Friction between the normal component and vortices in rotating superfluid helium

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We discuss the Hall-Vinen theory of the damping of second sound in rotating helium, its derivation, and the limits of its applicability. We show that the Iordanskiĭ force must be added to the transverse friction force between vortices and phonons only if their scattering cross section is given by the usual formula as the square of the scattering amplitude in the Born approximation. For rotons whose scattering cross-section is given by the classical scattering theory one does not need add the Iordanskiĭ force. The resulting transverse force then turns out to be the same for rotons and phonons and equal to the Iordanskiĭ force ($D' = -\kappa\rho_n$) in magnitude. We determine the correction to the friction force between vortices and phonons due to the natural oscillations of the vortex filament.

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It is well known that the appearance of vortices in rotating helium II leads to an additional damping of second sound due to the friction between the elementary excitation and the vortices. A theoretical analysis of this effect, first given by Hall and Vinen[1] reduces to solving two problems: firstly, a determination of the scattering cross sections of the elementary excitations and the change in the momentum of the excitations due to such a scattering and, secondly, solving the hydrodynamical problem of the connection between the quantities determining the change in the momentum of the excitations during the scattering (speed of the vortex motion and normal velocity in the vicinity of the vortex) with the averaged magnitude of the relative velocity $\mathbf{v}_n - \mathbf{v}_s$ ($\mathbf{v}_n$ and $\mathbf{v}_s$ are the velocities of the normal and the superfluid components), which occur directly in the hydrodynamic equation describing second sound.

The determination of the scattering cross section enables us to find the coefficients D and D′, which are proportional to it, in the expression for the friction force f acting upon a single vortex due to the normal component:

$$\mathbf{f}=D(\mathbf{v}_R-\mathbf{v}_L)+D'[\boldsymbol{\varkappa}(\mathbf{v}_R-\mathbf{v}_L)]/\varkappa, \tag{1}$$

where $\mathbf{v}_L$ is the velocity of the vortex motion, $\mathbf{v}_R$ the normal velocity near the vortex, and κ the circulation vector ($\kappa = h/m$). Lifshitz and Pitaevskiĭ[2] calculated D and D′ for the quasi-classical scattering of rotons.

The solution of the hydrodynamical problem led Hall and Vinen to the following two relations:

$$\mathbf{f}=[\boldsymbol{\varkappa}\rho_s(\mathbf{v}_s-\mathbf{v}_L)], \tag{2}$$

$$\mathbf{f}=-\frac{4\pi\eta}{\ln(r_m/l)}(\mathbf{v}_R-\mathbf{v}_n), \tag{3}$$

where η is the viscosity of the normal component, l the mean free path of the quasiparticles, and r_m a cutoff parameter of the order of the viscous length. Equation (2) is the expression for the Magnus effect in a superfluid liquid, and (3) is connected with the hydrodynamical drag effect of the normal liquid by the vortex, as a result of which the normal velocity $\mathbf{v}_R$ close to the vortex differs from the velocity $\mathbf{v}_n$ far from it.

Subsequently the problem of scattering by a vortex was solved also for phonons. Phonons determine the damping of second sound at rather low temperatures (<0.5 K), where there are so far no measurements.

Pitaevskiĭ[3] was the first to consider the problem of phonon scattering; he used the hydrodynamic equations of an inviscid liquid (hydrodynamic approximation). Later the correctness of the calculations of the phonon friction force in the hydrodynamic approximation was put in doubt.[4,5] In particular, Iordanskiĭ[5] concluded that one should add to the right-hand side of (1) a force which is independent of the scattering cross section. Having obtained this force for phonons he proposed to take it into account also in the roton region where it leads to a quantitatively appreciable effect. A number of authors[6,7] have discussed the physical meaning of the Iordanskiĭ force, but the problem of its existence has been put in doubt.[8,9]

The hydrodynamic part of Hall and Vinen's work, and especially the Magnus effect (Eq. (2)) has also been discussed in the literature. The Magnus effect leads to the fact that the damping of second sound which is connected with the vortices must vanish as $\rho_s \to 0$, but this has not been confirmed experimentally near the λ-point.[10] On the other hand, one gets the impression from the derivation of Eq. (1) as given by Hall and Vinen[1] that ρ_s in (2) is the superfluid density in the immediate vicinity of the vortex filament where the applicability of the concept of a two-component hydrodynamics would be doubtful, and this would make the derivation itself also doubtful, as emphasized by the authors themselves.

There are thus in the theory of the damping of second sound in rotating helium a number of problems which remain unexplained and this was the reason for the present paper. In it we propose a new derivation for the hydrodynamic relations (2) and (3) which shows that ρ_s in Eq. (2) is the superfluid density in the volume where it is well defined and that the Magnus effect must be taken into account. However, it follows from this derivation that Hall and Vinen's theory becomes inapplicable when one approaches the λ-point and this apparently causes the disagreement with experiments in its neighborhood. We show also that in the roton region one needs not add the Iordanskiĭ force to Eq. (1). However, the necessity to add the Iordanskiĭ force to (1) for phonons is caused by the fact that the scattering cross-section which determines the coefficient D′ is not the same as the one which is defined in the Born approximation as the square of the amplitude of scattering of a phonon by a vortex (we shall call such a scattering cross section a wave cross section). This is connected with the long-range character of the velocity field of the vortex and the inapplicability of the usual asymptotic representation of the scattered wave. As the calculation of the coefficient D' for rotons (Appendix I) gives the value $D' = -\kappa\rho_n$, which differs in sign from the one obtained in[2]1) this means that the transverse force is the same for phonons and rotons and is the same both in magnitude and sign as the Iordanskiĭ force.

We determine also the longitudinal friction force between phonons and vortices taking into account possible eigen oscillations of the vortex filament when sound propagates along and at right angles to the vortices.

1. TWO-COMPONENT HYDRODYNAMICS: MAGNUS EFFECT AND VISCOUS DRAG

To obtain Eqs. (2) and (3) from Hall and Vinen's theory we split off around the vortex filaments which penetrate the rotating helium cylinders of radius r_0 inside which two-component hydrodynamics is inapplicable. The regions outside and inside these cylinders we call, respectively, the hydrodynamic and the vortex regions. Let the mean free path l of the quasiparticles be much longer than the correlation radius which determines the size of the vortex core, and therefore $r_0 \sim l$.

In the equations of the two-component hydrodynamics we change to a rotating system of coordinates and we linearize the equations with respect to the second sound amplitude, denoting small deviations from the equilibrium values by primed quantities:

$$\mathbf{v}_s=\mathbf{v}_v+\mathbf{v}_s', \quad \mathbf{v}_n=\mathbf{v}_n', \tag{4}$$

and similarly for other quantities. Here $\mathbf{v}_V$ is the velocity field for the lattice of vortex lines in the incompressible fluid. In the rotating system curl $\mathbf{v}_V$ is non-vanishing not only on the vortex lines, but also in the volume, where curl $\mathbf{v}_V = -2\mathbf{\Omega}$; $\mathbf{\Omega}$ is the angular velocity vector. Second sound sets the vortex filaments in motion, and therefore

$$\partial\mathbf{v}_v/\partial t=-(\mathbf{v}_L\nabla)\mathbf{v}_v=-\nabla(\mathbf{v}_L\mathbf{v}_v)+[2\mathbf{\Omega}\times\mathbf{v}_L], \tag{5}$$

where the velocity $\mathbf{v}_L$ of the displacement of the vortex line is linear in the sound amplitude.

