Drag effects when there is resonance interaction between particles and a Langmuir wave in an inhomogeneous plasma

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We study the electron acceleration (deceleration) due their non-linear resonance interaction with a monochromatic Langmuir wave in an inhomogeneous plasma. We find the change Δv in the velocity of a particle when it passes through a region where it suffers resonance interaction with the wave. A fundamental role is then played by a dimensionless parameter β (Eq.(2.4)) which is proportional to the wave amplitude and inversely proportional to the acceleration of the phase velocity (i.e., the density gradient). We show that for untrapped particles $\overline{\Delta v}\neq 0$ only when $|\beta|>1$, while the sign of $\overline{\Delta v}$ is the opposite of the sign of the acceleration of the trapped particles which is the same as the phase acceleration (when $|\beta|\leq 1$ the wave cannot trap the particles). We obtain an expression for the average non-linear change in the distribution function, caused by the effects of the drag on the particles by the wave. We show that no additional particle flux arises then (in the stationary case) but that, in general, there is a density change produced by the wave. We also evaluate the integral energy flux of the particles caused by drag effects.

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1. INTRODUCTION. STATEMENT OF THE PROBLEM

Recently there have been both experimental and theoretical intensive studies (in laboratory as well as cosmic plasmas) of non-linear effects of the interaction between monochromatic waves and resonant particles. Of particular interest have then been the resonance processes in an inhomogeneous plasma where the phase velocity of the wave varies. In addition to the very peculiar behavior of the growth rate[1-5] and the evolution of waves,[3-8] resonant acceleration of particles, which has mainly been studied numerically,[9-11] can happen in this case.

In the present paper we give an analytical theory of effects of the dragging of resonant particles by a monochromatic Langmuir wave in an inhomogeneous plasma. After some modifications, the results obtained can be extended also to the case of whistlers (important in connection with experiments about the research in non-linear monochromatic waves of that band in the magnetosphere).

If we assume that a Langmuir wave is excited by an external stationary source at the point $x = 0$ and propagates into the region $x > 0$, we can write down the equation for the electric field in the form

$$E(x,t)=E(x)\cos\left[\int_0^x k(x')\,dx'-\omega t+\varphi(x)\right],\qquad(1.1)$$

where ω and $E(x = 0)$ are assumed to be given while $k(x)$ is determined from the dispersion equation $\epsilon(\omega, k, x) = 0$. If, for the sake of argument, we assume the unperturbed distribution function to be Maxwellian, $f_M(v, x) = n(x)\exp(-v^2/2v_e^2)/(2\pi)^{1/2}v_e$, then the dispersion equation takes the form

$$\omega^2=\omega_p^2+3k^2(x)v_e^2,\quad \omega_p^2=4\pi e^2n(x)/m\qquad(1.2)$$

(we shall assume the thermal velocity v_e to be constant). For not too large amplitudes the non-linear evolution of the wave $E(x)$ and the non-linear advance of the phase $\varphi(x)$ are determined by the interaction of the wave with resonant particles, i.e., particles with a velocity v which is sufficiently close to the local value of the phase velocity $v_\varphi(x) = \omega/k(x)$. The width of the resonance region is determined by the condition

$$|v-v_\varphi(x)|\leq[k(x)\tau(x)]^{-1}.\qquad(1.3)$$

where τ is the non-linear time:

$$\tau=(m/eEk)^{1/2},\qquad(1.4)$$

which, as to order of magnitude, is equal to the period of the oscillations of the trapped particles.

Assuming the ions to be immobile we write down the kinetic equation for the electrons in the field of the wave (1.1) in the form

$$\frac{\partial f}{\partial t}+v\frac{\partial f}{\partial x}-\left(\frac{eE}{m}+\frac{\partial\Psi}{\partial x}\right)\frac{\partial f}{\partial v}=0,\qquad(1.5)$$

where we have introduced the potential $\Psi(x)$ of the external forces that sustain the inhomogeneity of the plasma (without loss of generality we can put $\Psi(0) = 0$). When there is no wave field the stationary solution of Eq. (1.5), i.e., the unperturbed distribution function, takes the form

$$f_0(v,x)=F(v^2+2\Psi(x)).\qquad(1.6)$$

In the case when f_0 is the Maxwell distribution function we can write

$$\Psi(x)=-v_e^2\ln\frac{n(x)}{n(0)},$$
$$\frac{dk}{dx}=-\frac{\omega_p^2}{6v_e^2}\frac{1}{k}\frac{d}{dx}\ln\frac{n(x)}{n(0)}\qquad(1.7)$$

(the last relation follows from (1.2)).

