Explosive instability in inhomogeneous media

N. N. Filonenko and L. P. Mel'nik

L. V. Kirenskiĭ Institute of Physics, Siberian Division, USSR Academy of Sciences
(Submitted January 12, 1975; resubmitted April 29, 1975)
Zh. Eksp. Teor. Fiz. 69, 1183–1194 (October 1975)

We study resonance nonlinear wave interactions in which negative energy waves take part in nonequilibrium inhomogeneous media. The inhomogeneity of the medium leads to a detuning of the phase synchronism of the waves and may turn out to be a factor suppressing the explosive instability. We obtain criteria for the suppression of the explosive instability for various inhomogeneities, including small-scale ones. We obtain for a strongly inhomogeneous medium a solution with an oscillatory character. We study especially the regions near points where the phase synchronism is satisfied exactly. Depending on the phase relation in these regions, there occurs an appreciable growth in the intensities of the waves or, on the contrary, an efficient suppression of the instability. We show that the explosive instability may develop at definite phase relations also in a strongly inhomogeneous medium and the development of the explosive instability then has a stepwise character while the development length of the "explosion" is increased by the inhomogeneity. We consider examples of the non-linear interaction in a nonequilibrium inhomogeneous plasma.

PACS numbers: 52.35.Gq, 52.35.En

1. INTRODUCTION

Nonequilibrium media have interesting properties which are connected with the possibility that in such media negative-energy waves may exist.[1] Media in which negative-energy waves exist include plasma-beam systems,[2-4] a semiconducting and magnetoactive plasma with inhomogeneities of a well-defined form,[5-7] and a plasma with an admixture of particles with inversion of level populations.[8] Interactions in a system of particles and waves, amongst which are negative-energy waves, is often accompanied by a removal of energy from negative-energy modes with the simultaneous increase of the amplitude of the latter, which leads to an instability. It is interesting that resonance three-wave interactions involving negative-energy waves develop. Depending on the relation between the frequencies in the system of waves the nonlinear interaction process can then proceed in two ways. If the negative-energy wave has the lowest frequency, there occurs a strong exchange of energy between the low-frequency and the high-frequency waves, i.e., an effective generation of high-frequency waves is possible in an nonequilibrium medium through low-frequency pumping. In the opposite case the nonlinear interaction leads to the simultaneous increase in the amplitudes of all three waves.

In the simplest models[7,9] which describe such an interaction one observes that the amplitudes of all interacting waves increase to infinity after a finite time—the so-called explosive instability. Of course, depending on the actual physical situation, the model must be supplemented by factors which remove this singularity. For instance, the linear damping of the waves leads to a strong excitation of the explosive instability.[10] If the oscillations reach appreciable intensities as a result of the nonlinear interaction it becomes extremely important to take into account the reaction of the oscillations on the properties of the medium. The influence of such kind of factors on the development of the explosive instability in the framework of the weak-nonlinearity approximation has been considered for very different situations (see, e.g.,[11]) when, as a rule, the explosive instability is suppressed and its further development is periodic or quasi-periodic in nature with a characteristic amplitude that depends on the stabilizing parameter.

In the present paper we consider another situation when the wave amplitudes do not increase to such values that it becomes important to take into account higher order nonlinearity while the stabilization and the whole development of the explosive instability is determined by the inhomogeneity of the medium. The explosive instability, as the result of resonance interaction of waves, is extremely sensitive to the detuning of the phase synchronism of the waves: Rabinovich and Reutov[11] have shown that for not too high initial wave intensities even a small constant detuning stabilizes the explosive instability and establishes a periodic regime. In this connection it is qualitatively clear that the inhomogeneity of the medium will first of all lead to a detuning of the wave phase synchronism with a transfer of oscillations from the region where they interact efficiently and, as a result, to a suppression of the explosive instability.[12]

One must note that formally such a problem leads to the study of a nonlinear Hamiltonian system under the influence of an external force: in that case the motion of the system is infinite when there is no external force so that even taking into account the influence of a "smooth" inhomogeneity causes well-defined difficulties.[12] In the present paper we give a rather complete analysis of such a problem for different forms of inhomogeneities using the approach developed earlier.[13,14] Below we obtain a solution for the case of a strongly inhomogeneous medium. The nature of this solution is appreciably different near points where the phase synchronism holds exactly and far away from those points. Near the phase synchronism points a peculiar phase effect is displayed: depending on the phase with which the solution approaches such a point the region near the latter turns out to be a region of strong instability or, to the contrary, a region of strong stabilization of the explosive instability. The development of the "explosion" turns out to be possible for well-defined conditions and in strongly inhomogeneous media where there are only narrow isolated regions of wave phase synchronism. The nature of the singularity of the solution is then independent of the form of the inhomogeneity, for instance, also if it is random. We obtain the criteria for the suppression of the explosive instability for different scales of inhomogeneities, for instance, also when it is small-scale. In conclusion we discuss some examples of nonlinear wave interaction in a nonequilibrium inhomogeneous plasma.

