Transverse relaxation of phonons and kinetic phenomena in helium II at low temperatures

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(Submitted December 18, 1974; revised May 19, 1975)
Zh. Eksp. Teor. Fiz. 69, 1230–1242 (October 1975)

Relaxation phenomena are studied in He II at low temperatures, when they are determined only by phonons, and no rotons are excited. In accordance with the latest experimental data, the dispersion law of the phonons is assumed to be a decaying one but not much different from linear. The principal relaxation process in this case consists of three-phonon collisions which involve phonons moving at small angles with respect to one another. As a result, two relaxation processes are prominent in the phonon gas: 1) fast relaxation during a time $\tau_{\parallel}$ along a given direction, which leads to an equilibrium distribution function with a temperature and a drift velocity $v_n - v_s$ which depend on direction; 2) a slow leveling out of the temperature and drift velocity for the various directions, with a characteristic time $\tau_{\perp} \gg \tau_{\parallel}$. An expression is obtained for the differential transverse phonon-phonon relaxation operator for an arbitrary decaying phonon-dispersion law and also arbitrary value of the external perturbation that causes the disequilibrium of the phonons. The contribution of multiphonon processes to this operator is estimated.

PACS numbers: 67.40.Hf, 67.20.Cp

1. INTRODUCTION AND STATEMENT OF THE PROBLEM

The aim of the present paper is the development of a mathematical technique for describing kinetic phenomena in He II at a temperature below 0.6°K, when they are determined only by phonons. The theory of kinetic phenomena in He II has been developed in detail by Landau and Khalatnikov[1] (see also the book of Khalatnikov[2]). In the consideration of the temperature range below 0.6°K, where only phonons make a contribution to the kinetics, the above-mentioned authors have assumed a non-decaying phonon spectrum. On the other hand, comparatively recently, the results of the measurement of the absorption and the sound-velocity dispersion at temperatures below 0.6°K in the frequency range from 10 to 2000 MHz,[3,4] and also measurements of the dependence of the absorption on the pressure[5], have been found to agree with theory only if we assume that the phonon spectrum undergoes decay at not too high pressures:[6,7]

$$\omega(\mathbf{k}) = ck(1+\gamma k^2), \quad \gamma > 0 \tag{1.1}$$

(ω is the frequency and k the wave vector of the phonon). Investigation of the dependence of the specific heat on the temperature leads to the same condition.[8] Various experiments give a value $\gamma \approx (4-10) \times 10^{-17}$ cm.^{2} The equality

$$\gamma k_T^2 \ll 1 \tag{1.2}$$

holds for such small values of γ: here $k_T = T/\hbar c$ is the wave vector of the thermal phonon (for $T \approx 0.6$°K we have γk_T^2 of the order of 10^{-3}).

This form of the dispersion law is responsible for the peculiarity of the relaxation processes in a gas of thermal phonons. The basic role is played by three-phonon processes, in which, by virtue of the inequality (1.2), phonons which are propagated in only one direction take part. More accurately, the angles between the wave vectors of the phonons that take part in the three-phonon collisions are of the order of $\theta \approx \gamma^{1/2} k_T$. For this reason, we can separate two relaxation processes in the phonon gas in He II. The first is a rapid relaxation of the phonons which propagate along the given direction. It leads to an incomplete thermodynamic equilibrium; that is, it leads to the establishment of a Planck distribution function for the phonons along the given direction:

$$N_1 \equiv n_0\left[\hbar\frac{\omega(\mathbf{k}) - \mathbf{k}\mathbf{V}(\varkappa)}{T(\varkappa)}\right], \quad n_0(z) = (e^z - 1)^{-1}, \tag{1.3}$$

with temperature T and drift velocity1). **V**, which depend on the unit vector κ characterizing the direction.2) The time of establishment of equilibrium along a given direction will be called the longitudinal relaxation time and be denoted by $\tau_{\parallel}$. A rough estimate gives

$$\frac{1}{\tau_{\parallel}} \cong 10^2 \frac{(1+u)^2}{\rho\hbar^4 c^5} T^5,$$

where $u = (\rho/c)(\partial c/\partial\rho) = 2.84$,[4] and ρ is the density of He II at $T = 0$.