The basic problem solved in the present paper consists in evaluating the change in the distribution function in an arbitrary point x due to the resonant interaction of the wave with particles in the whole interval from the source $x = 0$ up to the given point x. We must then bear in mind that a particle trapped in the potential well of the wave moves with an average velocity equal to the phase velocity $v_\varphi(x)$. An untrapped particle, however, with a velocity v at the point x, interacts resonantly with the wave only in the vicinity of some point x_r where the phase velocity $v_\varphi(x_r) = \omega/k(x_r)$ becomes equal to the particle velocity (Fig. 1). As the width of the resonance region is small, and outside it we may assume that $v^2 + 2\Psi(x) =$ constant, the equation

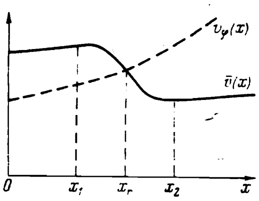

FIG. 1. Change in the average velocity of an untrapped particle under the action of the potential $\Psi(x)$ and the field of the wave; x_r is the point where there is resonant interaction ($v_\varphi(x_r) = v(x_r)$), $x_2 - x_1 = \Delta x_r$ is the width of the resonant interaction region.

determining the point x_r can be written in the form

$$v^2+2\Psi(x)=v_\varphi^2(x_r)+2\Psi(x_r). \quad (1.8)$$

As to the width Δx_r of the region around x_r where the resonant interaction between particle and wave takes place one can estimate it from the following considerations. The value of the phase velocity changes in the interval Δx_r by an amount $\Delta v_\varphi = -(\Delta x_r \omega/k^2)dk/dx$. On the other hand, the change in the particle velocity v as a result of the resonance interaction and under the influence of the field Ψ is of the order of

$$\Delta v \approx (k_r\tau_r)^{-1}+(k_r/\omega)(\partial\Psi/\partial x)_r \Delta x_r,$$

where r indicates that the corresponding quantities are taken in the point x_r. We then get from (1.3) that

$$\Delta x_r \approx v_\varphi(x)/(2\tau\alpha)\,|_{x=x_r}, \quad (1.9)$$

where the quantity α is defined as

$$\alpha=-\frac{\omega^2}{2k^2}\frac{dk}{dx}+\frac{k}{2}\frac{\partial\Psi}{\partial x}. \quad (1.10)$$

α has a simple physical meaning, namely: $-2\alpha/k$ is the acceleration of the particle in the reference frame moving with the phase velocity of the wave. Indeed, as this frame of reference is a non-inertial one, the acceleration of the particle in it consists of two parts: the phase acceleration with reversed sign, equal to $-v_\varphi dv_\varphi/dx$, which comes from the first term in (1.10), and the acceleration $-\partial\Psi/\partial x$ caused by the potential of the external forces (second term in (1.10)). In the case of a Maxwellian plasma we can use (1.7) to write α in the form

$$\alpha(x)=-\frac{\omega^2}{2k^2}\frac{dk}{dx}\left(1-6\frac{v_e^4k^4}{\omega^4}\right). \quad (1.11)$$

As we assume that $v_e \ll \omega/k$, the second term in the brackets is small. Generally speaking, the terms in (1.8) which contain $\Psi(x)$ are also small (provided the ratio of the densities in (1.7) is not too large). This justifies the neglect of the terms with $\Psi(x)$ in the studies[1,3-5,7,8] of effects at distances Δx which corresponds to $\Delta k/k \ll 1$. Since, however, in the present paper we consider effects that arise when $\Delta k = k(x) - k(0) \gtrsim k(0)$, we prefer to retain the terms with $\Psi(x)$ (for whistlers these terms are, in general, always important). It follows from((1.9) that $\Delta v_\varphi(x) = (dv_\varphi/dx)\Delta x_r \ll v_e$ when

$$\omega\tau \gg v_\varphi/v_e, \quad \omega \gg \sqrt{|\alpha|} \quad (1.12)$$

We assume that these conditions are satisfied everywhere in what follows. Moreover, we shall assume that the width of the resonance interaction region is much smaller than the spatial width of the packet, i.e.,

$$\frac{\Delta x_r}{\tau}\frac{d\tau}{dx} \ll 1 \quad (1.13)$$

Substituting (1.9) into (1.13) and using (1.10) we get the last condition in the form

$$d\ln\tau/d\ln k \ll \omega\tau \quad (1.14)$$

2. SOLUTION OF THE EQUATIONS OF MOTION FOR RESONANCE PARTICLES

We consider in more detail the motion of particles in the resonance region of phase space, i.e., when

$$|x-x_r|\leq\Delta x_r, \quad |v-v_r|\leq\Delta v_r, \quad v_r=v_\varphi(x_r)=\omega/k(x_r), \quad (2.1)$$

where Δx_r is defined in (1.9) and $\Delta v_r \sim 1/k_r\tau_r$. We assume here that conditions (1.12) to (1.14) are satisfied. It is convenient to change to a reference frame in which the phase of the wave is independent of the time. To do this we change in the kinetic equation (1.5) and in the equations of motion corresponding to it from the variables t, x, v to the new (dimensionless) variables z, u, θ defined as follows:

$$z=\int_0^x k(x')dx'-\omega t+\varphi(x)+\pi, \quad u=(kv-\omega)/2\sqrt{|\alpha|},$$