2. INTEGRALS OF MOTION. NATURE OF SINGULARITIES IN INHOMOGENEOUS MEDIA

If we excite in a stationary manner a negative-energy wave at the boundary of a nonlinear nonequilibrium medium, the development of the nonlinear interaction will have the stationary ($\partial/\partial t = 0$) nature of a spatial explosion (see also [11,12]). Following this formulation we shall consider in what follows the development of this instability in inhomogeneous media. In a weakly nonlinear and weakly inhomogeneous medium ($kL \gg 1$, $kl \gg 1$), each of the interacting waves will have the form of a narrow packet in k-space (the z-axis is at right angles to the boundary of the nonlinear medium):

$$a_j(z,t)\exp\{i(\omega_j t-k_j z)+i\varphi_j(z,t)\}. \quad (1)$$

Here $a_j(z,t)\exp\{i\varphi_j(z,t)\}$ are the slowly varying complex wave amplitudes; L and l are characteristic dimensions of the inhomogeneity and of the nonlinear interaction, ω_j and k_j are the frequencies and wave vectors of the waves and satisfy the phase synchronism conditions:

$$\omega_1=\omega_2+\omega_3,\quad k_1=k_2+k_3. \quad (2)$$

The wave with frequency ω_1 is the negative-energy wave.

We assume also, in agreement with all what has been said above, that the damping or growth rates of the waves are small and that the properties of the medium depend only on z; we then get the following set of equations for the moduli of the complex wave amplitudes $a_j(z)$ and their relative phase difference $\Phi(z) = \varphi_2(z) + \varphi_3(z) - \varphi_1(z)$:

$$V_1\frac{da_1}{dz}=\beta a_2a_3\cos\Phi,\quad V_2\frac{da_2}{dz}=\beta a_1a_3\cos\Phi,$$
$$V_3\frac{da_3}{dz}=\beta a_1a_2\cos\Phi, \quad (3)$$
$$\frac{d\Phi}{dz}=\Delta k(z)-\beta\left[\frac{a_2a_3}{V_1a_1}+\frac{a_1a_3}{V_2a_2}+\frac{a_1a_2}{V_3a_3}\right]\sin\Phi.$$

Here β is the coefficient of the nonlinear wave interaction, V_j are the group velocities of the waves (we shall assume that the group velocities of the waves have the same sign $V_j > 0$), and the quantity $\Delta k(z)$ determines the detuning of the phase synchronism which is connected with the inhomogeneity of the medium.

The set (3) has integrals of motion (of the kind of the Manley-Rowe relations in the theory of parametric amplifiers) with the physical meaning of the simultaneous growth or damping of the intensities of all three waves:

$$V_1n_1-V_2n_2=m_1,\quad V_1n_1-V_3n_3=m_2,$$
$$n_j(z)=a_j^2(z). \quad (4)$$

After introducing the dimensionless variables

$$x=z/l,\quad n(x)=n_1(xl)/n_1(0),\quad \varkappa(x)=l\Delta k(xl),$$
$$r_1=m_1/V_1n_1(0),\quad r_2=m_2/V_2n_2(0),\quad l=(V_2V_3)^{1/2}/\beta n_1^{1/2}(0) \quad (5)$$

and using Eqs. (4) we can reduce the set of Eqs. (3) to a Hamiltonian system of two equations for the relative intensity of the waves n(x) with negative energy and the relative phase difference of the waves $\Phi(x)$:

$$\frac{dn}{dx}=2[n(n-r_1)(n-r_2)]^{1/2}\cos\Phi\equiv\frac{\partial\mathscr{H}}{\partial\Phi},$$
$$\frac{d\Phi}{dx}=\varkappa(x)-\left[\left(\frac{(n-r_1)(n-r_2)}{n}\right)^{1/2}+\left(\frac{n(n-r_1)}{n-r_2}\right)^{1/2}+\left(\frac{n(n-r_2)}{n-r_1}\right)^{1/2}\right]\sin\Phi\equiv-\frac{\partial\mathscr{H}}{\partial n}. \quad (6)$$