We write down the equations for the total current and the superfluid velocity:

$$\partial\mathbf{j}'/\partial t+\nabla(P'+\rho_s(\mathbf{v}_s'-\mathbf{v}_L)\mathbf{v}_v)+\nabla\hat{\tau}-[2\mathbf{\Omega}\rho_s(\mathbf{v}_s'-\mathbf{v}_L)]=0, \tag{6}$$

$$\partial\mathbf{v}_s'/\partial t+\nabla(\mu'+(\mathbf{v}_s'-\mathbf{v}_L)\mathbf{v}_v)-[2\mathbf{\Omega}(\mathbf{v}_s'-\mathbf{v}_L)]+[2\mathbf{\Omega}\mathbf{v}_s']=0, \tag{7}$$

where the vector $\nabla\hat{\tau}$ has components $\partial\tau_{ik}/\partial x_k$,

$$\tau_{ik}=-\eta(\partial v_{ni}/\partial x_k+\partial v_{nk}/\partial x_i).$$

We have dropped in (6) and (7) terms of higher order in $\mathbf{v}_V$ or $\mathbf{\Omega}$, for instance, the Coriolis forces $[2\mathbf{\Omega}\times(\rho_s\mathbf{v}_s'+\rho_n\mathbf{v}_n')]$ in Eq. (6) for the current, as in zeroth approximation in $\mathbf{\Omega}$ when there is no pinning of the first and second sound the total current is

$$\mathbf{j}'=\rho_s\mathbf{v}_s'+\rho_n\mathbf{v}_n'=0.$$

The solution of Eqs. (6) and (7) near the vortex can in the long-wavelength limit be obtained by analogy with the well-known problem of hydrodynamics of a point force in a two-dimensional incompressible viscous fluid.[11] The solution is of the form

$$\mathbf{v}_s'=\text{const}, \quad \mathbf{v}_n'(\mathbf{r})-\mathbf{v}_L=\text{rot}\,\boldsymbol{\psi}, \quad \boldsymbol{\psi}=\frac{(\mathbf{fr})}{4\pi\eta}\ln\frac{r}{r_m}+[(\mathbf{v}_n'-\mathbf{v}_L)\mathbf{r}],$$

$$P'=-\frac{(\mathbf{fr})}{2\pi r^2}-\rho_s(\mathbf{v}_s'-\mathbf{v}_L)\mathbf{v}_v, \quad \mu'+(\mathbf{v}_s'-\mathbf{v}_L)\mathbf{v}_v=\text{const}, \tag{8}$$

where $\mathbf{v}_n'$ without argument is the normal velocity far from the vortex. We choose as the cutoff radius r_m the smaller of two lengths: the viscous length $(\eta/\rho_n\omega)^{1/2}$ (ω is the sound frequency) and the distance between the vortices, while the force is determined by a surface integral over a cylinder of radius r_0:

$$\mathbf{f}=-\int(P'+\rho_s(\mathbf{v}_s'-\mathbf{v}_L)\mathbf{v}_v+\hat{\tau})\,d\mathbf{S}, \tag{9}$$

where $d\mathbf{S}$ is in the direction of the outward normal to the cylinder.

Averaging Eqs. (6) and (7) over a cell of the vortex lattice we get

$$\partial\mathbf{j}'/\partial t+\nabla P'+2\Omega\kappa^{-1}\mathbf{f}-[2\mathbf{\Omega}\rho_s(\mathbf{v}_s'-\mathbf{v}_L)]=0, \tag{10}$$

$$\partial\mathbf{v}_s'/\partial t+\nabla\mu'-[2\mathbf{\Omega}(\mathbf{v}_s'-\mathbf{v}_L)]+[2\mathbf{\Omega}\mathbf{v}_s']=0. \tag{11}$$

Setting the total momentum flux through a cylinder which surrounds the vortex and which lies in the hydrodynamic region equal to zero we get Eq. (2) for the force $\mathbf{f}$ which, according to (11) determines the force $\mathbf{F}_{ns} = 2\Omega\mathbf{f}/\kappa$ which together with the Coriolis force $-[2\mathbf{\Omega}\rho_s \times \mathbf{v}_s']$ acts upon the superfluid component.

We obtain Eq. (3) from (8) if we choose for $\mathbf{v}_R$ the value $\mathbf{v}_n'(\mathbf{r})$ at a distance $r_0 \sim l$ from the vortex filament. Such a choice of $\mathbf{v}_R$ is based on the assumption that the flux of quasiparticles scattered by a vortex is determined by the quasiparticle distribution function at mean free path distances from it without taking into account the perturbation connected with the flux of scattered quasiparticles. However, an estimate shows that taking into account the flux of quasiparticles which immediately after scattering collide with other quasiparticles and as a result of such a collision return again to the scatterer leads to the fact that we must take for the lower limit under the logarithm sign not the mean free path but a quantity of the form $l^{1-\alpha}\sigma^{\alpha}$, where σ is the effective scattering diameter of the vortex. The relative error of the coefficient in Eq. (3) is thus equal to2) $\ln(l/\sigma)/\ln(r_m/l)$. The viscous corrections of order unity to the logarithm in Eq. (2) are thus an excess of accuracy. However, this does not refer to the imaginary additional term no matter how small it is. Such an additional term means a mismatch in phase of the force $\mathbf{f}$ and the vector $\mathbf{v}_R - \mathbf{v}_L$ and taking it into account enabled Lynall and Mehl[12] to explain the experimentally observed change in the second sound velocity.

2. CONNECTION BETWEEN THE FRICTION FORCE AND THE QUASIPARTICLE SCATTERING CROSS SECTION

We now consider the vortex region where we can neglect the interaction of the quasiparticles moving in the velocity field of the vortex. If the mean free path is much longer than the quasiparticle de Broglie wavelength, inside this region, not too close to the vortex filament we can use the quasi-classical approximation and express the momentum flux tensor in terms of the classical distribution function $f(\mathbf{p})$ of the excitations:

$$\Pi_{ij}=P\delta_{ij}+\rho v_{si}v_{sj}+j_{0i}v_{sj}+j_{0j}v_{si}+\int d\mathbf{p}\,f(\mathbf{p})\,p_i\,\partial\varepsilon(\mathbf{p})/\partial p_j, \tag{12}$$

where $\epsilon(\mathbf{p})$ is the quasi-particle energy in a system moving with the superfluid velocity, and $j_{0i} = \int d\mathbf{p} f(\mathbf{p}) p_i$.

