+\frac{e\hat{a}\hat{\varkappa}\Delta(\eta)}{2(1-u)\varkappa P},\quad B=e^{i\beta},\quad C=e^{i\eta P^2}.\quad(3.10)$$
The identity
$$[X_\mu,ABC]=[X_\mu,A]BC+A[X_\mu,B]C+AB[X_\mu,C],\quad(3.11)$$
reduces the problem to finding the commutator of X_μ with each of the operators (3.10). One checks easily that the commutator $\gamma^\mu[X_\mu A]\propto\kappa^2=0$. The remaining commutators are evaluated in the Appendix (see (A.31), (A.35)). Substituting them into (3.11) and, using (A.35), transferring $e^{i\beta}$ to the right we get
$$\gamma^\mu[X_\mu,Q_l^{(1/2)}]=\left\{2\eta\left[\hat{P}+\frac{e(\gamma fP)}{\varkappa P}\int_0^1\Delta(\eta y)dy+\frac{\hat{\varkappa}\xi^2m^2}{2\varkappa P}\int_0^1\Delta^2(\eta y)dy\right]+\hat{\varkappa}\left(\frac{\partial\beta}{\partial(\varkappa P)}+2\eta\frac{\partial\beta}{\partial\varphi}\right)\right\}Q_l^{(1/2)},\quad(3.12)$$
where $(\gamma fP)=\gamma^\mu f_{\mu\nu}P^\nu$; we bear in mind that $f_{\mu\nu}=\kappa_\mu a_\nu-\kappa_\nu a_\mu$ is the electromagnetic field tensor $F_{\mu\nu}=f_{\mu\nu}\psi'(\varphi)$. Substituting (3.6), (3.7), (3.9), and (3.12) into (3.3) we have
$$M^{(1/2)}=e^{-ism^2u}\gamma^\mu\left\{m+\hat{P}(1-u)-u\frac{e(\gamma fP)}{\varkappa P}\int_0^1\Delta(\eta y)dy-\frac{\hat{\varkappa}\xi^2m^2u}{2(1-u)\varkappa P}\left[\int_0^1\Delta^2(\eta y)dy-2u\left(\int_0^1\Delta(\eta y)dy\right)^2\right]\right\}Q_l^{(1/2)}\gamma_\mu.\quad(3.13)$$
Performing in (3.13) the usual algebraic operations with the γ-matrices and using
$$\hat{P}\hat{a}\hat{\varkappa}=i\gamma^5(\gamma f^*P)+(\gamma fP),\quad(3.14)$$
where $f^*_{\alpha\beta}=1/2\,\epsilon_{\alpha\beta\gamma\delta}f^{\gamma\delta}$ and separating off on the right the operator $e^{i\eta\hat{P}^2}$ we have the mass operator (3.2) in the field of a linearly polarized wave:
$$M_l^{(1/2)}=\frac{\alpha}{2\pi}\int_0^\infty\frac{ds}{s}\int_0^1du\,e^{-isum^2}\{2(m-\hat{P})(1-\hat{a}\hat{\varkappa}H)+\hat{P}(1+u)+(\gamma fP)E-i\gamma^5(\gamma f^*P)uH+\hat{\varkappa}\xi^2G\}e^{i\beta}\exp\{i\eta\hat{P}^2\},\quad(3.15)$$
where we have used the notation
$$H=\frac{e\Delta(\eta)}{2\varkappa P},\quad E=\frac{e}{\varkappa P}\left[u\int_0^1\Delta(\eta y)dy-\Delta(\eta)\left(1+\frac{u}{2}\right)\right],$$
$$G=\frac{m^2}{2(\varkappa P)}\left[\Delta^2(\eta)+\frac{u}{1-u}\left(u\Delta(\eta)\int_0^1\Delta(\eta y)dy+\int_0^1\Delta^2(\eta y)dy-2u\left(\int_0^1\Delta(\eta y)dy\right)^2\right)\right].\quad(3.16)$$
The result obtained can easily be generalized to the general case of an elliptically polarized wave (see Appendix (A.20), (A.21)). The integral (3.6) for $Q^{(1/2)}$ is calculated using (A.26), as before (cf. (3.7)):
$$Q_{el}^{(1/2)}=\exp\left[\frac{e\hat{a}_1\hat{\varkappa}\Delta_1(\eta)}{2(1-u)(\varkappa P)}\right]\exp\left[\frac{e\hat{a}_2\hat{\varkappa}\Delta_2(\eta)}{2(1-u)(\varkappa P)}\right]Q_{el}^{(0)}$$
$$=-\frac{i\pi^2}{s^2}\left(1+\frac{e\hat{a}_1\hat{\varkappa}\Delta_1(\eta)}{2(1-u)(\varkappa P)}+\frac{e\hat{a}_2\hat{\varkappa}\Delta_2(\eta)}{2(1-u)(\varkappa P)}\right)\exp[i(\beta_1+\beta_2)]\exp[i\eta P^2].\quad(3.17)$$
where $Q^{(0)}_{el}$ is given by Eq. (2.15) with the substitution (2.23). We can use Eq. (3.9) to evaluate the second integral, where we must substitute (3.17). The method of calculation is similar to the one used above. As a result we find the mass operator in the field of an elliptically polarized wave:
$$M_{el}^{(1/2)}=\frac{\alpha}{2\pi}\int_0^\infty\frac{ds}{s}\int_0^1du\,e^{-isum^2}\{2(m-\hat{P})(1-\hat{a}_1\hat{\varkappa}H_1-\hat{a}_2\hat{\varkappa}H_2)\quad(3.18)$$
$$+\hat{P}(1+u)+(\gamma f_1P)E_1+(\gamma f_2P)E_2-i\gamma^5(\gamma f_1^*P)uH_1-i\gamma^5(\gamma f_2^*P)uH_2-i\gamma^5\Lambda N+\hat{\varkappa}(\xi_1^2G_1+\xi_2^2G_2)\}\exp[i(\beta_1+\beta_2)]\exp[i\eta\hat{P}^2],$$
where $E_{1,2}$, $G_{1,2}$, and $H_{1,2}$ are given by Eqs. (3.16) with the substitution of Δ_1 and Δ_2, respectively; the tensors
$$f_k^{\mu\nu}=\varkappa^\mu a_k^\nu-\varkappa^\nu a_k^\mu\quad f^*_{k\alpha\beta}={}^1/_2\varepsilon_{\alpha\beta\gamma\delta}f_k^{\gamma\delta},$$
$$\Lambda=\varepsilon_{\alpha\beta\gamma\delta}\gamma^\alpha a_1^\beta a_2^\gamma\varkappa^\delta.\quad(3.19)$$
$$N=\frac{e^2u}{2(\varkappa P)}\left(1+\frac{1}{1-u}\right)\left[\Delta_2(\eta)\int_0^1\Delta_1(\eta y)dy-\Delta_1(\eta)\int_0^1\Delta_2(\eta y)dy\right].$$
We draw attention to the appearance of a term which contains $\gamma^5\Lambda$ and which does not occur in (3.15).