The boundary conditions have the form

$$n(0)=1,\quad \Phi(0)=\Phi_0,\quad r_{1,2}\leqslant 1. \quad (7)$$

The Hamiltonian $\mathscr{H}(n,\Phi)$ of the set of Eqs. (6) is the function:[12]

$$\mathscr{H}=2\sin\Phi[n(n-r_1)(n-r_2)]^{1/2}-\varkappa n. \quad (8)$$

The set of Eqs. (6) has also the following integral of motion:

$$\Gamma_0=2\sin\Phi[n(n-r_1)(n-r_2)]^{1/2}-\int_0^x dx_1\,\varkappa(x_1)\frac{dn}{dx_1}, \quad (9)$$

where the constant Γ_0 is determined by the boundary conditions (7). We note that the Hamiltonian is connected with the integral Γ_0 as follows:

$$\mathscr{H}=\Gamma_0-\int_0^x dx_1\,n(x_1)\frac{d\varkappa}{dx_1} \quad (10)$$

and is a rigorous integral of motion only in a medium with a constant detuning of the phase synchronism.

We use (9) and substitute $\sin\Phi$ into the second of Eqs. (6) and integrate the resultant equation by parts, using the boundary conditions (7). Substituting the integral expression for $\Phi(x)$ obtained in this way into the first of Eqs. (6) we reduce the set (6) to a single integro-differential equation for n(x):

$$\frac{dn}{dx}=2[n(n-r_1)(n-r_2)]^{1/2}\cos\left[\Phi_0+\int_0^x dx_1\,\varkappa(x_1)-\frac{1}{2}\int_0^x dx_1\left(\frac{1}{n}+\frac{1}{n-r_1}+\frac{1}{n-r_2}\right)\left(\Gamma_0+\int_0^{x_1}dx_2\,\varkappa(x_2)\frac{dn}{dx_2}\right)\right]. \quad (11)$$

The argument of the cosine is the phase $\Phi(x)$.

By studying (9) and (11) near the singularity we can show that if the explosive instability develops in an inhomogeneous medium the form of the singularity for the relative wave intensity n(x) is the same as in a homogeneous medium (including also the case of random inhomogeneities):

$$n(x)\sim(x-x_0)^{-2},\quad |x-x_0|\ll 1.$$

This fact follows at once from expanding (9) and (11) near the singularity x_0, as $\sin\Phi(x_0)$ is independent of the form of the inhomogeneity.

3. STRONG INHOMOGENEITY

We shall take an inhomogeneity to be a strong one in the case when the detuning of the phase synchronism due to the inhomogeneity of the medium $\kappa(x)$ leads to an appreciable advance of the phase $\Phi(x)$, which is the argument of the cosine in Eq. (11), over lengths of the order of l. The asymptotic solution of Eq. (11) turns out for that case to be quite clear and is also very useful for a qualitative analysis for the problem in the general case. We restrict our considerations to the case where Eq. (1) together with the boundary conditions is of the form

$$\frac{dn}{dx}=2n^{3/2}\cos\left[\int_0^x dx_1\,\varkappa(x_1)-\frac{3}{2}\int_0^x\frac{dx_1}{n(x_1)}\int_0^{x_1}dx_2\frac{dn}{dx_2}\varkappa(x_2)\right], \quad (12)$$

$$\Gamma_0=\Phi_0=r_1=r_2=0,\quad n(0)=1. \quad (13)$$

It is convenient to look for n(x) in the form

$$n(x)=\left[1-\int_0^x dx_1\cos\left(\int_0^{x_1} dx_2\varkappa(x_2)+\Pi(x_1)\right)\right]^{-2}, \qquad (14)$$

when the phase advance $\Pi(x)$ satisfies the following equation:

$$\Pi(x)=-\frac{3}{2}\int_0^x\frac{dx_1}{n(x_1)}\int_0^{x_1}dx_2\varkappa(x_2)\frac{dn}{dx_2}, \qquad (15)$$

where we must substitute expression (14) for $n(x)$. The physical meaning of using Eqs. (14) and (15) to introduce the new unknown $\Pi(x)$ instead of $n(x)$ consists of the following: When $\kappa(x)$ is large the phase advance $\Pi(x)$ turns out to be a slowly varying function and Eq. (14) is in that case the solution where one can easily estimate the integral in it using the stationary phase method.