We shall look for the change in Π_{ij}, which is linear in the second sound amplitude, at large distances from the vortex filament in a coordinate system moving with velocity $\mathbf{v}_L$. The change in the distribution function $f(\mathbf{p})$ is, first of all, connected with the change in the equilibrium distribution function $\{\exp[(\epsilon(\mathbf{p}) + \mathbf{p}(\mathbf{v}_V + \mathbf{v}_S' - \mathbf{v}_R')/k_B T] - 1\}^{-1}$. Expanding it we get the correction linear in $\mathbf{v}_S'$ and $\mathbf{v}_R'$:

$$\Delta_0 f = \frac{\partial n_0}{\partial \varepsilon}\mathbf{p}(\mathbf{v}_s' - \mathbf{v}_R') + \frac{\partial^2 n_0}{\partial \varepsilon^2}(\mathbf{p}(\mathbf{v}_s' - \mathbf{v}_R'))(\mathbf{p}\mathbf{v}_v),$$
$$n_0(\varepsilon) = [\exp(\varepsilon(\mathbf{p})/k_B T) - 1]^{-1}. \tag{13}$$

Secondly, the distribution function changes due to scattering, if $\mathbf{v}_R \neq \mathbf{v}_L$. Were the scattering by the vortex field to occur in a limited region, the scattered quasiparticles would move at large distances from the vortex filament in planes that pass through the vortex filament, and the correction to the distribution function would have the form

$$\Delta_\sigma f = -\frac{\partial n_0}{\partial \varepsilon}\left(-(\mathbf{v}_R' - \mathbf{v}_L)\mathbf{p}\frac{\sigma_T(\mathbf{p})}{r} + \left[\frac{\boldsymbol{\varkappa}}{\varkappa}(\mathbf{v}_R' - \mathbf{v}_L)\right]\mathbf{p}\frac{\sigma_s(\mathbf{p})}{r}\right)\delta(\vartheta - \vartheta_r), \tag{14}$$

where ϑ and ϑ_r are the angles in the cylindrical system of coordinates for the vector $\mathbf{p}$ and the radius vector $\mathbf{R}$ drawn in the given point at a distance r from the vortex filament, while the cross sections σ_T and σ_S are given in terms of the differential scattering cross section $\sigma(\gamma)$:

$$\sigma_T = \int_{-\pi}^{\pi}\sigma(\gamma)(1 - \cos\gamma)\,d\gamma, \quad \sigma_s = \int_{-\pi}^{\pi}\sigma(\gamma)\sin\gamma\,d\gamma, \tag{15}$$

where γ is the difference between the angles ϑ for the vector $\mathbf{p}$ after and before the collision.

However, the velocity field of the vortex is long-range and one can not even at very large distances assume the quasiparticle momentum to be unchanged. However, to a good approximation one can assume a trajectory with a large impact parameter to be rectilinear and determine the change in the momentum along the trajectory by solving the quasi-classical equations of motion in first order in κ:

$$\frac{d\mathbf{R}}{dt} = \frac{d\varepsilon}{dp}\frac{\mathbf{p}}{p} + \frac{[\boldsymbol{\varkappa}\mathbf{r}]}{2\pi r^2}, \quad \frac{d\mathbf{p}}{dt} = \frac{[\boldsymbol{\varkappa}\mathbf{p}]}{2\pi r^2} - \frac{([\boldsymbol{\varkappa}\mathbf{p}]\mathbf{r})\mathbf{r}}{\pi r^4}. \tag{16}$$

We find that $\mathbf{p} = \mathbf{p}_\infty - p\mathbf{v}_V/v_G$ where v_G and $\mathbf{r}$ are the projections of the group velocity $(\partial\epsilon/\partial p)\mathbf{p}/p$ and the radius vector $\mathbf{R}$ on the plane at right angles to the vertex filament and $\mathbf{p}_\infty$ is the momentum $\mathbf{p}$ as $r \to \infty$.

Such a change in momentum under the conditions that the distribution function at infinity is the equilibrium one and that $\mathbf{v}_R \neq \mathbf{v}_L$ leads to the following correction to the distribution function:[3)]

$$\Delta_p f = +\frac{\partial n_0}{\partial \varepsilon}(\mathbf{v}_R' - \mathbf{v}_L)(\mathbf{p} - \mathbf{p}_\infty) = -\frac{\partial n_0}{\partial \varepsilon}\frac{p}{v_G}(\mathbf{v}_R' - \mathbf{v}_L)\mathbf{v}_v. \tag{17}$$

Terms of higher order in κ operate already in a limited region of space and can be taken into account through the cross sections σ_T and σ_S which determine the correction $\Delta_\sigma f$.

The total change in the distribution function $\Delta f = \Delta_0 f + \Delta_\sigma f + \Delta_p f$ satisfies in first approximation in $\mathbf{v}_V$ the kinetic equation:

$$(\mathbf{v}_s' - \mathbf{v}_L)\frac{\partial}{\partial \mathbf{r}}\left(\frac{\partial n_0}{\partial \varepsilon}(\mathbf{v}_v\mathbf{p})\right) + \frac{\partial \varepsilon}{\partial p}\frac{\mathbf{p}}{p}\frac{\partial \Delta f}{\partial \mathbf{r}} - \frac{\partial}{\partial \mathbf{r}}(\mathbf{v}_v\mathbf{p})\frac{\partial \Delta f}{\partial \mathbf{p}} = 0, \tag{18}$$

if we choose as the zeroth approximation $\Delta f = (\partial n_0/\partial\epsilon)(\mathbf{v}_S' - \mathbf{v}_R')\cdot\mathbf{p}$. Substituting Δf into (12) we get the required correction to Π_{ij}:

$$\Delta\Pi_{ij} = \rho_s(v_{si}' - v_{Li})v_{vj} + \rho_s(v_{sj}' - v_{Lj})v_{vi} - \rho_s(\mathbf{v}_v(\mathbf{v}_s' - \mathbf{v}_L))\delta_{ij}$$
$$-\frac{r_i r_j}{\pi r^4}\left(D((\mathbf{v}_R' - \mathbf{v}_L)\mathbf{r}) + D'\left(\left[\frac{\boldsymbol{\varkappa}}{\varkappa}(\mathbf{v}_R' - \mathbf{v}_L)\right]\mathbf{r}\right)\right), \tag{19}$$

where

$$D = -\frac{1}{2}\int\frac{\partial n_0}{\partial \varepsilon}p_\perp^2\sigma_T(\mathbf{p})v_G\,d\mathbf{p},$$
$$D' = \frac{1}{2}\int\frac{\partial n_0}{\partial \varepsilon}p_\perp^2\sigma_s(\mathbf{p})v_G\,d\mathbf{p}, \tag{20}$$

and $p_\perp$ is the component of $\mathbf{p}$ at right angles to κ.

Knowing $\Delta\Pi_{ij}$ and equating to zero the momentum flux through the cylinder surrounding the vortex filament we get Eq. (1) without the Iordanskiĭ force. However, for phonons the differential cross section $\sigma(\gamma)$ which determines D and D' is not the same as the wave scattering cross section determined by the square of the scattering amplitude. We shall see in Sec. 3 that for phonons there appears, when we take the corresponding change in the wave scattering cross-section, in the quantity of first order in κ the transverse Iordanskiĭ force which is the same as the transverse force for rotons which is determined in Appendix I.