The expressions (3.15) and (3.18) which we have found for the mass operator must be renormalized in the standard way (cf. [6] Eq. (3.15)):
$$M_R^{(1/2)}=M^{(1/2)}-M^{(1/2)}(\hat{P}=m,A_\mu=0)-(\hat{P}-m)\frac{\partial M^{(1/2)}}{\partial\hat{P}}(\hat{P}=m,A_\mu=0).\quad(3.20)$$
As a result we obtain the renormalized mass operator of an electron in the field of a plane wave:
$$M_R^{(1/2)}=M^{(1/2)}-\frac{\alpha}{2\pi}\int_0^\infty\frac{ds}{s}\int_0^1du\,e^{-isu^2m^2}\{m(1+u)-(\hat{P}-m)(1-u)[1-2ism^2u(1+u)]\},\quad(3.21)$$
where we must substitute (3.18) (or (3.15) in the case of a linearly polarized wave) for $M^{(1/2)}$.

The average of the mass operator over the mass shell is of major interest for physical applications. We introduce the operator
$$\pi^\mu=\pi_0^\mu+i\varkappa^\mu\frac{e\hat{\varkappa}\hat{A}'}{2(\varkappa i\partial)},\quad\pi_0^\mu=i\partial^\mu-\varkappa^\mu\left(\frac{e(Ai\partial)}{(\varkappa i\partial)}-\frac{e^2A^2}{2(\varkappa i\partial)}\right),\quad(3.22)$$

which satisfies the following relations:

$$[\pi^\mu, \pi^\nu]=0, \quad \pi^2=\hat{P}^2, \quad [\hat{P}^2-m^2, \pi^\mu]=0. \tag{3.23}$$