The second relaxation process is a slow establishment of equilibrium between different directions and leads to equalization of the temperature and drift velocity at different values of κ. The characteristic time over which the transverse relaxation manages to take place will be designated $\tau_{\perp}$. Our task is to calculate the transverse-relaxation operator. Since each single scattering act is through a small angle, this operator should be of differential form.

What sort of problems can be stated and solved with the help of the transverse operator obtained by us? The simplest problem of such a type is the calculation of the first viscosity coefficient. It reduces to a study of the relaxation of the second spherical harmonic. This problem was solved (by another method than the one developed by us) in a recent paper of Benin.[9] 3) For illustration, we give the solution of this problem by our method.

There is a whole group of problems on the calculation of the spreading out of the shadow created by a screen of some shape in the phonon flux which is emitted by a heated solid immersed in helium. A very interesting paper[10] was recently published in which this effect was observed. There are two limiting cases here, (1) linear, where the temperature of the source exceeds the temperature of He II only slightly, and linearization of the equations of the theory in terms of this difference is possible, and (2) nonlinear, where the temperature of the He II can be assumed to be small in comparison with the high temperature of the source, as in[10].

It is interesting to construct a theory of sound absorption in the frequency range $\tau_{\parallel}^{-1} \gg \omega \gtrsim \tau_{\perp}^{-1}$. Here the equilibrium of the phonons is disrupted in a range of angles of the order of $(\omega\tau_{\perp})^{-1/6}$ and the problem of the calculation of the nonequilibrium distribution function in this interval requires the solution of the differential equation. Here both the linear case (small sound intensity) and the nonlinear case, where the sound intensity "heats" the phonons in a narrow cone of directions. We hope to consider the problem of the acoustical effects in He II at low temperatures in a separate paper.

The last two problems, it seems to us, are practically impossible to solve by the methods of[9], since a great number of spherical harmonics of the nonequilibrium distribution function arise in this case, and to study the phonon relaxation it is necessary to solve the corresponding differential equation directly.

One could name still another series of important problems, including nonlinear ones, for the solution of which one must know the transverse relaxation operator. Keeping this in mind, we have not limited ourselves to the linear approximation in the calculation of this operator in the present paper, but have considered the general case of an arbitrary nonequilibrium condition.

The first calculation of the viscosity coefficient η at $T < 0.6$°K was made in the extremely interesting paper of Maris.[11] In it, the viscosity coefficient was determined by numerical calculation. Its results are in excellent agreement with the experimental data. One of the important qualitative results of Maris' work was the conclusion that $\tau_{\perp}^{-1}$ is proportional not to γ, as was to be expected if the relaxation process represented its usual diffusion in the space of directions κ, but to γ^2. Correspondingly, the transverse relaxation operator that we have calculated is not a diffusion operator, i.e., an operator of second order, but is a differential operator of fourth order. We shall call this the superdiffusion operator.

We shall attempt to explain the physical reason for such a structure of the transverse relaxation operator. The phonon-phonon collision operator (2.2) consists of two components. The first describes the process of attachment of a given phonon to another phonon (and its inverse), and the second, the process of the decay of a given phonon (and its inverse). Each of these components would have led separately to a second-order operator. If we consider the interaction of phonons propagating in a narrow cone of directions, then the first processes lead to a narrowing of this cone, and the second to its broadening. The contributions of both components in first order in the small parameter γk_T^2 cancel each other. Thus, the appearance of the superdiffusion operator is essentially a consequence of the small dispersion in a system of particles whose number is not conserved in the collisions.

It must also be noted that the phonon dispersion law takes the form (1.1) only for sufficiently small k. The characteristic value of k at which significant deviations from the simple formula (1.1) appear depends on the pressure.[5,7] Therefore, in the derivation of the transverse relaxation operator, do not specify the form of the dispersion law, but give it in the form

$$\omega(\mathbf{k}) = ck[1+\xi(k)], \tag{1.4}$$

where $\xi(k) \ll 1$ is an arbitrary function of k.