3. SCATTERING OF PHONONS BY A VORTEX. IORDANSKIĬ FORCE

We consider the problem in the hydrodynamic approximation[4)], i.e., the sound wave propagates in an inviscid liquid in the presence of a vortex. The liquid density and velocity at any time can be written in the form

$$\rho = \rho_0 + \rho_{ph}(t), \quad \mathbf{v} = \mathbf{v}_v(t) + \mathbf{v}_{ph}(t), \tag{21}$$

where ρ_0 is the average liquid density, ρ_{ph} and $\mathbf{v}_{ph}$ are the changes in the liquid density and velocity connected with the oscillations,

$$\mathbf{v}_v(t) = \frac{\varkappa}{4\pi}\int\frac{[d\mathbf{R}_v(t)(\mathbf{R} - \mathbf{R}_v(t))]}{|\mathbf{R} - \mathbf{R}_v(t)|^3}$$

is the velocity field of the incompressible liquid around the curved vortex filament, and $\mathbf{R}_V$ is the radius vector of a point on the vortex filament which is time-dependent in such a way that

$$\frac{\partial \mathbf{v}_v}{\partial t} = -\frac{\varkappa}{4\pi}\nabla\int\frac{[d\mathbf{R}_v(\mathbf{R} - \mathbf{R}_v)]}{|\mathbf{R} - \mathbf{R}_v|^3}\frac{\partial \mathbf{R}_v}{\partial t}.$$

We write down the hydrodynamic equations, linearized in ρ_{ph} and $\mathbf{v}_{ph}$:

$$\frac{\partial \rho_{ph}}{\partial t} = -\rho_0\,\mathrm{div}\,\mathbf{v}_{ph} - \mathbf{v}_v\nabla\rho_{ph}, \quad \frac{\partial \mathbf{v}_{ph}}{\partial t} = -\frac{c^2}{\rho_0}\nabla\rho_{ph} - \nabla(\mathbf{v}_v\mathbf{v}_{ph}) - \frac{\partial \mathbf{v}_v}{\partial t}, \tag{22}$$

where c is the sound velocity. Although Eqs. (22) are inapplicable close to the core of the vortex, their contribution to the scattering is unimportant in the long-wavelength limit and we can choose a solution of (22) which is bounded at zero.

At low temperatures most of the natural oscillation modes of the vortex are slower than the sound modes as the frequency of the first is proportional to $k^2|\ln kr_c|$ and of the second to k (r_c is the vortex core radius). We can thus neglect the filament velocity which is induced through its oscillations, i.e., the vortex filament moves with the phonon velocity $\mathbf{v}_{ph}$ and taking its natural oscillations into account reduces to the fact that its average position is curved, but close to the z-axis.

In the Born approximation for a curved vortex filament Eq. (22) leads to the following wave scattering cross-section:

$$\sigma(\mathbf{k} \to \mathbf{k}_1) = \frac{1}{L}\left(\frac{k\varkappa}{4\pi c}\right)^2\left(\frac{2}{q^2} - \frac{1}{k^2}\right)^2\iint([\mathbf{k}\mathbf{k}_1]d\mathbf{R}_v(z_1))$$
$$\times([\mathbf{k}\mathbf{k}_1]d\mathbf{R}_v(z_2))\exp\{iq[\mathbf{R}_v(z_1) - \mathbf{R}_v(z_2)]\}, \tag{23}$$

where $\mathbf{k}$ and $\mathbf{k}_1$ are the wavevectors of the incident and the scattered waves, $\mathbf{q} = \mathbf{k}_1 - \mathbf{k}$, and L is the length of the vortex filament.

If we neglect the curvature of the vortex filament we have

$$\sigma(\mathbf{k}\to\mathbf{k}_1)=\left(\frac{k}{c}\right)^2\frac{\delta(q_z)}{8\pi}\left\{\frac{2}{q^2}-\frac{1}{k^2}\right\}^2([\mathbf{k}\mathbf{k}_1]\varkappa)^2. \quad (24)$$

In Appendix II we determine the cross section, taking the curvature of the vortex due to its natural oscillations into account.

We must note that in our method for separating the vortex field connected with the moving vortex filament no corrections to the $\mathbf{v}_{ph}$ field that diverge like $1/r^2$ because of the scattering, arise, such corrections appear if we split off in (22) the $\mathbf{v}_V$ field for an immobile vortex filament. This has at times been the reason for doubts about the correctness of the hydrodynamic approximation.[4,5] However, the correct value for the cross section is obtained for any method of separating $\mathbf{v}_V$, and (24) therefore is the same as obtained by Pitaevskiĭ[3] and by Iordanskiĭ[5].[5)]

We now show that the correction to the momentum flux tensor Π_{ij} for $\mathbf{v}'_R - \mathbf{v}_L \neq 0$ has the same form as in (19), but that the differential cross section differs from the wave cross section by an amount which is caused by the appearance of the transverse Iordanskiĭ force. This quantity, like the Iordanskiĭ force, is of first order in κ and when evaluating it we can drop terms $\propto\kappa^2$, such as the wave scattering cross section. It is, however, necessary to take into account terms of second order in the phonon amplitude ρ_{ph} and in $\mathbf{v}_{ph}$ and to average over times longer than the inverse phonon frequencies. We have then

$$\Delta\Pi_{ij}={}^1/_2\delta_{ij}(\langle\rho_{ph}{}^2\rangle-\langle v_{ph}{}^2\rangle)+\langle\rho_{ph}v_{phi}\rangle v_{vj}+\langle\rho_{ph}v_{phj}\rangle v_{vi}+\rho_0\langle v_{phi}v_{phj}\rangle, \quad (25)$$

where the averages are determined by means of the phonon distribution function $-(\partial n_0/\partial\epsilon)\hbar\mathbf{k}\cdot(\mathbf{v}'_R - \mathbf{v}_L)$.

The current $\langle\rho_{ph}\mathbf{v}_{ph}\rangle$ in (25) can be evaluated in zeroth approximation in v_V (plane sound wave), i.e.,

$$\langle\rho_{ph}\mathbf{v}_{ph}\rangle=-\int\hbar^3\,d\mathbf{k}\frac{\partial n_0}{\partial\varepsilon}(\hbar\mathbf{k}(\mathbf{v}_R'-\mathbf{v}_L))\hbar\mathbf{k}=\rho_n(\mathbf{v}_R'-\mathbf{v}_L), \quad (26)$$

and to determine $\langle\rho^2_{ph}\rangle$, $\langle v^2_{ph}\rangle$, and $\langle v_{phi}v_{phj}\rangle$ we must find the asymptotic values of ρ_{ph} and v_{ph} as $r\to\infty$ in first approximation in $\mathbf{v}_V$.