We first of all consider the solution (14), (15) in the region where there are no stationary phase points. The main contribution to expression (14) is then connected with the integration limits, when the conditions

$$|\varkappa^{-1}|\ll 1, \qquad \left|\frac{d\varkappa^{-1}}{dx}\right|\ll 1 \qquad (16)$$

are satisfied, and the solution takes the form

$$n(x)\approx\left[1-\frac{1}{\varkappa(x)}\sin\left(\int_0^x\varkappa(x_1)dx_1+\Pi(x)\right)\right]^{-2}, \qquad (17)$$

$$\Pi(x)\approx 3\int_0^x\frac{dx_1}{\varkappa(x_1)}. \qquad (18)$$

Expression (18) shows that $\Pi(x)$ leading to a slow phase advance must be taken into account in (17) if we are interested in the exact phase of the solution, since $\Pi(x)$ can give a phase advance of the order of unity over appreciable lengths.

We now turn to the form of the solution near the stationary-phase points x_c for which

$$\varkappa(x_c)=-\frac{d\Pi}{dx}(x_c)\approx 0. \qquad (19)$$

The stationary-phase points are the points where, as we see from (19), the phase synchronism conditions are satisfied exactly. When the inequalities

$$(|\varkappa'|^{-1/2})\ll 1, \qquad \left|\frac{d|\varkappa'|^{-1/2}}{dx}\right|\ll 1, \qquad \varkappa'=\frac{d\varkappa}{dx}(x_c) \qquad (20)$$

hold, the expression for the integral occurring in Eq. (14) which is asymptotic in the parameters (20) can in the region near the point $x = x_c$ be obtained by the stationary phase method. Let $\Phi_c \equiv \Phi(x_c)$ be the phase difference with which the interacting waves arrive in the stationary phase point.

Since estimates show that the contribution of the quantity $\Pi(x)$ to the solution in the region near $x \approx x_c$ is small, the approximate solution takes in that region the form

$$n(x)\approx[1-\psi(x)]^{-2}, \qquad |x-x_c|\leqslant(|\varkappa'|)^{-1/2}, \qquad (21)$$

$$\psi(x)=\left(\frac{\pi}{|\varkappa'|}\right)^{1/2}\cos\Phi_c\left[1+\operatorname{sign}(x-x_c)C\left(\left(\frac{|\varkappa'|}{2}\right)^{1/2}|x-x_c|\right)\right]$$
$$-\frac{1}{2}\left(\frac{\pi}{|\varkappa'|}\right)^{1/2}\operatorname{sign}(\varkappa')\sin\Phi_c\left[1+\operatorname{sign}(x-x_c)S\left(\left(\frac{|\varkappa'|}{2}\right)^{1/2}(x-x_c)\right)\right] \qquad (22)$$

Here $C(x)$ and $S(x)$ are Fresnel integrals, defined by the equations

$$C(x)=\left(\frac{2}{\pi}\right)^{1/2}\int_0^x dt\cos t^2, \qquad S(x)=\left(\frac{2}{\pi}\right)^{1/2}\int_0^x dt\sin t^2.$$

The nature of the solution (21), (22) is essentially determined by the sign of the function $\psi(x)$, which determines whether the region close to x_c turns out to be a region of strong stability or, on the other hand, a region in which the wave intensity increases significantly. The maximum value ψ_m is easily found from (22):

$$\psi_m=\left(\frac{\pi}{|\varkappa'|}\right)^{1/2}[2\cos\Phi_c-\operatorname{sign}(\varkappa')\sin\Phi_c]. \qquad (23)$$

The sign of ψ_m, and hence the nature of the solution, is determined by the magnitude of Φ_c in (23). For instance, when sign $\kappa' = -1$ we have

$$\operatorname{sign}\psi_m=1, \quad -\operatorname{arctg}2<\Phi_c<\pi-\operatorname{arctg}2, \qquad (24)$$

$$\operatorname{sign}\psi_m=-1, \quad \pi-\operatorname{arctg}2<\Phi_c<2\pi-\operatorname{arctg}2. \qquad (25)$$

Equations (21) to (23) show that even in a strongly inhomogeneous medium the explosive instability can develop near points of exact phase synchronism. The condition for the suppression of the instability near these points is the following:

$$3(\pi/|\varkappa'|)^{1/2}<1, \quad \operatorname{sign}\psi_m=1. \qquad (26)$$

It is clear from (17) that far from the stationary phase points in strongly inhomogeneous media the solution is oscillating in nature with small periods and amplitudes while oscillations of the intensity occur near some average value which is determined by the contribution from the preceding stationary phase points to the solution. The solution, after passing through p stationary-phase points, will thus take the form

$$n(x)=\left[1-\sum_{j=1}^{p}\psi_j-\frac{1}{\varkappa(x)}\sin\left(\int_0^x\varkappa(x_1)dx_1+\Pi(x)\right)\right]^{-2}, \qquad (27)$$

where ψ_j is the contribution to the solution from the j-th stationary-phase point and is given by Eq. (23). The solution (27) shows that even when the local stability condition (26) is satisfied a build-up $\sum_{j=1}^{p}\psi_j$ with positive sign may occur, and as a result the explosive instability may develop at some large distance from entering the nonlinear medium.