In accordance with (3.23) we choose states which satisfy the squared Dirac equation $(\hat{P}^2 - m^2)\Phi = 0$, and also the equation

$$\pi^\mu\Phi=p^\mu\Phi, \tag{3.24}$$

where p_μ is a constant 4-vector ($p^2 = m^2$), i.e., there are three independent vector components (p^1, p^2, $p^3 - p^0$ in the special system). We can clearly write the solution of the Dirac equation in the form

$$\Psi=(\hat{P}+m)\Phi. \tag{3.25}$$

We perform a unitary transformation on Ψ:

$$\Psi_U=U\Psi, \quad U=\exp[-e\hat{\varkappa}\hat{A}/2\varkappa P], \tag{3.26}$$

which changes the Dirac equation $(\hat{P} - m)\Psi = 0$ into the equation

$$(\hat{\pi}^0-m)\Psi_U=0, \quad \pi_\mu{}^0\Psi_U=p_\mu\Psi_U. \tag{3.27}$$

We can now use (3.23) and (3.27) to describe spin states as in the case of free particles. We introduce then the spin 4-vector S^μ ($S^2 = -1$, $pS = 0$, in operator form $p \to \pi^0$) and use the operator $-\gamma^5\hat{S}$ to classify the spin states. Performing on the operator $-\gamma^5\hat{S}$ the transformation which is the inverse of (3.26) we get for the Dirac equation spin operator:

$$R=\exp\left[\frac{e\hat{\varkappa}\hat{A}}{2\varkappa P}\right](-\gamma^5\hat{S})\exp\left[-\frac{e\hat{\varkappa}\hat{A}}{2\varkappa P}\right]$$
$$=-\gamma^5\left[\hat{S}+\frac{e}{\varkappa P}(\hat{\varkappa}(AS)-\hat{A}(\varkappa S))-\frac{e^2}{2(\varkappa P)^2}(\varkappa S)\hat{\varkappa}\hat{A}\right], \tag{3.28}$$

with eigenstates

$$R\Psi_p{}^{\pm s}=\pm\Psi_p{}^{\pm s}, \tag{3.29}$$

where p is the constant 4-vector (3.24).

We note that all relations obtained become completely obvious if we use the well-known form of the wavefunction of an electron in the field of a wave (see, e.g., [5]). One checks easily that the following relations hold for the states (3.29):

$$\overline{\Psi}_p{}^s\gamma^5(\gamma f_{1,2}^*P)\Psi_p{}^s=(Sf_{1,2}^*P), \quad \overline{\Psi}_p{}^s\gamma^5\Lambda\Psi_p{}^s=(Sa_1a_2\varkappa),$$
$$\overline{\Psi}_p{}^s\hat{\varkappa}\Psi_p{}^s=\varkappa p/m, \quad \overline{\Psi}_p{}^s(\gamma f_{1,2}P)\Psi_p{}^s=0, \tag{3.30}$$

where $(a_1a_2a_3a_4) \equiv \epsilon^{\alpha\beta\gamma\delta}a_{1\alpha}a_{2\beta}a_{3\gamma}a_{4\delta}$; in deriving the last equation one can start from $(\gamma fP) = 1/2[\hat{P}, \sigma f]$. Substituting these relations into (3.21) we have for the renormalized mass operator in the field of an elliptically polarized plane wave on the mass shell:

$$\langle M_R^{1/2}\rangle_{el}=\int d^4x\,\overline{\Psi}_p{}^sM_{Rel}^{(1/2)}\Psi_p{}^s=\frac{\alpha}{2\pi}\int\frac{d^4x}{V}\int_0^\infty\frac{ds}{s}\int_0^1 du\,e^{-isu^2m^2}\Big\{\Big[m(1+u)$$
$$+\frac{\varkappa p}{m}(\xi_1{}^2G_1+\xi_2{}^2G_2)-iu(H_1(Sf_1{}^*p)+H_2(Sf_2{}^*p))$$
$$-i(Sa_1a_2\varkappa)N\Big]\exp\{-i(\beta_1+\beta_2)\}-m(1+u)\Big\}, \tag{3.31}$$

where we have used the same notation as in (3.18). The last two terms in the square bracket depend on the electron spin; they drop out for unpolarized electrons. For the monochromatic plane wave (2.26) we find the final form for the average of the renormalized mass operator of an electron:

$$\langle M_R^{(1/2)}\rangle_{el}=\frac{\alpha}{2\pi}m\int_0^\infty\frac{dt}{t}\int_0^\infty\frac{dv}{(1+v)^2}\Big\{e^{-iz_1}\Big[\Big(1+\frac{v}{1+v}+g_1\Big(\frac{\xi_1{}^2+\xi_2{}^2}{2}\Big)$$
$$+i\frac{e^2}{m^2}\frac{(Sa_1a_2\varkappa)}{m}g_3\Big)J_0(Z_0)+ig_2\frac{(\xi_1{}^2-\xi_2{}^2)}{2}J_1(Z_0)\Big]-\Big(1+\frac{v}{1+v}\Big)e^{-2itv/\lambda}\Big\}. \tag{3.32}$$