$$\theta(x)=\int_0 \frac{\sqrt{|\alpha|}\,k}{\omega}dx'. \quad (2.2)$$

Retaining only the main terms with respect to the small parameters $(\omega\tau)^{-1}$ and $(\omega\tau)^{-1}\, d\ln\tau/d\ln k$, we get the equations of motion in the form[1)]

$$\frac{dz}{d\theta}=2u, \quad \frac{du}{d\theta}=|\beta|(\cos z-\beta^{-1}), \quad (2.3)$$

where the dimensionless parameter β which plays an important role in what follows has the form

$$\beta=(2\alpha\tau^2)^{-1}. \quad (2.4)$$

Apart from unimportant coefficients, z plays in the set (2.3) the role of the coordinate, θ that of the time, and u that of the velocity, while the force consists of two parts: a periodic part $|\beta|\cos z$ (from the wave field) and a constant part $-|\beta|\beta^{-1}$ which in fact is the inertial force in the frame of reference moving with the phase velocity of the wave (in the chosen units).[2)] In obtaining the set (2.3) we dropped terms with derivatives of β (which are small by virtue of (1.14)). We can thus assume that in (2.3) β = constant and write down the corresponding energy integral ϵ = constants:

$$\varepsilon=u^2+y(z), \quad y(z)=|\beta|(z\beta^{-1}-\sin z), \quad (2.5)$$

where ϵ and y(z) are the dimensionless total and potential energies of the particle in the frame of reference moving with the phase velocity of the wave. The quantity y(z) is, apart from an arbitrary constant, determined by Eqs. (2.5) and (2.2). It is convenient to choose the constant such that at the resonance point $x = x_r$ (where u = 0) the quantities z and ϵ are of the order of unity. Outside the resonance region we have then $z\beta^{-1} \to -\infty$ (Fig. 2). It also follows from the form of the effective potential y(z) that potential wells and, hence, trapped particles, exist only when $|\beta| > 1$. Indeed, $dy/dz = |\beta|(\beta^{-1} - \cos z)$ cannot vanish when $|\beta| < 1$, i.e., it is in that case a monotonic functions. This fact will play a principal role in what follows.

We now write down the solution of Eqs. (2.3) for a particle which, moving from the point $\{x_1(\theta_1), z_1, u_1\}$ to

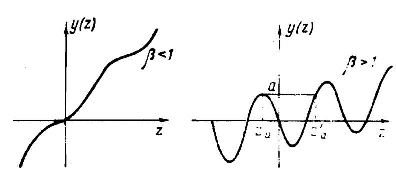

FIG. 2. The effective potential energy (2.5) for $\beta > 0$.

the point $\{x_2(\theta_2), z_2, u_2\}$, underwent a reflection from the potential $y(z)$ in the point $\{x_r(\theta_r), z_r\}$.[3] Using the fact that in the point of reflection $u_r = 0$, we get from (2.5) and (2.3)

$$\beta u_1+|\beta|(\theta_1-\theta_r)=-\frac{\beta|\beta|}{2}\left\{\Phi(\varepsilon)-\int\limits_{-\infty\beta}^{z_1}\frac{dz\cos z}{\sqrt{\varepsilon-y(z)}}\right\},$$

$$\beta u_2+|\beta|(\theta_2-\theta_r)=-\frac{\beta|\beta|}{2}\left\{\Phi(\varepsilon)-\int\limits_{-\infty\beta}^{z_2}\frac{dz\cos z}{\sqrt{\varepsilon-y(z)}}\right\}, \quad (2.6)$$

where

$$\Phi(\varepsilon)=\int\limits_{-\infty\beta}^{z_r(\varepsilon)}\frac{dz\cos z}{\sqrt{\varepsilon-y(z)}}, \quad (2.7)$$

and $z_r(\epsilon)$ is the point of reflection of a particle with energy ϵ which is determined by the equation

$$\operatorname{sign}\beta[z_r(\varepsilon)-\beta\sin z_r(\varepsilon)]=\varepsilon, \quad (2.8)$$

which is obtained from (2.5) if we put there $u = 0$, $z = z_r$.

It follows from (2.7) and (2.8) that $\Phi(\epsilon + 2\pi) = \Phi(\epsilon)$ so that $\Phi(\epsilon)$ can be expanded in a Fourier series

$$\Phi(\varepsilon)=-b+\sum_{n=1}^{\infty}(b_n\cos n\varepsilon+c_n\sin n\varepsilon). \quad (2.9)$$

The coefficients of this expansion were found in[5]. In view of their importance for what follows we give them in full:

$$b=\frac{1}{\pi\beta}\int\limits_{z_a}^{z_a'}\sqrt{a-y(z)}\,dz; \quad (2.10)$$

$$b_n=\sqrt{\frac{2\pi}{n}}\frac{J_n(\beta n)}{\beta}-\sqrt{\frac{2}{\pi n}}\frac{1}{\beta}\int\limits_{z_a}^{z_a'}dz\,\{\cos ny(z)C(\sqrt{n}\,u_a)-\sin ny(z)S(\sqrt{n}\,u_a)\}, \quad (2.11)$$

$$c_n=\sqrt{\frac{2\pi}{n}}\frac{J_n(\beta n)}{\beta}-\sqrt{\frac{2}{\pi n}}\frac{1}{\beta}\int\limits_{z_a}^{z_a'}dz\,\{\cos ny(z)S(\sqrt{n}\,u_a)+\sin ny(z)C(\sqrt{n}\,u_a)\}.$$