A numerical analysis of the problem (12), (13) was made on an electronic computer and it showed that the analytical results given here describe the singularities of the solution for possible situations very well. For instance, one can clearly see in Fig. 1 the regions of fast oscillations of the solution and also the narrow regions near the stationary phase points $x_c = \pi/2, 3\pi/2$. Curves 1 and 2 demonstrate the phase effect described by Eqs. (21) to (23) where curve 2 corresponds to the stable case near the stationary-phase points, sign $\psi_m = -1$, and the phase Φ_c incident onto the interval is given by inequalities (25). The curve 1, on the other hand, corresponds to the case sign $\psi_m = 1$ and as a result to the development, ultimately, of the explosive instability.

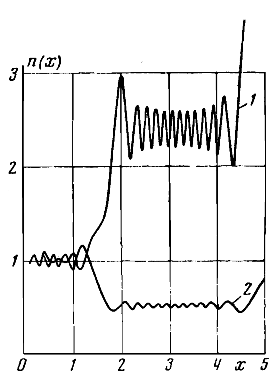

FIG. 1. Numerical solution of (12), (13) for $\kappa(x) = a\cos bx$. 1) a = 40; 2) a = 35.8.

4. STABILIZATION CRITERIA

We shall distinguish between local stability of the solution n(x) at lengths of the order l, and the asymptotic stability as $x \to \infty$ of the solution at large distances. Equation (27) shows that a locally stable solution (see (26)) can, nonetheless, turn out to be unstable at sufficiently large distances. We must note that the problem of the asymptotic stability (as $x \to \infty$) of the solution n(x) is, apparently, in the general case rather complicated, since it is clear from the considerations given above that the stability may depend on various detailed characteristics of the solution such as the phase Φ_c. At the same time, of most practical interest is the consideration of stability over distances of the order l.

We restrict ourselves in what follows to a study of the local stability. We turn to Eq. (11). A sufficient condition for the stabilization of the instability at a dimensionless distance of the order unity is the occurrence of a maximum in the function n(x), or

$$\left.\frac{dn}{dx}\right|_{x\approx 1}=0,\quad \left.\frac{d^2n}{dx^2}\right|_{x\approx 1}<0. \tag{28}$$

One can show that the stabilization criterion (28) depends weakly on the initial conditions Γ_0, Φ_0, r_1, and r_2. When $\Gamma_0 = \Phi_0 = r_1 = r_2 = 0$, Eqs. (28) become

$$\left|\int_0^1 \varkappa(x_1)\,dx_1-\frac{3}{2}\int_0^1\frac{dx_1}{n(x_1)}\int_0^{x_1}dx_2\,\varkappa(x_2)\frac{dn}{dx_2}\right|\geqslant\frac{\pi}{2}, \tag{29}$$

$$-\varkappa(1)+3n^{1/2}(1)<0. \tag{30}$$

The upper limit of the integrals in Eq. (29) is unity as we consider the boundary of the stability region and the maximum of the solution lies close to unity. The meaning of the first condition (29) is that it is necessary for the stabilization of an instability that the inhomogeneity of the medium leads to an appreciable advance in the phase difference of the waves. The second condition (30) leads to the important conclusion that the maximum value n_{max} of a stable solution in an inhomogeneous medium can not be too large:

$$n_{max}<\varkappa^2(1)/9. \tag{31}$$

One must, of course, choose in each actual case the stronger of the inequalities (29) and (30). We turn to the condition (29). The unknown solution n(x) occurs in the integral in (29). Since n(x) increases near the boundary of the stability to large values, we can obtain a rather good estimate of the boundary of the stability region by substituting in (29) the unperturbed solution $n(x) = (1-x)^{-2}$. For the case $\kappa(x) = \kappa' x$ we get in this way from (29) the following condition for the suppression of the explosive instability:

$$\varkappa'>3\pi. \tag{32}$$

The results of a numerical analysis (see Fig. 2) show that the solution turns out to be stable when

$$\varkappa'>10.5. \tag{33}$$

we consider the case of a small-scale inhomogeneity

$$L/l\equiv b^{-1}\ll 1, \tag{34}$$

where L is the scale of the inhomogeneity.