Because of the long range the usual asymptotic expression for short-range potentials diverges at small angles ψ between the projections of the radius vector $\mathbf{R}$ and the wavevector $\mathbf{k}$ on the plane perpendicular to the vortex (the two-dimensional vectors $\mathbf{r}$ and $\mathbf{k}_\perp$) as the scattering amplitude diverges as $1/\psi$. The correct asymptotic expression for the velocity potential φ_{ph} ($\mathbf{v}_{ph} = \nabla\varphi_{ph}$) for small $|\psi| \ll 1$ has the form (see Appendix III)

$$\varphi_{ph}(\mathbf{R})\sim e^{i\mathbf{k}\mathbf{R}}\left(1+i\frac{\varkappa k}{2\pi c}\left[\pi\Phi\left(\psi\left(\frac{k_\perp r}{2i}\right)^{1/2}\right)-\psi\right]\right). \quad (27)$$

Using (27) to determine the density ρ_{ph} and the velocity $\mathbf{v}_{ph}$ and substituting these into (25) we find that $\langle\rho^2_{ph}\rangle - \langle v^2_{ph}\rangle = 0$ and the term $\rho_0\langle v_{phi}v_{phj}\rangle$ is equal to

$$\rho_0\langle v_{phi}v_{phj}\rangle=-\hbar^3\int d\mathbf{k}\frac{\partial n_0}{\partial\varepsilon}(\hbar\mathbf{k}(\mathbf{v}_R'-\mathbf{v}_L))\hbar c\,\mathrm{Re}\left\{\frac{k_ik_j}{k}\left(1+i\frac{\varkappa k}{\pi c}\times\left[\pi\Phi\left(\psi\left(\frac{k_\perp r}{2i}\right)^{1/2}\right)-\psi\right]\right)-\frac{1}{c}(v_{vi}k_j+v_{vj}k_i)(1-(2\pi k_\perp r)^{1/2}\times\exp[i(k_\perp r\psi^2/2-\pi/4)])\right\}. \quad (28)$$

Evaluating this integral we see that we get from (25) Eq. (19) with $v'_S = v_L$, $D = 0$, $D' = -\kappa\rho_n$. There thus acts a Iordanskiĭ force on the vortex which must be added to the force determined through the wave scattering cross-section and omitted in our calculation.

If

$$|\psi|\gg\frac{1}{(k_\perp r)^{1/2}},\ \text{we have}\ \Phi\left(\psi\left(\frac{k_\perp r}{2i}\right)^{1/2}\right)=\frac{\psi}{|\psi|}$$

and Eq. (27) is the solution of the problem in the geometric optics approximation while the imaginary part of the factor for a plane wave in (27)

$$\frac{\varkappa k}{2\pi c}\left(\frac{\psi}{|\psi|}\pi-\psi\right)$$

is the phase shift of the wave in the velocity field of the vortex which is small for long-wavelength phonons. In the small angle region $|\psi| \lesssim (k_\perp r)^{-1/2}$ the geometric optics approximation becomes inapplicable but just this region (diffraction region) is responsible for the occurrence of the Iordanskiĭ force.

In the diffraction region phonon wavepackets are incident with impact parameters small compared to the distance from the vortex for which the momentum flux tensor is determined. It is therefore not surprising that the contribution from the diffraction region to the distribution function has the same form as $\Delta_\sigma f$ and can be taken into account by including in the differential scattering cross-section the quantity $\sigma_I = \kappa\delta(\gamma)/c\gamma$ which leads to the value $\sigma_S = \kappa/c$ for phonons (see (14)).

To explain why the Iordanskiĭ force does not need to be added in the case of rotons we consider the connection between it and the phase shift $\Delta S(b)/\hbar$ of the wavepacket describing some quasi-particle. Here S(b) is the change in the classical action due to the interaction with the vortex field after passing through the whole of the classical trajectory while b is the impact parameter with a sign which is the same as the sign of the component of the angular momentum of the quasiparticle along the circulation vector κ. If the quasiparticle trajectories are nearly rectilinear and the change in the transverse momentum $\delta p_\perp$ along them is small we get, using the relation $\mathbf{p} = \partial S/\partial\mathbf{r}$, for the cross section σ_S that determines the transverse force[6)]

$$\sigma_s=\int_{-\infty}^{\infty}\delta p_\perp\frac{db}{p}=\frac{\Delta S(\infty)-\Delta S(-\infty)}{p}=\frac{\varkappa}{v_g}, \quad (29)$$

where v_g is the total group velocity and is equal to the sound velocity c for phonons. To determine $\Delta S(b)$ for $b = \pm\infty$ it is sufficient to solve the classical equation of motion (16) in first approximation in the vortex field, i.e., in κ, whence $\Delta S(\pm\infty) = \pm\kappa p/2v_g$.

Equation (28) is true for both phonons and rotons. However, for rotons one can use Appendix 1 to verify that the change $\Delta S(b)$ from $b = -\infty$ to $b = +\infty$ occurs continuously in the region where the quasi-classical (geometric optics) approximation is applicable and therefore the quasi-classical scattering cross-section already includes the effect responsible for the appearance of the transverse force, and we need not add the Iordanskiĭ force. For phonons, however, the whole change $\Delta S(b)$ occurs discontinuously in the diffraction region and the effect of this on the transverse force is not taken into account by the usual Born wave scattering cross section.

We must note that both for rotons and for phonons we can neglect the coefficient D as compared to D′. The quantity $D' = -\kappa\rho_n$ means that the vortex in the superfluid liquid moves with the average mass velocity[7] $(\rho_s v_s + \rho_n v_n)/\rho$.

4. COMPARISON WITH EXPERIMENTS.

In deriving the equations of the Hall-Vinen theory in §§1 and 2 we dropped a number of terms and assumed ρ_s in the hydrodynamic region to be constant. For this it is necessary that the correlation length be much smaller than $r_0 \sim l$. Moreover, using the theory for the scattering of non-interacting quasiparticles by the vortex field assumed that the region in which appreciable scattering occurs has a size much smaller than the mean free path l. For rotons the size of the scattering region is of the order of $b^* \sim \kappa p_0/k_B T$ (see Appendix I) and the necessary condition for the correctness of the theory considered above is the condition $b^* \ll l$. All these conditions are violated when the temperature increases. Thus, already for T = 1.4 K, $b^* \sim l$. The theory can therefore, strictly speaking be compared with experiment only at sufficiently low temperatures.

This is possibly the reason that the conclusion following from Hall and Vinen's theory that the damping of second sound by vortices as $\rho_s \to 0$ is not confirmed by experiments near the λ-point.

We show in Fig. 1 the magnitude of the coefficient D′ obtained experimentally.[15] The value $D' = -\kappa\rho_n$ obtained in the present paper differs by a factor two or three from the experimental data. The cause of the discrepancy, apart from what has been mentioned above, may be a contribution to the scattering by the vortex core; this was already pointed out by Pitaevskiĭ.[8] One should expect a better agreement at lower temperatures.

In conclusion I want to thank S. V. Iordanskiĭ, V. L. Gurevich, L. P. Pitaevskiĭ, G. E. Pikus, and A. G. Aronov for discussing and considering this work.

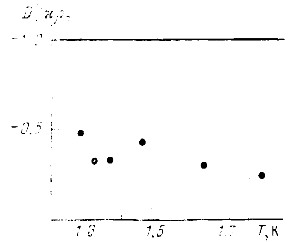

FIG. 1. Experimental and theoretical values of the coefficient D′ for rotons. The line corresponds to the magnitude $D' = \kappa\rho_n$ obtained in the present paper. Earlier known theoretical values of D′ are: $\kappa\rho_n$, [2] 0, [5], and $-\kappa\rho_n$.[8] The points are calculated from the coefficients B and B′ obtained experimentally [15] using Eqs. (25) and (26) of ref. [7] in which the contribution from the Iordanskiĭ force was dropped. To determine the roton mean free path $l = t_r(2k_B T/\mu)^{1/2}$ the time t_r was determined from Eq. (15.7) from [16].