Here $J_n(w)$ is a Bessel function, $S(w)$ and $C(w)$ are Fresnel integrals, and $u_a = \sqrt{a - y(z)}$. The meaning of the quantities a, z_a, and z_a' is clear from Fig. 2. When $|\beta| \le 1$, $z_a = z_a'$ and, hence,

$$b=0,\quad b_n=c_n=\sqrt{\frac{2\pi}{n}}\frac{J_n(\beta n)}{\beta}. \quad (2.12)$$

When $|\beta| \gg 1$, clearly $z_a \to -\pi/2$, $z_a' \to 3\pi/2$, $a \to \beta$. In that case the main term in the expansion (2.9) is the first one which is (asymptotically) equal to

$$b\approx\frac{4}{\pi}\frac{\sqrt{2|\beta|}}{\beta}, \quad (2.13)$$

while the coefficients $b_n \sim c_n$ are proportional to $|\beta|^{-3/2}$.

3. RESONANCE ACCELERATION OF PARTICLES

In accordance with the definitions of the preceding section we denote by v_1 the velocity of an untrapped particle in the point $x_1(\theta_1)$ (i.e., before it enters the region Δx_r of resonance interaction with the wave) and by v_2 its velocity in the point $x_2(\theta_2)$, after it has passed through the resonance region. We can then, on the one hand, always assume that $|v_2 - v_1| \gg (k_r\tau_2)^{-1}$ and $|x_2 - x_1| \gg \Delta x_r$, and, on the other hand, neglect the change in the wave parameters in the region Δx_r. The quantity $\Delta v = v_2 - v_1$ characterizes the acceleration of the particle when it passes through the resonance region. Outside this region the particle velocity oscillates weakly and changes somewhat under the influence of the external potential $\Psi(x)$ which maintains the inhomogeneity.

To evaluate Δv we note, first of all, that it follows from the definitions of θ and α that

$$\theta(x_i)-\theta(x_r)=\frac{1}{\omega\sqrt{|\alpha|}}\int\limits_{x_r}^{x_i}|\alpha|k(x')\,dx'$$
$$\approx\frac{\omega}{2\sqrt{|\alpha|}}\left\{\frac{k_r-k_i}{k_r}+\frac{1}{v_r^2}[\Psi(x_i)-\Psi(x_r)]\right\}\operatorname{sign}\alpha, \quad (3.1)$$

where $i = 1, 2$ and where we assume that $|k_r - k_i| \ll k_r$. By virtue of (1.12) we can then always assume that $|\theta_i - \theta_r| \gg 1$, i.e., $|u_i| \gg |\beta|^{1/2}$, $|\epsilon - z_1 \operatorname{sign}\beta| \gg |\beta|$. Hence we can write the last terms in the right-hand sides of (2.6) approximately in the form

$$\int\limits_{-\infty\beta}^{z_i}\frac{dz\cos z}{\sqrt{\varepsilon-y(z)}}\approx\frac{\sin z_i}{\sqrt{\varepsilon-z_i\operatorname{sign}\beta}}\approx\frac{\sin z_i}{|u_i|}. \quad (3.2)$$

Using the second of Eqs. (2.2) we see that these terms correspond to the usual linear approximation for the equations of motion (2.3) and, accordingly, for the kinetic equation. However, the terms $\beta\Phi(\epsilon)/2$ in (2.6) describe essentially non-linear effects when the particle passes through the resonance region. Using Eqs. (2.2) to change from u_i to v and bearing in mind that $u_r = 0$ and $v_r = \omega/k(x_r)$ we get after simple calculations

$$v_i=v_r+\frac{1}{v_r}\left\{[\Psi(x_r)-\Psi(x_i)]+(-1)^iA(x_r)\left[\Phi(\varepsilon)-\frac{\sin z_i}{\sqrt{\varepsilon-z_i\operatorname{sign}\beta}}\right]\right\}, \quad (3.3)$$

with the notation

$$A(x)=v_\varphi^2(x)\sqrt{2|\beta(x)|}/2\omega\tau(x). \quad (3.4)$$

The first term in the braces in (3.3) describes the acceleration of a particle caused by the external potential and the terms with $\Phi(\epsilon)$ the acceleration due to non-linear resonance effects. Finally, the last term, corresponding to the interaction far from the resonance point, vanishes as $\beta z_i \to -\infty$. Expression (3.3) contains as a parameter the quantity ϵ which, while staying constant in the region $x_2 - x_1$ (in the approximation used) depends on the details of the initial conditions in the point x_i. Averaging (3.3) over ϵ and using (2.9) we get:

$$\overline{v_i}=v_r+\frac{1}{v_r}[\Psi(x_r)-\Psi(x_i)-(-1)^iA(x_r)b(x_r)]. \quad (3.5)$$