It must be noted that in constructing the theory we cannot assume the nonlinear increment $\Delta\omega = ck\xi(k)$ to the phonon dispersion law to be arbitrarily small. It should be greater than the indeterminacy of the frequency of the phonon, which is equal to $\tau_{\parallel}^{-1}$:

$$\Delta\omega \gg \tau_{\parallel}^{-1}. \tag{1.5}$$

Such an inequality will occur in any event for sufficiently low temperatures, since the characteristic value $\Delta\omega \sim c\gamma(T/\hbar c)^3$ is proportional to T^3, whereas $\tau_{\parallel}^{-1} \sim T^5$. However, for comparatively high temperatures, this inequality can become reversed. Then, in writing down the phonon-phonon interaction operator, we should, within the limits of accuracy of the calculation, discard the contribution $\Delta\omega$ (in this case the sign of the contribution generally does not play a role) and the processes of three-phonon collisions are considered with account of the indeterminacy of $\tau_{\parallel}^{-1}$ as was done in the well-known work of Simons.[12] The study of phonon relaxation in the Simons case is one of the interesting problems of kinetics. In the present work, however, we shall not touch on it.

2. DERIVATION OF THE OPERATOR OF TRANSVERSE RELAXATION OF PHONONS

We write down the kinetic equation in the form

$$P\{N\} = J\{N\}. \tag{2.1}$$

The left side $P\{N\}$ includes the time derivative of the phonon distribution function and the Poisson brackets. The right side is the three-phonon collision operator and is of the form

$$J\{N\} = \frac{\pi\hbar}{4\rho}\int\frac{d^3k'}{(2\pi)^3}\Big\{b^2(\mathbf{k},\mathbf{k}',\mathbf{k}'')f_+(\mathbf{k},\mathbf{k}',\mathbf{k}'')\delta(\omega''-\omega-\omega') + \frac{1}{2}b^2(\mathbf{k},\mathbf{k}',\mathbf{k}''')f_-(\mathbf{k},\mathbf{k}',\mathbf{k}''')\delta(\omega-\omega'-\omega''')\Big\}. \tag{2.2}$$

Here

$$\mathbf{k}''=\mathbf{k}+\mathbf{k}', \quad \mathbf{k}'''=\mathbf{k}-\mathbf{k}', \quad \omega'=\omega(\mathbf{k}'), \quad \omega''=\omega(\mathbf{k}''), \quad \omega'''=\omega(\mathbf{k}'''), \tag{2.3}$$

$$\begin{gathered} f_+(\mathbf{k},\mathbf{k}',\mathbf{k}'') = N''(N+1)(N'+1)-(N''+1)NN', \\ f_-(\mathbf{k},\mathbf{k}',\mathbf{k}''') = (N+1)N'N'''-N(N'+1)(N'''+1), \end{gathered} \tag{2.4}$$

$$b(\mathbf{k},\mathbf{k}',\mathbf{k}'') = \frac{\mathbf{k}'\mathbf{k}''}{k'k''}k\left(\frac{\omega'\omega''}{\omega}\right)^{1/2} + \frac{\mathbf{k}''\mathbf{k}}{k''k}k'\left(\frac{\omega''\omega}{\omega'}\right)^{1/2} + \frac{\mathbf{k}\mathbf{k}'}{kk'}k''\left(\frac{\omega\omega'}{\omega''}\right)^{1/2} + (2u-1)c^2\frac{kk'k''}{(\omega\omega'\omega'')^{1/2}}. \tag{2.5}$$

For the derivation of the kinetic equations in the considered state of incomplete thermodynamic equilibrium, we should use a procedure of the Chapman-Enskog type. The peculiarity of the given case is that we have here not a single small parameter associated with the relaxation time, but one more small parameter ξ. We represent the operator of the phonon-phonon collisions in the form of a power series in ξ:

$$J\{N\} = J_0\{N\} + J_1\{N\} + J_2\{N\} + \ldots \tag{2.6}$$

In the zeroth approximation, the distribution function satisfies the equation $J_0\{N\} = 0$, which describes the relaxation along the given direction κ. Therefore, its solution is the Planck distribution $N_0 = n_0(\hbar ck/\Theta)$. In this approximation, there is no sense in introducing the two functions V and T, since the function (1.3) reduces to N_0 through the substitution $(1 - \kappa\cdot\mathbf{V}/c)/T = 1/\Theta$.