Let $\kappa(x) = af(bx)$, where f(bx) is a periodic alternating function of period L. It is clear that the first integral in (29) in that case gives a contribution of the order a/b. An estimate shows that the second integral also makes a contribution of the same order of magnitude, and the

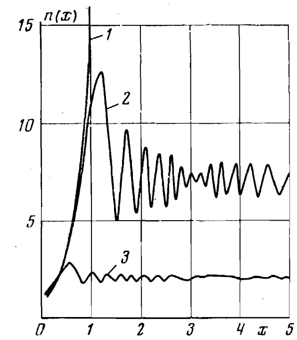

FIG. 2. Numerical solution of (12), (13) for $\kappa(x) = \kappa' x$. 1) $\kappa' = 10.5$; 2) $\kappa' = 10.7$; 3) $\kappa' = 18$.

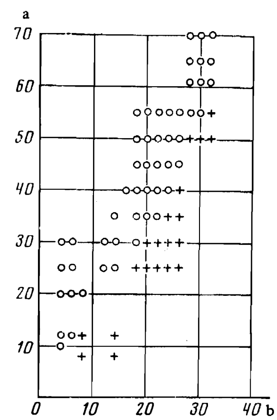

FIG. 3

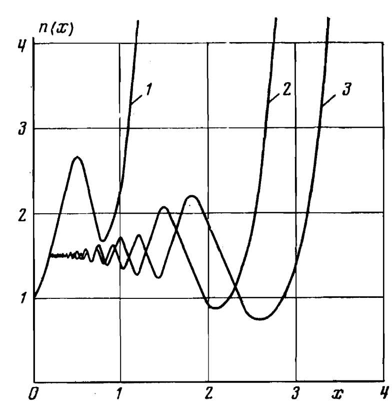

FIG. 4

FIG. 3. $\kappa(x) = a \cos bx$. ○) parameters corresponding to stable solutions; +) parameters corresponding to unstable solutions.

FIG. 4. Curve 1) $\kappa(x) = ax$ for $0 < x < 0.8$ and $\kappa(x) = 0$ for $x \geqslant 0.8$; a = 20. Curves 2 and 3) $\kappa(x) = a/x^2$ for $x \geqslant 0.2$ and $\kappa(x) = 0$ for $0 < x < 0.2$. a = 20 for curve 2, a = 25 for curve 3.

condition for the stabilization of the solution takes in that case the form

$$\alpha a/b\equiv\alpha aL/l\geqslant\pi/2, \tag{35}$$

where α is a numerical coefficient of the order of unity. The diagram showing the stability of the solution for the particular case $\kappa(x) = a \cos bx$ as function of the parameters a and b illustrates the stability criterion (35) well (see Fig. 3). In conclusion we note that the result of the passage of interacting waves through a layer of a strongly inhomogeneous medium will be an increase in the length l of the explosive instability to a quantity of the order of the dimensions of that layer. The solution can, generally speaking, then be of the form of growing intensity oscillations with a subsequent development of the explosion (see Fig. 4).

5. SMOOTH INHOMOGENEITY

We make a few remarks that refer mainly to the region where the smooth-inhomogeneity approximation is applicable; the basic results for this case were obtained by Davydova and Oraevskiĭ.[12] If we neglect in (12) in the cosine the quantity

$$\frac{3}{2}\int_0^x\frac{dx_1}{n(x_1)}\int_0^{x_1}dx_2\,n\frac{d\varkappa}{dx_2}, \tag{36}$$

we get the solution of the problem (12), (13) for $\kappa(0) = 0$ for a smooth inhomogeneity:[12]

$$n(x)=\left[1-\int_0^x dx_1\cos\left(\frac{1}{2}\int_0^{x_1}dx_2\,\varkappa(x_2)\right)\right]^{-2}, \tag{37}$$

which differs formally from (14) for a strong inhomogeneity by the coefficient ½ and the absence of the phase $\Pi(x)$. The Hamiltonian $\mathscr{H}$ in (10) is then assumed to be an integral of motion; this is valid when we can neglect the quantity

$$\int_0^x dx_1\, n(x_1)\frac{d\varkappa}{dx_1}, \tag{38}$$

which is equivalent to dropping the phase advance (36).