APPENDIX I

TRANSVERSE FORCE FOR ROTONS IN THE QUASI-CLASSICAL APPROXIMATION

We consider the projection of the roton trajectory onto the xy-plane. Let the roton before the collision move parallel to the y-axis starting from negative values and at a distance b from it. Therefore, b is the impact parameter and for $y = -\infty$ and $x = b$.

Equations (16) for a roton with energy $\epsilon(p) = \Delta + (p - p_0)^2/2\mu$ has for small values of $(p - p_0)/p_0$ the form

$$\frac{dy}{dt}=\frac{p-p_0}{\mu}\sin\vartheta,\quad \frac{dp_x}{dt}=\frac{\varkappa p_0(b^2-y^2)}{2\pi r^4}\sin\vartheta,$$
$$\frac{dp}{dt}=\frac{\varkappa p_0 by}{\pi r^4}\sin\vartheta^2. \qquad (I.1)$$

For small $(p - p_0)/p_0$ the main contribution to the friction force comes from deflections at small angles, i.e., the trajectories are almost parallel to the y-axis at all ranges and we put therefore in the right-hand sides of the equations of motion everywhere $p_y \approx p_0 \sin\vartheta$, $p_x = 0$, $x = b$, where ϑ is the angle between **p** and the z-axis.

Let $p - p_0 > 0$. The angle γ over which the projection of the momentum **p** rotates in the xy-plane is determined differently in the interval $0 < b < b^* = \kappa p_0 \sin\vartheta/2\pi E$ and outside it. Here E is the roton energy, calculated with respect to the roton minimum and equal to $(p - p_0)^2/2\mu$ far from the vortex. Let initially $b > b^*$ or $b < 0$. The roton then moves after the collision in the direction $y = \infty$ and the deflection angle γ is obtained after eliminating t from (I.1) and integrating these equations along the trajectory:

$$\gamma=-\frac{p_x(\infty)}{p_0\sin\vartheta}=-\frac{\varkappa}{2\pi v_G}\int_{-\infty}^{\infty}\frac{(b^2-y^2)\,dy}{(b^2+y^2)^2}\left(1-\frac{bb^*}{b^2+y^2}\right)^{-1/2}$$
$$=-\frac{\varkappa}{\pi v_G}\int_0^{\pi/2}\frac{d\varphi\cos 2\varphi}{(b^2-bb^*\cos\varphi^2)^{1/2}}. \qquad (I.2)$$

If, however, $0 < b < b^*$ the roton moving before the collision from $y = -\infty$ progresses only up to the point $y^* = -(bb^* - b^2)^{1/2}$ where the momentum becomes equal to p_0. This is a turning point, after which the roton moves again in the opposite direction towards $y = -\infty$. However, on the reverse path we have already $p < p_0$ and the group velocity and the momentum are directed in the opposite direction. For the angle γ we get

$$\gamma=-\frac{p_x(-\infty)}{p_0\sin\vartheta}=\frac{-\varkappa}{2\pi v_G}\left\{\int_{-\infty}^{y^*}\frac{(b^2-y^2)\,dy}{(b^2+y^2)^2}\left(1-\frac{bb^*}{b^2+y^2}\right)^{-1/2}\right.$$
$$\left.+\int_{y^*}^{-\infty}\frac{-(b^2-y^2)\,dy}{(b^2+y^2)^2}\left(1-\frac{bb^*}{b^2+y^2}\right)^{-1/2}\right\}=-\frac{\varkappa}{\pi v_G b^*}\int_0^{\pi/2}\frac{d\varphi\,(2b\cos\varphi^2-b^*)}{(b^*b-b^2\cos\varphi^2)^{1/2}}, \qquad (I.3)$$

where the two terms in the integration over y correspond to the change in p_x along the path to the vortex and away from it.

We can now determine the value of the cross section σ_s which determines the transverse force:

$$\sigma_s=\int_{-\pi}^{\pi}\sigma(\gamma)\sin\gamma\,d\gamma\approx\int_{-\infty}^{\infty}\gamma\,db=+\frac{\mu\varkappa}{|p-p_0|}. \qquad (I.4)$$

The same value is also obtained when $p < p_0$ before the collision. Substituting σ_s into (20) we get $D' = -\kappa\rho_n$.

In the calculation given here the cross section was determined in terms of the angle γ between the initial and final momenta of the roton at infinite distance from the vortex. In actual fact we are interested in the momentum with which the rotons enter and leave the cylinder of large radius r around the vortex. The angle between these two momenta differs from γ by an amount of order 1/r which is connected with the change in momentum along the trajectory which arises in first order in κ from the quasi-classical equations of motion (16). This effect was already taken into account in the correction $\Delta_p f$ to the distribution function (see Sec. 2). Goodman[8] included this effect in the scattering cross section, and due to this the cross-section turned out to be divergent. Moreover, in[8] a contribution to D′ from the terms $j_{0j}v_{si}$ to the tensor Π_{ij} (see (12)) was neglected which equals $-\kappa\rho_n/2$ which cancels the contribution $\kappa\rho_n/2$ from $\Delta p f$. Goodman therefore obtained by means of numerical calculations at low temperatures a value close to $-\kappa\rho_n/2$ instead of the correct value $-\kappa\rho_n$.

APPENDIX II

EFFECT OF THE NATURAL OSCILLATIONS OF THE VORTEX FILAMENT ON THE SCATTERING OF PHONONS BY A VORTEX

To determine the corrections caused by the oscillations of the vortex filament we average Eq. (23) over all states of this filament which are a consequence of its natural oscillations.

The integrals arising when averaging (23) can be evaluated by changing to a Fourier representation for small displacements of the points of the vortex filament from the original position (the z-axis):

$$x_v(z)=\int_{-\infty}^{\infty} dp\, x(p)\, e^{ipz},\quad y_v(z)=\int_{-\infty}^{\infty} dp\, y(p)\, e^{ipz}, \tag{II.1}$$

$x_v(z)$, $y_v(z)$ together with z are three Cartesian coordinates for $\mathbf{R}_v$. We determine the average values $\langle x(p)^2\rangle$ and $\langle y(p)^2\rangle$ from the classical Rayleigh-Jeans distribution for the natural oscillations of the vortex filament:

$$\langle x(p)^2\rangle=\langle y(p)^2\rangle=k_BTL/\pi\varkappa^2\rho_0 p^2|\ln pr_c|, \tag{II.2}$$

where r_c is the size of the vortex core. The problem reduces to evaluating the generating function

$$J=\left\langle \exp\left\{iq[\mathbf{R}_v(z_1)-\mathbf{R}_v(z_2)]+\mathbf{t}_1\frac{\partial\mathbf{r}(z_1)}{\partial z_1}+\mathbf{t}_2\frac{\partial\mathbf{r}(z_2)}{\partial z_2}\right\}\right\rangle$$
$$=\exp\left\{iq_z(z_1-z_2)-\frac{2k_BT}{\varkappa^2\rho_0}\int_{-\infty}^{\infty}dp\left[q_\perp^2\frac{1-\cos p(z_1-z_2)}{p^2|\ln pr_c|}+iq_\perp(\mathbf{t}_1+\mathbf{t}_2)\right.\right.$$
$$\left.\left.\times\frac{\sin p(z_1-z_2)}{p|\ln pr_c|}-\mathbf{t}_1\mathbf{t}_2\frac{\cos p(z_1-z_2)}{|\ln pr_c|}\right]\right\}, \tag{II.3}$$

where $\mathbf{q}_\perp(q_x, q_y)$, $\mathbf{t}_1$, and $\mathbf{t}_2$ are two-dimensional vectors.