The first-order correction n_1 to the distribution function is found from the equation

$$P\{N_0\}=J_0'n_1+J_1\{N_0\}, \tag{2.7}$$

where J_0' is the zeroth-approximation operator linearized with respect to the increment n_1.

The operator J_0 conserves the energy along the specified direction (or the momentum, which is the same thing in our approximation). And therefore, the following relation must be satisfied for Eq. (2.7) to have a solution:

$$\int_0^\infty [P\{N_0\}-J_1\{N_0\}]kk^2\,dk=\int_0^\infty J_0'n_1kk^2\,dk=0. \tag{2.8}$$

It turns out that

$$\int_0^\infty J_1\{N_0\}k^3\,dk=0.$$

This statement will be proved below. The solvability condition of (2.8) is not satisfied here, since

$$\int_0^\infty P\{N_0\}k^3\,dk$$

is generally speaking different from zero. The way out of this contradiction is to write P in an equation not of first, but of second approximation. Then n_1 should be determined from the equation

$$J_0'n_1+J_1\{N_0\}=0. \tag{2.9}$$

On the other hand, this equation means that the function $N_1 = N_0 + n_1$ makes the collision operator vanish with accuracy to terms of first order in the dispersion, inclusive. Inasmuch as it does not contain external forces, it is natural to expect its solution to be the Planck function. But it can no longer have the simple form N_0 in first order in ξ. We shall seek it in the form (1.3) and verify that it really causes the collision operator to vanish with accuracy to terms of first order in ξ if T and $v \equiv \boldsymbol{\kappa}\cdot\mathbf{V}/c$ are connected in this case by some additional relation. Thus, in place of the function n_1 we have another function $v(\boldsymbol{\kappa})$.

In order to calculate $J\{N_1\}$ with accuracy to terms of first order in ξ, we must expand the functions f_+ and f_- in powers of the small angle at which the scattering occurs. We represent them in the form

$$f_+=(N_1''+1)NN_1(e^{\hbar A_+}-1),\quad f_-=(N_1+1)N_1'N_1'''(1-e^{\hbar A_-}), \tag{2.10}$$

where

$$A_+=\frac{\omega-ckv}{T}+\frac{\omega'-ck'v'}{T'}-\frac{\omega''-ck''v''}{T''},$$

$$A_-=\frac{\omega'-ck'v'}{T'}+\frac{\omega'''-ck'''v'''}{T'''}-\frac{\omega-ckv}{T}, \tag{2.11}$$

$$T'=T(\varkappa')\ \text{etc.}$$

We introduce the differential rotation operator[4)]

$$l_i=e_{ijm}k_j\frac{\partial}{\partial k_m},$$

where e_{ijm} is a completely antisymmetric tensor.

We expand A_+ and A_- in powers of the angles with accuracy to terms of second order:

$$\begin{aligned}A_+&=\frac{\omega-ckv}{T}+\exp(\theta'\mathbf{nl})\frac{\omega'-ck'v}{T}-\exp(\theta''\mathbf{nl})\frac{\omega''-ck''v}{T}\\&=\left[(\omega'\theta'-\omega''\theta'')\mathbf{nl}+\frac{\omega'\theta'^2+\omega''\theta''^2}{2}(\mathbf{nl})^2\right]\frac{1}{T}\\&\quad-c\left[k+k'-k''+(k'\theta'-k''\theta'')\mathbf{nl}+\frac{k'\theta'^2+k''\theta''^2}{2}(\mathbf{nl})^2\right]\frac{v}{T},\end{aligned} \tag{2.12}$$