Since b has the sign of β (see (2.10)), we conclude that untrapped particles (at $|\beta| > 1$) when passing through the resonance region are on average accelerated by the wave when $\beta < 0$ (i.e., when $\alpha < 0$ or, which amounts to the same, $dn/dx < 0$) and are slowed down by the wave when $\beta > 0$ ($\alpha > 0$). The sign of the acceleration of the trapped particles is thus the opposite of the sign of the change in the phase velocity and, accordingly, the sign of the acceleration of the trapped particles. It also follows from (3.5) and (2.12) that when $|\beta| \le 1$ the average change in the particle velocity due to the resonance interaction vanishes.

4. RELATIONS BETWEEN PHASE VOLUMES

We consider the relations between the phase volumes of the particles $D\Omega_1 = dv_1dx_1$ and $D\Omega_2 = dv_2dx_2$, i.e., at different sides of the resonance interaction region. Using (2.2) we can write

$$D\Omega=dv\,dz/k=v\,dv\,dz/\omega. \quad (4.1)$$

For a comparison of $D\Omega_1$ and $D\Omega_2$ it is convenient to express them in terms of the variables v_r and ϵ. Using (3.3) to change from the quantities v_i and z_i to v_r and ϵ and using as the second pair of equations the relations $u_i^2 + \operatorname{sign}\beta(z_i - \beta\sin z_i) = \epsilon$ (where u_i is expressed in terms of v_i) we get approximately

$$D\Omega_i=\frac{v_r\,dv_r\,d\varepsilon}{\omega}\left\{1+\frac{1}{v_r}\left[\frac{\partial\Psi(x_r)}{\partial v_r}+(-1)^i\frac{\partial(A(x_r)\Phi(\varepsilon,x_r))}{\partial v_r}\right]\right\}. \quad (4.2)$$

We shall in what follows be interested in the average integral phase volumes which we define as follows:

$$d\Omega_i=\int_{-\pi}^{\pi}d\varepsilon\left(\frac{D\Omega_i}{d\varepsilon}\right). \quad (4.3)$$

Substituting (4.2) into (4.3) and using (2.9) we get

$$d\Omega_i=\frac{2\pi v_r\,dv_r}{\omega}\left\{1+\frac{1}{v_r}\left[\frac{\partial\Psi(x_r)}{\partial v_r}-(-1)^i\frac{\partial A(x_r)b(x_r)}{\partial v_r}\right]\right\}. \quad (4.4)$$

It is now important that, in general, $d\Omega_1 \neq d\Omega_2$. This fact does, however, not violate Liouville's theorem. Indeed, into the phase volume $d\Omega_2$ enter not only those untrapped particles which were in $d\Omega_1$ before the interaction with the wave, but also particles which were trapped in the vicinity of the point x_r and afterwards left the phase volume of trapped particles and became untrapped (or vice versa) because of a change in the depth of the potential wells due to the change in the parameters β and τ. For the balance of the phase volumes, required by Liouville's theorem, it is thus necessary also to take into account the change in the phase volume of the trapped particles (cf. the analogous situation considered in[3,4] for $|\beta| \gg 1$). This can be done as follows.

It follows from (2.2) that the total phase volume of trapped particles at a given point is equal to

$$\Omega_T\equiv\iint_T dv\,dx=\frac{2\sqrt{|\alpha|}}{k_r^2}\iint_T\frac{d\varepsilon\,dz}{\sqrt{\varepsilon-y(z)}}, \quad (4.5)$$

where T is the region in which the variables ϵ and z vary, corresponding to trapped particles. From Fig. 2 it is clear that

$$\iint_T\frac{d\varepsilon\,dz}{\sqrt{\varepsilon-y(z)}}=\int_{z_0}^{z_0'}dz\int_{y(z)}^{\bullet}\frac{d\varepsilon}{\sqrt{\varepsilon-y(z)}}=2\pi\beta b.$$

As a result the phase volume of trapped particles in a single potential well equals

$$\Omega_T=(4\pi/\omega)A(x_r)\,|b(x_r)|. \quad (4.6)$$

Accordingly when v_r is changed by an amount dv_r the increase in the trapped particle phase volume is equal to

$$d\Omega_T=\frac{4\pi}{\omega}\frac{\partial}{\partial v_r}[A(x_r)\,|b(x_r)|]dv_r. \quad (4.7)$$

We can easily check that the law of conservation of average phase volume

$$d\Omega_1=d\Omega_2+(\operatorname{sign}\beta)\,d\Omega_T \quad (4.8)$$

follows in accordance with Liouville's theorem from Eqs. (4.4) and (4.7).