For the particular case $\kappa(x) = \kappa' x$ the stabilization criterion following from (37) has the form

$$|\varkappa'|>2\pi. \tag{39}$$

The difference between (39) and (32), (33) is caused by the fact that in the stable case an estimate shows that the omitted terms (36), (38) are no longer small. To obtain the solution in that case for large x we must use Eqs. (14), (17), and (18) for a strong inhomogeneity, since the criterion (16), which takes the form

$$1/\varkappa' x \ll 1, \tag{40}$$

turns out to be satisfied.

When the criterion (20)

$$(\pi/|\varkappa'|)^{1/2} \ll 1 \tag{41}$$

is satisfied, Eqs. (21) and (22) correctly reflect the solution near x = 0 and Eq. (23) determines the asymptotic value near which n(x) oscillates when $x \gg |\kappa'|^{-1}$.

We note, finally, that in the stable case, when n(x) oscillates near some average value, the solution (37) ceases to be valid when (see (8) to (10), (38)):

$$\left|\int_0^x dx_1\, n(x_1)\frac{d\varkappa}{dx_1}\right| \approx \left|\bar{n}\frac{d\varkappa}{dx}x\right| \gtrsim |\varkappa(x)\bar{n}|, \tag{42}$$

i.e., at distances x for which

$$x \gtrsim \varkappa(x)\,|\varkappa'|^{-1}. \tag{43}$$

6. CONCLUSIONS

The analysis given above shows that the nature of the development of the explosive instability in inhomogeneous media is determined by the following important parameter: the magnitude of the advance of the relative difference in the phases of the waves due to the inhomogeneity,

$$\int_0^l dz\, \Delta k(z)$$

over a length l. If the inhomogeneity of the medium is sufficiently large so that (depending on the scale of the inhomogeneity) one of the following conditions is satisfied:

$$\int_0^l dz\, \Delta k(z) \gtrsim 3\pi/2, \quad L/l \gtrsim 1, \tag{44}$$

$$\int_0^l dz\, \Delta k(z) \gtrsim \pi/2, \quad L/l \ll 1, \tag{45}$$

the explosion will not develop and the nonlinear interaction of the waves has an oscillatory nature. Conditions (36) and (37) show that the strongest stabilization of the explosive instability is produced by inhomogeneities of scale l and that inhomogeneities with larger or smaller scales turn out to be less stabilizing factors.

The stabilizing effect of different parameters (inhomogeneities in the density and temperature of the plasma, of external fields, etc.) is in actual cases determined by how these parameters occur in the dispersion laws of the interacting waves. We consider two characteristic examples. In a strongly anisotropic magnetoactive plasma where the average kinetic energy of the ions in directions at right angles to the magnetic field is appreciably larger than the longitudinal kinetic energy a cyclotron instability develops on the branch of oscillations with negative energy with a characteristic linear growth rate $\gamma_{lin} \approx (m/M)^{1/2}\omega_i$ (ω_i is the ion cyclotron frequency).[15] The characteristic growth rate for the nonlinear interaction γ_{nl} will then be of the order of magnitude of

$$\gamma_{nl} \approx (\varepsilon/n_0T_e)^2 (m/M)^{1/2}\omega_i,$$

(ϵ is the initial energy density of the wave, n_0 the plasma density, and T_e the electron temperature) and when $\gamma_{nl} \gg \gamma_{lin}$ the explosive instability can develop.[6]

We estimate under what conditions the inhomogeneity in the magnetic field prevents the development of the explosive instability. We shall for the sake of simplicity assume that the magnetic field changes linearly with a characteristic inhomogeneity length $L \sim (\omega_i^{-1} d\omega_i/dz)^{-1}$. Recalling the dispersion law for the case considered, $\omega(k) \approx p\omega_i \pm (T_e/M)^{1/2}k_{\parallel}$ (p is an integer)[6,15] we get $\Delta k(z)$ and l in the following form:

$$\Delta k(z) \approx p\omega_i (M/T_e)^{1/2} z/L, \tag{46}$$

$$l \approx (T_e/M)^{1/2}\gamma_{nl}^{-1} = (n_0T_e/\varepsilon)^2 (T_e/m)^{1/2}/\omega_i. \tag{47}$$

Substituting (38), (39) into (36) we get finally the following condition for the suppression of the explosive instability due to the inhomogeneity of the magnetic field:

$$\varkappa' \equiv p(n_0T_e/\varepsilon)^4 (M/m)^{1/2} r_e/L \gtrsim 3\pi. \tag{48}$$