$$\begin{aligned}A_-&=\exp(\theta'\mathbf{nl})\frac{\omega'-ck'v}{T}+\exp(-\theta'''\mathbf{nl})\frac{\omega'''-ck'''v}{T}-\frac{\omega-ckv}{T}\\&=\left[(\omega'\theta'-\omega'''\theta''')\mathbf{nl}+\frac{\omega'\theta'^2+\omega'''\theta'''^2}{2}(\mathbf{nl})^2\right]\frac{1}{T}\\&\quad-c\left[k'+k'''-k+(k'\theta'-k'''\theta''')\mathbf{nl}+\frac{k'\theta'^2+k'''\theta'''^2}{2}(\mathbf{nl})^2\right]\frac{v}{T}.\end{aligned} \tag{2.13}$$

Here $\mathbf{n} = \mathbf{k}\times\mathbf{k}'/|\mathbf{k}\times\mathbf{k}'|$; the angle θ''' has a minus sign because the vector $\boldsymbol{\kappa}'''$ turns away from the vector $\boldsymbol{\kappa}$ to a direction opposite to the vector $\boldsymbol{\kappa}'$.

According to the law of sines, we have

$$k'\sin\theta'=k'''\sin\theta''',\quad k''\sin\theta''=k'\sin\theta'.$$

The terms of first order in (2.12) and (2.13) cancel each other. There are then left in (2.10) only terms of first order in A_+ and A_-. The resultant expression must be substituted in (2.2), and integration carried out over the angle φ', which determines the location of the plane $(\boldsymbol{\kappa}, \boldsymbol{\kappa}')$, on which the vector n depends. Here we use the relations $[\mathbf{n}\cdot\boldsymbol{l}, n_j] = 0$ and

$$\frac{1}{2\pi}\int_0^{2\pi} n_in_k\,d\varphi'=\frac{1}{2}\delta_{ik}^{\perp},\quad \delta_{ik}^{\perp}=\delta_{ik}-\varkappa_i\varkappa_k. \tag{2.14}$$

Expanding further in terms of the small angle in the arguments of the δ functions, we find

$$\begin{aligned}\delta(\omega-\omega'-\omega''')&=c^{-1}\vartheta(k-k')\delta[(kk'/k''')\theta'^2/2-\beta(k,k',k''')],\\\delta(\omega+\omega'-\omega'')&=c^{-1}\delta[(kk'/k'')\theta'^2/2-\beta(k'',k,k')],\end{aligned} \tag{2.15}$$

where

$$\begin{aligned}&\vartheta(x)=1\ \text{if}\ x>0,\quad \vartheta(x)=0\ \text{if}\ x<0,\\&\beta(k,k',k''')=k\xi(k)-k'\xi(k')-k'''\xi(k''').\end{aligned} \tag{2.16}$$

We obtain, finally,

$$\begin{aligned}J\{N_1\}=\frac{\pi\hbar^2c(1+u)^2}{2(2\pi)^3\rho}&\left\{\int_0^\infty dk'k'^2k''^2(N_0''+1)N_0N_0'\beta(k'',k,k')\right.\\&\left.-\frac{1}{2}\int_0^k dk'k'^2k'''^2\beta(k,k',k''')(N_0+1)N_0'N_0'''\right\}\left(l^2\frac{1-v}{T}-2\frac{v}{T}\right)+O(\xi^2).\end{aligned} \tag{2.17}$$

It is clear from this expression that, with accuracy to terms of order ξ, the collision operator vanishes as soon as the condition

$$l^2\frac{1-v}{T}-2\frac{v}{T}=0 \tag{2.18}$$

is satisfied. This is also the additional condition which connects the functions v and T.