5. CHANGE IN THE DISTRIBUTION FUNCTION DUE TO THE RESONANCE INTERACTION

Using the general relations of the last two sections we can get first a relation between the values of the average distribution function in the points x_1, v_1 and x_2, v_2. To do this we start from the law of conservation of the particle number

$$\bar f(2)\,d\Omega_2=\bar f(1)\,d\Omega_1-dN_T, \quad (5.1)$$

where dN_T is the change in the number of trapped particles in a single potential well when the resonance velocity changes by dv_r (and, hence, the coordinate of the resonance point changes by dx_r), and the average distribution functions in the points 1 and 2 are defined as follows:

$$\bar f(i)=\frac{1}{d\Omega_i}\int_0^{2\pi}d\varepsilon\,f(v_i,z_i,x_i)\frac{D\Omega_i}{d\varepsilon}. \quad (5.2)$$

We assume here that v_i and z_i are expressed in terms of v_r and ϵ; when integrating we imply that v_r = constant. We now consider two cases.

(1) When x increases the phase volume of the trapped particles decreases: $d\Omega_T(x)/dx < 0$. The capture of particles by the wave occurs then only in the immediate vicinity of the source, i.e., when x = 0; farther away however, the particle escapes capture. In that case, the average trapped-particle distribution function clearly remains unchanged and is equal to the unperturbed distribution function at x = 0 and $v_\varphi = v_\varphi(0) = \omega/k(0)$. The number of trapped particles in a single potential well equals $N_T(x) = f_0(v_\varphi(0), 0)\Omega_T(x)$.

(2) $d\Omega_T/dx > 0$; in that case the wave captures new particles when x increases. It is clear that in that case the average trapped particle distribution function changes.

In any of these cases we can define the average trapped particle distribution function as $\bar f_T(x, v) = n_T(x)\delta(v - v_\varphi(x))$, where $n_T(x) = k(x)N_T(x)/2\pi$ is the trapped-particle density. We introduce further the average number of trapped particles per unit volume

$$\bar f_T(x)=N_T(x)/\Omega_T(x). \quad (5.3)$$

In case (1) we have then

$$\bar f_T(x)=f_0\left(\frac{\omega}{k(0)},0\right),\quad dN_T(x)=f_0\left(\frac{\omega}{k(0)},0\right)d\Omega_T(x). \quad (5.4)$$

Expressing $n_T(x)$ in terms of $\bar f_T(x)$ and using Eq. (4.6) for $\Omega_T(x)$ we get

$$\bar f_T(x,v)=\frac{2}{v}|b(x)|A(x)\bar f_T(x)\delta(v-v_\varphi(x)). \quad (5.5)$$

For the average change in the distribution function in the trapped particle phase space, caused by the field of the wave, we have

$$\overline{\delta f_T}(x,v)=\frac{2}{v}|b(x)|A(x)[\bar f_T(x)-f_0(v_\varphi(x),x)]\delta(v-v_\varphi(x)). \quad (5.6)$$

In the square brackets in this expression we have the average deviation of the number of trapped particles per unit phase volume, from the equilibrium value. The quantity (5.6) satisfies the normalization condition

$$(2\pi/k)\int_{-\infty}^{\infty}dv\,\overline{\delta f_T}(v,x)=\Omega_T(x)[\bar f_T(x)-f_0(v_\varphi(x),x)].$$

We turn now to the calculation of the untrapped particle distribution function. To do this we start from Eqs. (5.1) and (5.2). Since particles with a velocity v_1 at the point x_1 no longer interact resonantly with the wave, their distribution function can be expressed simply in terms of the unperturbed distribution function

$$\bar f(1)=f_0(v_1,\ x_1)=F[v_1^2+2\Psi(x_1)]. \quad (5.7)$$

Substituting (5.7) into (5.2), using (3.3) and (4.4), and neglecting terms $(\omega\tau)^{-2}$ we get:

$$\bar f(1)=f_0(v_r,x_r)+\frac{\partial f_0(v_r,x_r)}{\partial v_r}\frac{A(x_r)b(x_r)}{v_r}. \quad (5.8)$$

When substituting (5.8) into (5.1) we must consider separately the two cases discussed above. When $d\Omega_T/dx < 0$ we can substitute (5.4) instead of dN_T. Using (4.4) and (3.5) we get as a result

$$\bar f_{UT}(2)-f_0(v_2,x_2)=2\left[v_r+\frac{\partial\Psi}{\partial v_r}\right]^{-1}\frac{d}{dv_r}[A(x_r)b(x_r)]$$
$$\times\left[f_0(v_r,x_r)-f_0\left(\frac{\omega}{k(0)},0\right)\right]+\frac{2A(x_r)b(x_r)}{v_r}\frac{\partial f_0(v_r,x_r)}{\partial v_r} \quad (5.9)$$

where the index UT indicate that the quantity $\bar f_{UT}$ refers to untrapped particles.