Here

$$g_1=2\sin^2t+v\Big(1-\frac{\sin 2t}{2t}+\frac{v}{1+v}g\Big),$$
$$g_2=-2\sin^2t+v\Big[-\frac{1}{2}\Big(\cos 2t-\frac{\sin 2t}{2t}\Big)+\frac{v}{1+v}g\Big]$$
$$g=-\cos^2t+\frac{\sin 2t}{t}-\frac{\sin^2t}{t^2},$$
$$g_3=\frac{2v(2+v)}{\lambda(1+v)}\sin t\Big(\frac{\sin t}{t}-\cos t\Big), \quad \lambda=\frac{2(\varkappa p)}{m^2}; \tag{3.33}$$

the variables and the remaining notation is the same as in (2.28). The average of the mass operator of an electron in a field depends not only on the characteristics ξ_1^2 and ξ_2^2 of the intensity and the invariant integral of motion (κp), but also on the spin correlation term

$$\frac{e^2}{m^2}(Sa_1a_2\varkappa)=\frac{\xi_1{}^2+\xi_2{}^2}{2}\tilde{\xi}_2(\varkappa S), \tag{3.34}$$

where $\tilde{\xi}_2$ is the Stokes parameter characterizing the degree of circular polarization of the plane wave. For unpolarized electrons this term drops out. Putting $\xi_2 = 0$ we have the case of linear polarization and when $\xi_1^2 = \xi_2^2 = \xi^2$ the case of circular polarization. In the latter case this expression simplifies considerably and depends only on elementary functions:

$$\langle M_R^{(1/2)}\rangle_{cr}=\frac{\alpha}{2\pi}m\int_0^\infty\frac{dt}{t}\int_0^\infty\frac{dv}{(1+v)^2}\Big\{\Big(\exp[-iZ_1{}^{cr}]-\exp\Big[-\frac{2ivt}{\lambda}\Big]\Big)$$
$$\times\Big[1-\frac{\lambda\rho}{4}\Big(\frac{3}{(1+v)^2}-1\Big)\Big]+\xi^2\sin^2t\Big(2+\frac{v^2}{1+v}\Big)\exp[-iZ_1{}^{cr}]\Big\},$$
$$\rho=\tilde{\xi}_2\frac{(\varkappa S)}{(\varkappa p)}m, \tag{3.35}$$

where Z_1^{cr} is defined in (2.29) and where the integral has been transformed, using integration by parts.

The total probability for the emission in the field of a wave for a spin-1/2 particle is (see Eq. (3.19) in [6])

$$W^{(1/2)}=-\frac{2m}{\varepsilon}\operatorname{Im}\langle M_R^{(1/2)}\rangle, \tag{3.36}$$

where ϵ is the zeroth component of the average kinetic momentum (quasi-momentum) of an electron in the field of a wave. Equations (3.36) and (3.32) are a new representation for the probability, which is convenient in particular for an analysis of a situation when $\xi^2 \sim 1$. Another advantage of it is that for any polarization of the wave $W^{(1/2)}$ is represented by a double integral[4)] whereas in the traditional approach[1-5] the total probability is a triple integral for the case of an elliptically polarized wave (this problem has recently been considered in [9]).

We consider now the asymptotic expansion of the mass operator. In the case of low intensities, $\xi_{1,2} \ll 1$, when perturbation theory is applicable, we can expand the mass operator (for the sake of simplicity we restrict ourselves to the case of a circularly polarized wave) in powers[5)] of ξ^2:

$$\langle M_R^{(1/2)}(\xi^2\ll1)\rangle_{cr}=\frac{\alpha m\xi^2}{2\pi}\Big\{\Big(\frac{1}{\lambda}+\frac{\rho}{4}\Big)[S_2(1+\lambda)-S_2(1-\lambda)]$$
$$+\Big(\frac{1}{4}-\frac{\rho}{2\lambda}-\frac{2}{\lambda^2}\Big)\Big[\frac{\pi^2}{3}-S_2(1+\lambda)-S_2(1-\lambda)\Big]+\frac{3\lambda^2-\lambda\rho-4}{4(1-\lambda^2)}$$
$$+\frac{\lambda^4-2\lambda^3\rho-3\lambda^2}{4(1-\lambda^2)^2}\ln\lambda-i\pi\Big[\frac{1}{8}+\frac{\rho}{2}+\frac{2}{\lambda}+\frac{\lambda^2\rho-1}{8(1+\lambda)^2}\Big]\Big\}; \tag{3.37}$$

here

$$\lambda=\frac{2\varkappa p}{m^2}, \quad S_2(x)=-\int_0^1\frac{dt}{t}\ln(1-tx) \tag{3.38}$$

is a Spence integral. The imaginary part of $\langle M_R^{(1/2)}(\xi^2 \ll 1)\rangle_{cr}$ given by (3.37) is connected with the total cross section for the Compton scattering of a photon with circular polarization by an electron:

$$\sigma_{com}^{(1/2)}=\frac{4}{a^2\lambda}\,\mathrm{Im}\,\frac{\langle M_R^{(1/2)}(\xi^2\ll 1)\rangle_{cr}}{m}. \qquad (3.39)$$

Bearing in mind that when $\lambda > 0$, $\mathrm{Im}S_2(1+\lambda) = \pi\ln(1+\lambda)$, we get from (3.37):

$$\sigma_{com}^{(1/2)}=\frac{2\pi\alpha^2}{m^2\lambda}\left\{\left[1-\rho-\frac{2(\rho+2)}{\lambda}-\frac{8}{\lambda^2}\right]\ln(1+\lambda)+\frac{1+4\rho}{2}+\frac{8}{\lambda}+\frac{\lambda^2\rho-1}{2(1+\lambda)^2}\right\}. \qquad (3.40)$$

We can obtain the same cross section from the expressions which are given, e.g., in [5] through elementary integration. It is interesting that when $\lambda \gg 1$ the logarithmic term in (3.40) occurs with a factor $(1-\rho)$ and vanishes when $\rho = 1$. This fact can easily be understood. To do this we note that the value $|\rho| = 1$ is possible only when the vectors $\mathbf{n} = \boldsymbol{\kappa}/\kappa$ and $\boldsymbol{\zeta}$ (the direction of the spin quantization axis) are collinear (e.g., in the rest frame of the electron $\rho = -\xi_2(\boldsymbol{\zeta}\cdot\mathbf{n})$). A contribution which is logarithmic when $\lambda \gg 1$ comes from the diagram in which first the final photon is emitted and afterwards the initial one is absorbed, and is determined by the kinematic situation in which the final photon is scattered backward (this is most simply verified by analyzing the situation in the c.m.s.). If the vectors $\boldsymbol{\zeta}$ and $\mathbf{n}$ are collinear the process is only possible for a single value of the helicity of the initial photon.

The real part of (3.37) gives the correction to the mass in lowest order in ξ^2; its asymptotic expansions have the form

$$\mathrm{Re}\langle M_R^{(1/2)}(\xi^2\ll 1)\rangle_{cr}=\begin{cases}\dfrac{\alpha m\xi^2}{6\pi}\lambda^2\left(\ln\dfrac{1}{\lambda^2}-\dfrac{11}{24}\right) & \lambda\ll 1\\[2ex] \dfrac{\alpha m\xi^2}{8\pi}\left[\ln^2\lambda+\ln\lambda+\dfrac{\pi^2}{2}\left(\rho+\dfrac{1}{3}\right)-3\right], & \lambda\gg 1\end{cases} \qquad (3.41)$$

It is clear that for small λ this quantity increases with increasing λ from zero, while for large λ it is proportional to $\ln^2\lambda$ and depends weakly on ρ for all λ.