The second-approximation correction n_2 to the function N is determined from the equation

$$P\{N_0\}=J_0'n_2+J_2\{N_0\}. \tag{2.19}$$

The condition of solvability of this equation is

$$\int_0^\infty P\{N_0\}k^3\,dk=\int_0^\infty J_2\{N_0\}k^3\,dk; \tag{2.20}$$

(2.20) is the second equation which connects the variables v and T. It makes the set of equations of the problem complete, and our purpose is to calculate right side of (2.20)—the three-phonon collision operator.

The integral on the right side of (2.20) is equal, with accuracy to a constant factor $\hbar c/(2\pi)^3$, to the second-order term in ξ in the expansion

$$\left[\frac{\partial\mathscr{E}}{\partial t}\right]_{\text{coll}}=\int_0^\infty \hbar ckJ\{N_1\}\frac{k^2dk}{(2\pi)^3}. \tag{2.21}$$

This quantity constitutes the change in the energy

$$\mathscr{E}(\varkappa)=\int_0^\infty \hbar ckN_1\frac{k^2dk}{(2\pi)^3}, \tag{2.22}$$

of the phonons that propagate in the direction of κ as a consequence of the collisions. If we neglect dispersion, then $\mathscr{E} = (\pi/120)\Theta^4/(\hbar c)^3$.

For scattering at small angles, the contributions to (2.21) from the processes of decay (the second term of (2.2)) and association (the first term of (2.2)) cancel each other in the zeroth approximation. To verify this, we transform it so that the two terms containing $f_+(\mathbf{k}, \mathbf{k}', \mathbf{k}'')$ and $f_-(\mathbf{k}, \mathbf{k}', \mathbf{k}''')$ are reduced to one which contains a θ'-dependent operator that acts only on one of these functions, for example, on $f_-(\mathbf{k}, \mathbf{k}', \mathbf{k}''')$. For this purpose, we note that the vectors $\mathbf{k}, \mathbf{k}', \mathbf{k}'', \mathbf{k}'''$ lie in a single plane, so that the parallelogram of vectors $\mathbf{k}, \mathbf{k}', \mathbf{k}''$ differs from the parallelogram of vectors $\mathbf{k}''', \mathbf{k}', \mathbf{k}$ by a rotation through the angle $-\theta'''$ about an axis parallel to $\mathbf{n}$. We denote this rotation operator by $I(-\theta''')$. We transform in the first term in (2.2) from the integration variables k' and θ' to the variables k'' and $\theta' - \theta''$, and then make the following change of variables

$$\mathbf{k}\to\mathbf{k}''', \quad \mathbf{k}''\to\mathbf{k}, \quad \mathbf{k}'\to\mathbf{k}'.$$

The latter is done in order that the notation for the corresponding vectors be the same in both parallelograms.

We can verify that here

$$f_+(\mathbf{k},\mathbf{k}',\mathbf{k}'')\to I(-\theta''')f_+(\mathbf{k}''',\mathbf{k}',\mathbf{k}) = -I(-\theta''')f_-(\mathbf{k},\mathbf{k}',\mathbf{k}'''), \qquad (2.23)$$

$$b^2(\mathbf{k},\mathbf{k}',\mathbf{k}'')\to I(-\theta''')b^2(\mathbf{k}''',\mathbf{k}',\mathbf{k}) = b^2(\mathbf{k},\mathbf{k}',\mathbf{k}'''). \qquad (2.24)$$

The latter equality follows from the fact that b^2 is a symmetric function of its arguments, and depends only on the moduli of the vectors and their scalar products. As a result, the integrals of the corresponding decay and combination processes can be unified:

$$\left[\frac{\partial\mathscr{E}}{\partial t}\right]_{\text{coll}} = \frac{\pi\hbar^2 c}{4\rho(2\pi)^6}\int_0^{2\pi}d\varphi'\int_0^{\infty}dk\int_0^{\infty}dk'\int_0^{\pi}\sin\theta'\,d\theta'\,k^2k'^2 b^2(\mathbf{k},\mathbf{k}',\mathbf{k}''') \qquad (2.25)$$
$$\times\delta(\omega-\omega'-\omega''')\left[\frac{k}{2}f_-(\mathbf{k},\mathbf{k}',\mathbf{k}''') - k'''I(-\theta''')f_-(\mathbf{k},\mathbf{k}',\mathbf{k}''')\right].$$