In the second case when the particles can not leave the capture region when x increases we have

$$\bar f_{UT}(2)=\bar f_{UT}(1)=f_0(v_1,x_1) \quad (5.10)$$

(cf. the analogous situation in[3,4]). Using this relation and Eq. (3.5) we get in the approximation used

$$\bar f(2)=f_0(v_2,x_2)+\frac{\partial f_0(v_r,x_r)}{\partial v_r}\frac{2A(x_r)b(x_r)}{v_r}. \quad (5.11)$$

Substituting then (5.10) into (5.1) we get after simple calculations for the number of particles in a single potential well

$$N_T(x)=N_T(0)+\frac{4\pi}{\omega}\int_{v_\varphi(0)}^{v_\varphi}dv'f_0(v',x')\frac{d[A(x')b(x')]}{dv'}$$
$$(d\Omega_T/dx>0,\quad N_T(0)=f_0(\omega/k(0),0)\Omega_T(0)). \quad (5.12)$$

This expression together with (5.3) determines the trapped particle distribution function.

We now turn from x_2 to an arbitrary point $x > x_2$. Let the particle velocity then change from v_2 to v. In that case $v^2 + 2\Psi(x) = v_2^2 + 2\Psi(x_2)$, as the particle is outside resonance with the wave when moving from x_2 to x (see Fig. 1). Hence the deviation of the untrapped particle distribution function from the equilibrium one equals

$$\delta\bar f_{UT}(v,x)=\bar f_{UT}(v,x)-f_0(v,x)=\bar f_{UT}(2)-f_0(v_2,x_2). \quad (5.13)$$

Substituting (5.9) or (5.11) into (5.13) we get on the right-hand side of (5.13) a function of x_r of the order of $(\omega\tau)^{-1}$. As we neglect terms $(\omega\tau)^{-2}$ we can then assume that x_r is expressed in terms of x, v through Eq. (1.8). Differentiating (1.8) at constant x we get

$$v\frac{dv}{dv_r}=v_r+\frac{\partial\Psi(x_r)}{\partial v_r}. \quad (5.14)$$

Using this relation we get after simple transformations from (5.9), (5.11), and (5.8) for both cases

$$\overline{\delta f_{uT}}(v,x)=\frac{2\theta(x_r)}{v}\frac{\partial}{\partial v}\{A(x_r)b(x_r)[f_0(v,x)-\bar f_T(x_r)]\}, \quad (5.15)$$

where the function $x_r(x, v)$ is defined in (1.8):

$$\theta(x_r)=1\quad(0<x_r<x),\quad \theta(x_r)=0\quad(x_r>x). \quad (5.16)$$

Combining (5.15) and (5.6) we get for the average perturbation of the distribution function

$$\overline{\delta f}(v,x)=\frac{2\theta(x_r)}{v}\frac{\partial}{\partial v}\{A(x_r)b(x_r)[f_0(v,x)-\bar f_T(x_r)]\}$$
$$+\frac{2|b(x)|A(x)}{v}[\bar f_T(x)-f_0(v,x)]\delta\left(v-\frac{\omega}{k(x)}\right), \quad (5.17)$$

where $\bar f_T(x)$ is given either by (5.4) or by (5.3) and (5.12).

The first term in this expression, which is connected with the untrapped particles, consists of two parts that are proportional to Ab and $\partial(Ab)/\partial v$, respectively. The first part determines the change in the untrapped particle distribution function caused by their acceleration (slowing down); the second part is connected with the escape of particles from trapping (or vice versa). In the case when the wave propagates in the direction of increasing density (i.e., $dn/dx > 0$ and hence $\beta > 0$) the trapped particles accelerate and those from them which after some acceleration escape trapping (if $d\Omega/dx < 0$) can form a beam.[4)] In other words, in that case a region can be formed on the "tail" of the distribution function where $\partial\bar f(v,x)/\partial v = \partial(f_0 + \overline{\delta f})/\partial v > 0$. A more detailed study of such beams and of possible instabilities will be given in a separate paper.

6. INTEGRAL EFFECTS

We consider some simple consequences of Eq. (5.17). We saw above that in an inhomogeneous plasma a wave accelerates (decelerates) trapped particles while at the same time slowing down (accelerating) the resonance untrapped particles. It is thus natural to pose the question of the corresponding current density. After elementary calculations we find from (5.17) that

$$\overline{\delta j}=\int_{-\infty}^{\infty}v\overline{\delta f}(v,x)dv=0, \quad (6.1)$$

i.e., the drag effects discussed above do not lead to the appearance of an average current. We could, of course, have expected that result if we started from the exact kinetic equation (1.5) in the stationary case ($\partial\bar f/\partial t = 0$).