The behavior of $\langle M_R^{(1/2)}\rangle_{cr}$ for $\lambda \gg 1$ and arbitrary fixed value of the parameter ξ^2 is also of interest. The corresponding asymptotic expression has the form

$$\langle M_R^{(1/2)}\rangle_{cr}=\frac{\alpha m\xi^2}{8\pi}\Big\{\ln^2\tilde\lambda+\ln\tilde\lambda+\frac{\pi^2}{2}\left(\rho+\frac{1}{3}\right)+1-\frac{4}{\xi^2}\ln(1+\xi^2)$$
$$-i\pi\left[(1-\rho)\ln\tilde\lambda+\frac{1+5\rho}{2}\right]-4\int_0^\infty dt\ln\left[1-\frac{\sin^2 t}{t^2}\frac{\xi^2}{1+\xi^2}\right]$$
$$\times\left[\frac{\sin^2 t}{t}+i\rho\left(\frac{\sin^2 t}{t^2}-\frac{\sin 2t}{2t}\right)\right]\Big\}, \qquad (3.42)$$

where $\tilde\lambda = \lambda/(1+\xi^2) = 2\kappa p/m^2(1+\xi^2)$. Comparing (3.42) with (3.37) and (3.41) we see that taking the intensity exactly into account in the logarithmic terms, as compared to using perturbation theory, reduces in that limit to the substitution $\lambda \to \tilde\lambda$ or $m^2 \to m^{*2} = m^2(1+\xi^2)$, where m^* is the effective mass of the electron in the field of the wave (the square of the quasi-momentum equals m^{*2}). There is no such simple connection for the non-logarithmic terms and there occurs a complicated function of ξ^2 in (3.42).

The transition to the case $\xi^2 \gg 1$ which reduces to a consideration of the process in a constant crossed field (quasi-classical approximation) gives well-known results which are, e.g., given in [8]. We restrict our comments to those concerning the spin terms (see (3.31)). The spin correlation (3.34) in the field of a wave which we have studied is proportional to the frequency and vanishes in the limit of a constant field. However, in that limit there remains a term containing the electron spin in the combination (Sf^*p) which gives a well-known result (see [8], pp. 183, 184).

APPENDIX

We consider the transformations of the operator expressions occurring in the present paper. Let R be an operator. We define R(s) by the following formula (this procedure was widely used in [10]):

$$R(s)=\bar R=\exp(isP^2)R\exp(-isP^2), \qquad (A.1)$$

where $R(0) = R$. If $R = P_\mu$, we get, differentiating (A.1) with respect to s,

$$d\bar P_\mu/ds=i[\overline{P^2,P_\mu}]=2e\overline{P^\lambda F}_{\lambda\mu}. \qquad (A.2)$$

For a linearly polarized wave

$$A_\mu=a_\mu\psi(\varphi),\quad F_{\mu\nu}=f_{\mu\nu}\psi'(\varphi),\quad f_{\mu\nu}=\kappa_\mu a_\nu-\kappa_\nu a_\mu.$$

Substituting the explicit form of $F_{\mu\nu}$ into (A.2) we have

$$d\bar P_\mu/ds=-2e[\kappa_\mu(\overline{aP})-a_\mu(\kappa P)]\overline{\psi'}(\varphi), \qquad (A.3)$$

where we have used the fact that (κP) commutes with P_μ. Contracting (A.3) with a^μ we find

$$d(\overline{aP})/ds=2ea^2(\kappa P)\overline{\psi'(\varphi)}=2ea^2(\kappa P)\psi'(\bar\varphi). \qquad (A.4)$$

We also need to know the operator $X_\mu(s)$ ($[X_\mu, P_\nu] = -ig_{\mu\nu}$) for which an equation of the kind (A.2) has the form

$$d\bar X_\mu/ds=-2\bar P_\mu. \qquad (A.5)$$

Contracting that equation with κ^μ we have

$$d\bar\varphi/ds=-2(\kappa P). \qquad (A.6)$$

Solving (A.6) gives for $\bar\varphi$:

$$\bar\varphi=\varphi(s)=\varphi-2(\kappa P)s. \qquad (A.7)$$

Substituting (A.7) into (A.4) and solving the latter equation we get

$$(\overline{aP})=aP-ea^2\Delta(s), \qquad (A.8)$$

where

$$\Delta(s)=\psi(\varphi(s))-\psi(\varphi). \qquad (A.9)$$

Using (A.8) and (A.7) we solve Eq. (A.3):

$$P_\mu(s)=P_\mu+\frac{e}{\kappa P}(fP)_\mu\Delta(s)-\frac{e^2a^2\kappa_\mu}{2(\kappa P)}\Delta^2(s). \qquad (A.10)$$