Now the second term in the square brackets must be symmetrized relative to k' and k'''. For this purpose, we transform in the integration from the variables k' and θ' to the variables k' and θ''', and then make the transformations $k' \rightleftarrows k'''$, $\theta' \rightleftarrows \theta'''$ and take the half-sum of the resulting and initial expressions. We thus obtain

$$\left[\frac{\partial\mathscr{E}}{\partial t}\right]_{\text{coll}} = \frac{\pi\hbar^2 c}{8(2\pi)^6\rho}\int_0^{2\pi}d\varphi'\int_0^{\infty}dk\int_0^{\infty}dk'\int_0^{\pi}\sin\theta'\,d\theta'\,k^2k'^2$$
$$\times b^2(\mathbf{k},\mathbf{k}',\mathbf{k}''')\delta(\omega-\omega'-\omega''')\{kf_-(\mathbf{k},\mathbf{k}',\mathbf{k}''') \qquad (2.26)$$
$$-k'''I(-\theta''')f_-(\mathbf{k},\mathbf{k}',\mathbf{k}''') - k'I(\theta')f_-(\mathbf{k},\mathbf{k}',\mathbf{k}''')\}.$$

The last term in the curly brackets contains the operator $I(\theta')$ rather than $I(-\theta')$ because, in the derivation of the corresponding expression, the vector $\mathbf{k}'$ was replaced by $\mathbf{k}'''$, and $\mathbf{k}\times\mathbf{k}''' = -\mathbf{k}\times\mathbf{k}'$.

We make use of the expressions

$$I(\theta') = \exp(\theta' L), \quad I(-\theta''') = \exp(-\theta''' L), \qquad (2.27)$$

where

$$L = \mathbf{n}\mathbf{L}, \quad \mathbf{L} = \mathbf{l} + \mathbf{l}' + \mathbf{l}'''.$$

Expanding the rotation operators I in series in the small angles θ' and θ''' with accuracy to terms of second order, we obtain

$$\left[\frac{\partial\mathscr{E}}{\partial t}\right]_{\text{coll}} = \frac{\pi\hbar^2 c}{8(2\pi)^6\rho}\int_0^{2\pi}d\varphi'\int_0^{\infty}dk\int_0^{k}dk'\int_0^{\pi}\sin\theta'\,d\theta'\,k^2k'^2$$
$$\times b^2(\mathbf{k},\mathbf{k}',\mathbf{k}''')\delta(\omega-\omega'-\omega''')\{k-k'-k''' \qquad (2.28)$$
$$-(k'\theta'-k'''\theta''')L - {}^1/_2(k'\theta'^2+k'''\theta'''^2)L^2\}f_-(k,k',k''').$$

The expression in the curly brackets turns out to be of first order in ξ. This indicates a cancellation of the contributions from the decay and combination processes, of which we wrote earlier.

Substituting in (2.28) the expression (2.10) for f_-, calculated with accuracy up to first order in the small quantity A_-, we use formula (2.13) for A_- and expression (2.15) for the δ function.

As a result we obtain the following expression with account of (2.5):

$$\left[\frac{\partial\mathscr{E}}{\partial t}\right]_{\text{coll}} = \frac{4}{\pi}\int_0^{2\pi}d\varphi'(L^2+1)M(\Theta)\left\{(\mathbf{n}\mathbf{l})^2\frac{1}{T} - [(\mathbf{n}\mathbf{l})^2+1]\frac{v}{T}\right\}, \qquad (2.29)$$

where $M(\Theta)$ is the following function of the effective temperature $\Theta(\kappa) = T/(1 - v)$:

$$M(\Theta) = \frac{\hbar^3c^2(1+u)^2}{32(2\pi)^4\rho}\int_0^{\infty}dk\int_0^{k}dk'\beta^2(k,k',k-k') \qquad (2.30)$$
$$\times\vartheta(\beta)k^2k'^2(k-k')^2[n_0(\hbar ck/\Theta)+1]n_0(\hbar ck'/\Theta)n_0[\hbar c(k-k')/\Theta].$$