The average change in the density differs, however, from zero:

$$\overline{\delta n}(x)=2\int_{v_\varphi(0)}^{v_\varphi}dv_r\frac{v_r+\partial\Psi/\partial v_r}{v^3}A(x_r)b(x_r)[f_0(v_r,x_r)-\bar f_T(x_r)]. \quad (6.2)$$

To estimate this quantity we express it in terms of the trapped-particle density $n_T = kN_T/2\pi$. Using the fact that $N_T(x) \sim f_0(\omega/k(0), 0)\,\Omega_T(x)$ we easily get $\overline{\delta n} \sim n_T\{v_\varphi - v_\varphi(0)\}/v_\varphi$. In order that the theory developed above be valid it is necessary that $d\overline{\delta n}/dx \ll dn/dx$, i.e., $n_T/n \ll (v_e/v_\varphi)^2$. The change in the average density gradient due to drag effects must lead to the appearance of a constant electric field the potential of which must be small when compared with the potential $\Psi(x)$ which maintains the initial inhomogeneity in the framework of applicability of our theory. The magnitude of the electric field must depend on the boundary conditions of the problem which determine the possibility for charge compensation, and will be considered in a separate paper.

We consider, finally, the energy flux of the particles which is caused by drag effects. Using (5.17) and (5.14) we get

$$S(x)=\int_{-\infty}^{\infty}dv(mv^2/2)v\overline{\delta f}(v,x)=-2m\int_0^x dx_r\frac{dv_r}{dx_r}\left(v_r\right.$$
$$\left.+\frac{\partial\Psi(x_r)}{\partial v_r}\right)A(x_r)b(x_r)[f_0(v_r,x_r)-\bar f_T(x_r)]. \quad (6.3)$$

This expression determines, in particular, the average growth rate of the wave caused by particle drag effects

$$\bar\gamma(x)=\frac{8\pi e^2}{m}\frac{\tau(x)b(x)v_\varphi(x)}{\sqrt{2|\beta(x)|}}[f_0(v_\varphi(x),x)-\bar f_T(x)]. \quad (6.4)$$

A more general expression taking into account also coherent effects of the resonant interaction in the vicinity of the print x[5)] was obtained earlier in[5]. It can be written in the form

$$\gamma(x)=\bar\gamma(x)+(\gamma_L/\pi)\left[2b^2+\sum_{n=1}^{\infty}(b_n^2+c_n^2)\right], \quad (6.5)$$

where the coefficients b_n and c_n as functions of the parameter $\beta(x)$ are given by Eqs. (2.10) and (2.11), while γ_L is the growth rate in the linear approximation. It follows from (6.4) and (6.5) that the sign of the growth rate is independent of β, i.e. of dn/dx and is only determined by the sign of γ_L. It is also clear from (6.5) that Eq. (6.4) determines the main part of the growth rate when $f_0(v, x) - \bar{f}_T(x)$ is sufficiently large and $|\beta| > 1$ (when $|\beta| \leq 1$ the growth rate $\bar{\gamma}(x) = 0$ and (6.5) and (2.12) give the result of[1]).

In conclusion we note that Eq. (2.3) which is the basis of the theory proposed here is valid when $d^2\varphi/dx^2 \ll dk/dx$, where $\varphi(x)$ is the non-linear addition to the phase in (1.1). For an analysis of this condition we consider the rather simple case of a weakly inhomogeneous plasma, i.e., $|\beta| \gg 1$. In that case

$$v\frac{d\varphi(x)}{dx}=\frac{3.62}{\pi}\gamma_L\left(\frac{\omega}{kv_e}\right)^2\frac{1}{\omega\tau}+\frac{\bar{\gamma}(x)\beta}{3}, \qquad (6.6)$$

where v_g is the group velocity of the wave. The first term on the right-hand side is the non-linear frequency shift in a homogeneous plasma, found in ref.[12]. The second term is caused by non-linear effects in an inhomogeneous plasma.[5,8] For sufficiently large x this last term is the main one. Using (6.4) we get in that case the condition for the validity of the set (2.3) in the form

$$\frac{d^2\varphi}{dx^2}\Big/\frac{dk}{dx}\approx\frac{16}{3\pi^2}\gamma_L\tau\frac{v_e^2}{v_\varphi v_g}\ll 1 \qquad (6.7)$$

or, in the case (1.2), $\gamma_L\tau < 1$.

1)In obtaining the set (2.3) we neglected also the terms $(d^2\varphi/dx^2)/(dk/dx)$. The analysis given at the end of the paper shows that this is permissible when condition (6.7) holds.

2)We shall assume that the sign of α (and thus of β) does not change in the entire range of x. We note in passing that, in general, sign α = sign dn/dx, where n(x) is the plasma density.

3)In the laboratory frame of reference this reflection corresponds to the particle being overtaken by the wave (when $\alpha > 0$) or the other way round ($\alpha < 0$).

4)We note in passing that trapped particles must completely disappear as soon as the critical point is reached ($\omega_p(x_{cr}) = \omega$), nothwithstanding the fact that $E \to \infty$ as $x \to x_{cr}$. Indeed, the evolution of the wave proceeds in such a way that $E\sqrt{k} < \infty$. [7,8] Therefore in the critical point (where k = 0) $\tau = \infty$ and $\beta = b = 0$, and there can be no trapped particles.

5)These effects are determined by the difference between the particle velocities and the average values (3.5).

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