We now proceed to "disentangle" the expression $\exp(isP^2)$. For a linearly polarized wave we write P^2 in the form

$$P^2=P_a^2+P_\perp^2;\quad P_a=(aP)a/a^2,\quad P_\perp=P-P_a. \qquad (A.11)$$

In accordance with the method developed in Appendix B of [8] we write $\exp(isP^2)$ in the form

$$\exp(isP^2)=\exp(is(a+b))=L(s)\exp(isb)\exp(isa), \qquad (A.12)$$

where $a = P_\perp^2$ and $b = P_a^2$. Differentiating (A.12) with respect to s and multiplying the result from the left by L^{-1} and from the right by $e^{-isa}e^{-isb}$ we find

$$iL^{-1}dL/ds=b-e^{isb}f(s)e^{-isb}, \qquad (A.13)$$

where $f(s) = e^{isa} b\, e^{-isa}$. Using the fact that

$$f(s)=\sum_{n=0}^{\infty}\frac{(is)^n}{n!}\overbrace{[a,[a,\ldots[a,b]\ldots]]}^{n}=\sum_{n=0}^{\infty}\frac{(is)^n}{n!}\overbrace{[P^2\,[P^2,\ldots[P^2,b]\ldots]]}^{n}, \tag{A.14}$$

we have (cf. A(A.1))

$$f(s)=\exp\{isP^2\}\,b\exp\{-isP^2\}=\bar{b}. \tag{A.15}$$

We can now use directly the result (A.8):

$$f(s)=\bar{b}=(a\bar{P})^2/a^2=(aP-ea^2\Delta(s))^2/a^2. \tag{A.16}$$

As $[\bar{b}, b] = 0$, (A.13) takes the form

$$iL^{-1}dL/ds=b-\bar{b}. \tag{A.17}$$

The solution of this equation with the obvious initial condition $L(0) = 1$ will be

$$L(s)=\exp\left\{is\int_0^1[aP-ea^2\Delta(sy)]^2\frac{dy}{a^2}\right\}e^{-isb}. \tag{A.18}$$

Using (A.18) we have from (A.12)

$$\exp(iuP^2)=\exp\left\{iu\int_0^1[aP-ea^2\Delta(uy)]\frac{dy}{a^2}\right\}\exp(iuP_\perp^2). \tag{A.19}$$

In the general case of elliptical polarization of the wave (1.1), when

$$A_\mu(\varphi)=a_{1\mu}\psi_1(\varphi)+a_{2\mu}\psi_2(\varphi),$$

we write (cf. (A.11)) $P^2 = P_{a1}^2 + P_{a2}^2 + P_\perp^2$, where the operators

$$P_{a_1}^2=\frac{(a_1P)^2}{a_1^2}=b_1,\quad P_{a_2}^2=\frac{(a_2P)^2}{a_2^2}=b_2$$

commute with one another. Putting $b = b_1 + b_2$ in (A.12) we can immediately use the expressions obtained earlier, while in Eqs. (A.12) to (A.15) the terms containing $a_1\psi_1(\varphi)$ and $a_2\psi_2(\varphi)$ will enter additively. As a result we get for the general case of elliptical polarization:

$$\exp(iuP^2)=\exp\left\{iu\left[\int_0^1[a_1P-ea_1^2\Delta_1(uy)]^2\frac{dy}{a_1^2}+\int_0^1[a_2P-ea_2^2\Delta_2(uy)]^2\frac{dy}{a_2^2}\right]\right\}\exp(iuP_\perp^2), \tag{A.20}$$

where

$$\Delta_k(s)=\psi_k(\varphi(s))-\psi_k(\varphi)\qquad(k=1,\,2). \tag{A.21}$$

For spin-1/2 particles in the case of a linearly polarized wave $A_\mu(\varphi) = a_\mu\psi(\varphi)$ we must disentangle the expression

$$\exp\{iu\hat{P}^2\}=\exp\{iu(P_F^2+P_\perp^2)\}, \tag{A.22}$$

where

$$P_F^2=P_a^2+\frac{e}{2}\sigma F=P_a^2+ie\hat{a}\hat{\varkappa}\psi'(\varphi),$$

and the rest of the notation was given in (A.11). If we put $b = P_F$ in (A.12) the remaining steps will be the same as before (see (A.13) to (A.17)) and we have now for $f_F(s)$

$$f_F(s)=\frac{(a\bar{P})^2}{a^2}+ie\hat{a}\hat{\varkappa}\psi'(\varphi). \tag{A.23}$$

As a result we find for L_F

$$L_F(s)=\exp\left(\frac{e\hat{a}\hat{\varkappa}\Delta(s)}{2\varkappa P}\right)\exp\left\{is\int_0^1[aP-ea^2\Delta(sy)]^2\frac{dy}{a^2}\right\}\exp[-isP_F^2]. \tag{A.24}$$