Here $r_e \equiv (T_e/m)^{1/2}mc/eH$ is the electron Larmor radius. Using the results of Sec. 3 (Eqs. (21), (23)) we can easily estimate the maximum energy density of the waves ϵ_{max} in the stable case (see (40)):

$$\varepsilon_{max}/\varepsilon \sim (1-(\pi/\varkappa')^{1/2})^{-1}.$$

When a low density ion beam moves through a magnetized plasma an explosive instability may develop when there is an interaction between the low-frequency oscillation branches with a characteristic nonlinear growth rate[9]

$$\gamma_{nl} \sim 10^2 \left(\frac{\omega_{pi}}{ku}\right)^2 \left(\frac{n_0}{n'}\right)^{11/4} \left(\frac{\varepsilon}{n_0T_e}\right)\omega_{pi}.$$

Here ω_{pi} is the ion plasma frequency, u and n′ are the velocity and density of the ion beam. Estimates similar to the ones given above lead to the following criterion for the stabilization of such an instability due to the inhomogeneity in the plasma density or temperature:

$$10^{-4}\left(\frac{ku}{\omega_{pi}}\right)^3 \left(\frac{n'}{n_0}\right)^{5/2} \left(\frac{n_0T_e}{\varepsilon}\right)^2 \frac{u}{\omega_{pi}L} \gtrsim 3\pi.$$

In conclusion we make a few remarks referring to the nonlinear interaction of waves in strongly inhomogeneous media where the phase synchronism conditions are only satisfied in separate narrow regions of space. It was shown in Sec. 3 that in the regions near the phase synchronism points, depending on the phase relations, both an efficient development of the explosive instability (which is intuitively understandable) and a strong stabilization of it with an appreciable simultaneous damping of the intensities of all the waves may occur. Under conditions of favorable phase relations the explosive instability may develop also in a strongly inhomogeneous medium where there are only narrow regions of phase synchronism which are widely separated from one another. The development of the explosive instability then occurs stepwise and the development length of the "explosion" increases due to the inhomogeneity (see Fig. 1).

The authors are grateful to V. A. Ignatchenko for useful discussions.

[1] B. B. Kadomtsev, A. B. Mikhaĭlovskiĭ, and A. V. Timofeev, Zh. Eksp. Teor. Fiz. **47**, 2266 (1964) [Sov. Phys.-JETP **20**, 1517 (1965)].
[2] R. E. Aamodt and M. L. Sloan, Phys. Fluids **11**, 2218 (1968).
[3] K. S. Karplyuk, V. N. Oraevskiĭ, and V. P. Pavlenko, Ukr. Fiz. Zh. **15**, 859 (1970).
[4] C. T. Dum and E. Ott, Plasma Phys. **13**, 177 (1971).
[5] V. N. Oraevskiĭ, V. P. Pavlenko, and P. M. Tomchuk, Fiz. Tekh. Poluprov. **6**, 1647 (1972) [Sov. Phys.-Semicond. **6**, 1425 (1973)].
[6] A. V. Timofeev, ZhETF Pis. Red. **4**, 48 (1966) [JETP Lett. **4**, 32 (1966)].
[7] B. Coppi, M. N. Rosenbluth, and R. N. Sudan, Ann. Phys. (N.Y.) **55**, 207 (1969).
[8] V. N. Tsytovich, Nelineĭnye éffekty v plazme (Nonlinear effects in a plasma) Nauka, 1967, Ch. 5 [English translation published by Plenum Press, New York, 1970].
[9] V. I. Dikasov, L. I. Rudakov, and D. D. Ryutov, Zh. Eksp. Teor. Fiz. **48**, 913 (1965) [Sov. Phys.-JETP **21**, 608 (1965)].
[10] L. Stenflo, J. Weiland, and H. Wilhelmsson, Phys. Scripta **1**, 46 (1970).
[11] M. I. Rabinovich and V. P. Reutov, Izv. Vuzov Radiofizika **16**, 815 (1973) [English translation in Radiophysics and Quantum Electronics, Vol. 16].
[12] T. A. Davydova and V. N. Oraevskiĭ, Zh. Eksp. Teor. Fiz. **66**, 1613 (1974) [Sov. Phys.-JETP **39**, 791 (1974)].
[13] N. N. Filonenko, Abstracts of Contributions to the VIIth All-Soviet Conf. on Coherent and Nonlinear Optices, Moscow State Univ. Press, 1974.
[14] N. N. Filonenko, Abstracts of Contributions, IInd Internat. Conf. on Plasma Theory, Kiev, 1974.
[15] V. I. Pistunovich and A. F. Timofeev, Dokl. Akad. Nauk SSSR **159**, 779 (1964) [Sov. Phys.-Doklady **9**, 1083 (1965)].

Translated by D. ter Haar
130