The average cross-section is determined in terms of J and its derivatives with respect to the components of the vectors $\mathbf{t}_1$ and $\mathbf{t}_2$ in the limit as $\mathbf{t}_1\to 0$ and $\mathbf{t}_2\to 0$:

$$\sigma(\mathbf{k}\to\mathbf{k}_1)=\frac{1}{L}\left(\frac{k\varkappa}{4\pi c}\right)^2\left(\frac{2}{q^2}-\frac{1}{k^2}\right)^2\iint dz_1\,dz_2$$
$$\times\left[(k_xk_{1y}-k_{1x}k_y)^2J+(k_yk_{1z}-k_{1y}k_z)^2\frac{\partial^2J}{\partial t_{1x}\partial t_{2x}}\right.$$
$$\left.+(k_zk_{1x}-k_{1z}k_x)^2\frac{\partial^2J}{\partial t_{1y}\partial t_{2y}}\right]. \tag{II.4}$$

The term proportional to J in (II.4) gives the cross-section (24) where the δ-function is replaced by the resonance function $\tilde\delta(q_z)$ with a finite width:

$$\tilde\delta(q_z)=\frac{1}{\pi}\frac{\Delta q}{q_z^2+\Delta q^2};\quad \Delta q=\frac{2\pi k_BTq_\perp^2}{\varkappa^2\rho_0\ln(\varkappa^2\rho_0/r_cq_\perp^2k_BT)}. \tag{II.5}$$

The remaining terms give a correction to the cross-section which in the small-T limit is equal to

$$\Delta\sigma=\frac{k^2k_BT}{4\pi\rho_0c^2|\ln q_zr_c|}\left(\frac{2}{q^2}-\frac{1}{k^2}\right)^2\left[k_z^2k_{1\perp}^2+k_{1z}^2k_\perp^2-2k_zk_{1z}(k_xk_{1x}+k_yk_{1y})\right], \tag{II.6}$$

where $k_\perp^2=k_x^2+k_y^2$. This correction to the cross-section gives a correction to the coefficient D for a second sound wave directed at right angles to the vortex:

$$\Delta D=\frac{2.44(k_BT)^7}{\rho_0c^8\hbar^5\ln(hc/r_ck_BT)}. \tag{II.7}$$

In that case second sound will be damped and when it propagates along the vortex (along the z-axis) the corresponding coefficient D_z is equal to

$$D_z=\frac{3.25(k_BT)^7}{\rho_0c^8\hbar^5\ln(hc/r_ck_BT)}. \tag{II.8}$$

The quantities ΔD and D_z are small and at T = 0.5 K are, respectively, 1.3 and 1.7% of the quantity D which is determined by the cross-section (24) for the scattering by a rectilinear vortex. Even smaller corrections are obtained when we take into account the possible process of the absorption of a phonon with the creation of two eigen oscillation quanta of the vortex filament.

APPENDIX III

ASYMPTOTIC BEHAVIOR OF A SOUND WAVE INCIDENT UPON A VORTEX

Solving Eq. (22) in the Born approximation for a straight immobile vortex filament ($\partial v_v/\partial t = 0$) we get

$$\varphi_{ph}(\mathbf{R})\sim\exp(ik_zz)\left\{\exp(i\mathbf{kr})-\frac{ik}{2c}\int d\mathbf{r}_1\,H_0^{(1)}(k_\perp|\mathbf{r}-\mathbf{r}_1|)\mathbf{v}_v(\mathbf{r}_1)\mathbf{k}\exp(i\mathbf{kr}_1)\right\}$$
$$=\exp(i(k_zz+\mathbf{kr}))\left\{1-\frac{ik}{2c}\int d\mathbf{r}_1\,H_0^{(1)}(k_\perp|\mathbf{r}-\mathbf{r}_1|)\mathbf{v}_v(\mathbf{r}_1)\mathbf{k}\exp i\mathbf{k}(\mathbf{r}_1-\mathbf{r})\right\}, \tag{III.1}$$

where, as before, r is a two-dimensional vector in the xy-plane, the vortex filament lies along the z-axis, and $k_\perp$ is the projection of $\mathbf{k}$ on the xy-plane.

We determine the integral

$$A=-\frac{ik}{2c}\int d\mathbf{r}_1\,H_0^{(1)}(k_\perp|\mathbf{r}-\mathbf{r}_1|)\mathbf{v}_v(\mathbf{r}_1)\mathbf{k}\exp(i\mathbf{k}(\mathbf{r}_1-\mathbf{r}))$$

for large $\mathbf{r}$ and small angles $\psi\sim(k_\perp r)^{-1/2}$ by splitting the xy-plane into regions and choose in it variables in the way shown in Fig. 2.

We consider the contributions from the different regions.

Region I. Here we can assume the quantity ρ/r to be small and retain the first term in an expansion in terms of it:

$$A_1=\frac{i\varkappa kk_\perp}{4\pi cr}\sin\psi\int_0^{\rho_1}\rho\,d\rho\,H_0^{(1)}(k_\perp\rho)\int_0^{2\pi}\exp(-ik_\perp\rho\cos\varphi)\,d\varphi$$
$$=\frac{i\varkappa kk_\perp}{2cr}\sin\psi\int_0^{\rho_1}\rho\,d\rho\,H_0^{(1)}(k_\perp\rho)J_0(k_\perp\rho)$$
$$\approx\frac{i\varkappa kk_\perp}{2cr}\sin\psi\left(\int_0^{\infty}\rho\,d\rho\,H_0^{(1)}((k_\perp+\delta)\rho)J_0(k_\perp\rho)\right.$$
$$\left.-\frac{1}{\pi k_\perp\sqrt{1+\delta}}\int_{\rho_1}^{\infty}d\rho\,e^{i\delta\rho}\right)_{\delta\to0}=\frac{i\varkappa k\rho_1}{2\pi c}\frac{\sin\psi}{r}. \tag{III.2}$$

When changing the integral over $d\rho$ from zero to ρ_1 to

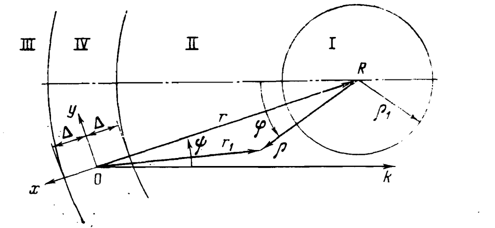

FIG. 2. Integration regions for evaluating the asymptotic values of the velocity potential for a sound wave incident upon a vortex. The point 0 is the vortex filament, R the point of observation $r\gg\rho_1\gg 1/k$, $r\gg\Delta\gg(r/k)^{1/2}$. The integration variables are: in I, II, III: the polar coordinates ρ and φ; in IV the coordinates $x=\rho-r$, $y=\pm\{r\rho(1-\cos(\varphi-\psi))\}^{1/2}$, where + and − correspond to $\varphi<\psi$ and $\varphi>\psi$.

the difference of two such integrals from 0 to ∞ and from ρ_1 to ∞ we used in the second integral asymptotic expressions for Bessel functions.