Substituting (A.24) into (A.12) with $L \to L_F$ and $P^2 \to \hat{P}^2$ we find (3.5). In the general case of an elliptically polarized wave (1.1) we shall have instead of (A.22)

$$\exp(iu\hat{P}^2)=\exp[iu(P_{F_1}^2+P_{F_2}^2+P_\perp^2)], \tag{A.25}$$

where

$$P_{F_{1,2}}^2=P_{a_{1,2}}^2+ie\hat{a}_{1,2}\hat{\varkappa}\psi_{1,2}'(\varphi).$$

Putting $b = P_{F1}^2 + P_{F2}^2$ we find through the same operations as before

$$\exp(iu\hat{P}^2)=\exp\left[\frac{e\hat{a}_1\hat{\varkappa}\Delta_1(u)}{2\varkappa P}\right]\exp\left[\frac{e\hat{a}_2\hat{\varkappa}\Delta_2(u)}{2\varkappa P}\right]\times\exp\left\{iu\left[\int_0^1[a_1P-ea_1^2\Delta_1(uy)]^2\frac{dy}{a_1^2}+\int_0^1[a_2P-ea_2^2\Delta(uy)]^2\frac{dy}{a^2}\right]\right\}\exp(iuP_\perp^2), \tag{A.26}$$

where we have used the same notation as in (A.20).

We derive for the case of a linearly polarized wave some operator expressions encountered in the calculations. We use (A.1) and (A.10) to write the combination

$$q=P^\mu\exp(isP^2)P_\mu \tag{A.27}$$

in the form

$$q=P^\mu P_\mu(s)\exp(isP^2)=\left[P^2-\frac{e^2a^2\Delta^2(s)}{2}\right]\exp(isP^2). \tag{A.28}$$

The commutator

$$B_\mu=[X_\mu,\exp(isP^2)]=[X_\mu-X_\mu(s)]\exp(isP^2). \tag{A.29}$$

can be evaluated by means of the solution of Eq. (A.5):

$$X_\mu(s)=X_\mu-2s\int_0^1 dy\,P_\mu(sy). \tag{A.30}$$

If we use the explicit form (A.10) of $P_\mu(s)$ we get

$$\hat{B}=\gamma^\mu B_\mu=2s\left[\hat{P}+\frac{e(\gamma fP)}{\varkappa P}\int_0^1\Delta(sy)\,dy-\frac{e^2a^2\hat{\varkappa}}{2\varkappa P}\int_0^1\Delta^2(sy)\,dy\right]. \tag{A.31}$$

To evaluate the commutator

$$D_\mu=[X_\mu,e^{i\beta}], \tag{A.32}$$

where β is the function (κP) we note that as

$$[X_\mu,\varkappa P]=-i\varkappa_\mu, \tag{A.33}$$

the commutator of X_μ with the function $\beta(\kappa P)$ is $-i\kappa_\mu\beta'(\kappa P)$, so that

$$D_\mu=[X_\mu,e^{i\beta}]=\varkappa_\mu\frac{d\beta}{d(\varkappa P)}e^{i\beta}. \tag{A.34}$$

The commutator of P_μ with a function of $\varphi = \kappa x$ can be evaluated directly, if we use the explicit form of $P_\mu = i\partial_\mu - eA_\mu$:

$$[e^{i\beta},P_\mu]=\varkappa_\mu\frac{\partial\beta}{\partial\varphi}e^{i\beta}. \tag{A.35}$$

If β is given by Eq. (2.13) we have

$$\frac{d\beta}{d\varphi}=-\frac{e^2a^2u\Delta(\eta)}{(\varkappa P)(1-u)}\left[\frac{\Delta(\eta)}{2}-\int_0^1\Delta(\eta y)\,dy\right]. \tag{A.36}$$

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[1]The consideration of spin-zero particles is of independent interest and is methodically very valuable.

[2]The parameter ξ is purely classical; the work done by the field over a wavelength $\lambda = 1/\kappa^0$ is ξm. If we introduce an "interaction volume" with transverse dimensions 1/m and longitudinal dimensions λ, $\xi^2 = n\alpha$ ($\alpha = e^2/4\pi = 1/137$, n is the number of photons in the interaction volume).

[3]It is clear that in that case terms in which the φ-dependence takes the form $e^{i\beta}d\beta/d\varphi = (1/i)(d/d\varphi)e^{i\beta}$ vanish.

[4]For circular polarization we have a single integral; see (2.30).

[5]A similar situation also occurs for scalar particles.

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