Regions II and III. Here we can at once replace $H_0^{(1)}(k_\perp \rho)$ by its asymptotic expression and integrate over φ by the saddle-point method:

$$A_{II} = \frac{i\varkappa k k_\perp}{4\pi c} \int_{\rho_1}^{r-\Delta} \rho \, d\rho \left(\frac{2}{i\pi k_\perp \rho}\right)^{1/2} \times \int_0^{2\pi} d\varphi \exp(ik_\perp \rho(1-\cos\varphi)) \frac{r\sin\psi - \rho\sin\varphi}{r^2+\rho^2-2r\rho\cos(\psi-\varphi)} \quad \text{(III.3)}$$

$$= \frac{i\varkappa k}{2\pi c} \int_{\rho_1}^{r-\Delta} d\rho \frac{r\sin\psi}{r^2+\rho^2-2r\rho\cos\psi} = \frac{i\varkappa k}{2\pi c}\left(\operatorname{arctg}\frac{r\sin\psi}{\Delta - r(1-\cos\psi)} - \psi - \frac{\rho_1}{r}\sin\psi\right),$$

$$A_{III} = \frac{i\varkappa k k_\perp}{4\pi c}\int_{r+\Delta}^{\infty} \rho \, d\rho \left(\frac{2}{i\pi k_\perp\rho}\right)^{1/2} \int_0^{2\pi} d\varphi \times \exp(ik_\perp\rho(1-\cos\varphi)) \frac{r\sin\psi - \rho\sin\varphi}{r^2+\rho^2-2r\rho\cos(\psi-\varphi)}$$

$$= \frac{i\varkappa k}{2\pi c} \operatorname{arctg} \frac{r\sin\psi}{\Delta + r(1-\cos\psi)}. \quad \text{(III.4)}$$

When making estimates in the regions I to III we have, in fact, not used the smallness of ψ; it was only necessary that $\Delta \ll r$. Higher order terms in $1/r$ do not contribute in the regions I to III (we checked this for the next two terms). For small ψ we have $r \sin \psi/\Delta \ll 1$ and the terms

$$\operatorname{arctg}\frac{r\sin\psi}{\Delta \pm r(1-\cos\psi)}$$

drop out.

Region IV. We use the condition that $\psi \sim (k_\perp r)^{-1/2} \ll 1$ is small and the asymptotic expansion for $H_0^{(1)}(k_\perp \rho)$. After changing to the variables x and y (see Fig. 2) we expand the integrand in terms of $1/r$ up to terms $1/r^2$ and take into account in this case that $x \sim y \sim r^{1/2}$, $\psi \sim r^{-1/2}$. For instance, the expansion of the exponent is:

$$\exp(ik_\perp\rho(1-\cos\varphi)) \approx \exp\left(i\frac{k_\perp r}{2}\left(\psi - \frac{y}{r}\right)^2\right)\left(1 + ik_\perp r\left(-\frac{\psi^4}{24} + \frac{y\psi^3}{6r} - \frac{y^2\psi^2}{4r^2} + \frac{\psi y^2}{8r^3} + \frac{x\psi^2}{2r} - \frac{xy\psi}{2r^2} + \frac{\psi x^2 y}{8r^3}\right)\right).$$

After integration

$$A_{IV} = \frac{\varkappa k}{2\pi c}\left\{i\pi\Phi\left(\left(\frac{k_\perp r}{2i}\right)^{1/2}\psi\right) - \frac{1}{8}\left(\frac{2\pi}{k_\perp r}\right)^{1/2} \times \exp\left(i\left(k_\perp r\frac{\psi^2}{2} - \frac{\pi}{4}\right)\right)\left(\psi + ik_\perp r\frac{\psi^3}{3}\right)\right\},$$

$$\Phi(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt. \quad \text{(III.5)}$$

Adding $A = A_I + A_{II} + A_{III} + A_{IV}$ we get the integral in (III.1), the main terms of which are retained in (27). The asymptotic expression for φ_Φ applicable when $|\psi| \gg 1/(k_\perp r)^{1/2} (-\pi < \psi < \pi)$ is of the form

$$\varphi_{ph}(\mathbf{R}) = \exp(ik_z z)\left\{\exp(ikr)\left(1 + \frac{i\varkappa k}{2\pi c}\left(\pi\frac{\psi}{|\psi|} - \psi\right)\right) + \exp\left(i\left(k_\perp r - \frac{\pi}{4}\right)\right)\frac{\varkappa k}{2\pi c}\left(\frac{2\pi}{k_\perp r}\right)^{1/2}\frac{\sin\psi}{2(1-\cos\psi)}\right. \quad \text{(III.6)}$$

It is the same as (27) when $1 \gg |\psi| \gg 1/(k_\perp r)^{1/2}$, as

$$\Phi\left(\left(\frac{k_\perp r}{2i}\right)^{1/2}\psi\right) \to \pm 1 + \exp\left(i\left(k_\perp r\frac{\psi^2}{2} - \frac{\pi}{4}\right)\right)\left(\frac{2}{\pi k_\perp r}\right)^{1/2}\frac{1}{\psi}$$

as $\psi(k_\perp r)^{1/2} \to \pm\infty$.

In conclusion we note that the problem of the scattering of a phonon by a vortex is completely analogous to the problem of the scattering of an electron by a magnetic field in a narrow cylindrical region (Aharonov-Bohm effect[17]).[9]

1)Goodman [8] and Titus [9] also obtained the negative sign of D′.

2)This estimate was obtained after critical remarks by V. D. Kagan, for which the author is grateful.

3)S. V. Iordanskiĭ pointed the existence of the correction Δp_f out to the author.

4)The nomenclature "hydrodynamic" is here not used at all in the same sense as in Sec. 1. There we were dealing with two-component hydrodynamics, applicable for characteristic lengths exceeding the mean free path of the quasiparticles; here we are dealing with the hydrodynamics of an inviscid liquid; quantizing the sound oscillations of this liquid we get phonons which are characteristic for a superfluid liquid with a mean free path which is infinite as long as we neglect interactions between the phonons.

5)The equation for the cross section in the paper by Fetter [4] differs by the absence of the second term in the braces in (24) which is connected with the motion of the filament. It is absent as in the partial wave method, used in [4], one must consider the wave for $l = 0$ separately, taking into account the momentum conservation law for the liquid in the vicinity of the vortex filament. In fact, Fetter [4] obtained the cross section for the scattering by a vortex filament which was rigidly pinned by some external forces.

6)This formula was obtained by Pitaevskiĭ. [13]

7)For phonons this statement follows from Efros' work [14] where a consideration of the hydrodynamical Hamiltonian was given without limitations to the degree of anharmonicity.

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