The expression (2.29) does not contain terms of first order in ξ, thus proving the vanishing of the integral

$$\int_0^{\infty}J_1\{N_0\}k^3\,dk.$$

We can now carry out the integration over ξ. For this purpose, we take out all the vectors $\mathbf{n}$ in sequence, beginning with the right extreme one, to the left of the operators I and $\mathbf{L}$, using the fact that $\mathbf{n}$ commutes not only with $\mathbf{n}\cdot\mathbf{I}$ but also with $\mathbf{L}$. We then replace the operator $\mathbf{L}$ by $\mathbf{l}$ and use the relation

$$\frac{1}{2\pi}\int_0^{2\pi}n_in_jn_kn_m\,d\varphi' = \frac{1}{8}a_{ijkm},$$

$$a_{ijkm} = \delta_{ij}{}^{\perp}\delta_{km}{}^{\perp} + \delta_{ik}{}^{\perp}\delta_{jm}{}^{\perp} + \delta_{im}{}^{\perp}\delta_{jk}{}^{\perp}.$$

This gives

$$\left[\frac{\partial\mathscr{E}}{\partial t}\right]_{\text{coll}} = a_{ijkm}(l_il_j+\delta_{ij})M(\Theta)\left[l_kl_m\frac{1}{T} - (l_kl_m+\delta_{km})\frac{v}{T}\right]. \qquad (2.31)$$

We carry out summation in (2.31), keeping it in mind that $\delta_{ij}^{\perp}$ commutes not with all the operators l_i. Using the relations $[k_i, l_j] = e_{imj}k_m$ and $\mathbf{k}\cdot\mathbf{l} = 0$, we can obtain the identity

$$a_{ijkm}l_il_j = l^2\delta_{km}{}^{\perp} + (\delta_{ik}\delta_{jm}+\delta_{im}\delta_{kj})l_il_j$$
$$+ l_j(e_{jnk}\varkappa_n\varkappa_m + e_{jnm}\varkappa_k\varkappa_n) + 2\varkappa_k\varkappa_m, \qquad (2.32)$$

and with its help and with allowance for (2.18) we obtain the transverse relaxation operator in a form that contains a single unknown function Θ:

$$\left[\frac{\partial\mathscr{E}}{\partial t}\right]_{\text{coll}} = 2l_il_jMl_il_j\frac{1}{\Theta} - l^2Ml^2\frac{1}{\Theta}. \qquad (2.33)$$

In accord with what was said in Sec. 1, this expression is proportional to ξ^2.

Let us see into what the expression (2.33) is transformed if the dispersion law has the very simple form (1.1). In this case, the function $\beta(k, k', k''')$ takes the form

$$\beta(k, k'\,k''') = 3\gamma kk'(k-k').$$

This allows us to introduce the dimensionless variables $\hbar ck/\Theta$ and $\hbar ck'/\Theta$ in the integrals over k and k' in (2.30) and by the same token to separate the dependence on Θ in explicit form. As a result we get

$$\left[\frac{\partial\mathscr{E}}{\partial t}\right]_{\text{coll}}=\frac{9\hbar F(1+u)^2\gamma^2}{32(2\pi)^4\rho(\hbar c)^{12}}\left(2l_il_j\Theta^{14}l_il_j\frac{1}{\Theta}-l^2\Theta^{14}l^2\frac{1}{\Theta}\right), \quad (2.34)$$

where the dimensionless constant F is equal to

$$F=\int_0^\infty dz\int_0^z dz'z^4z'^4(z-z')^4[n_0(z)+1]n_0(z')n_0(z-z'). \quad (2.35)$$

Its numerical value amounts to 1.0×10^7.

We call attention to the fact that the expression (2.34) is homogeneous in the effective temperature Θ.

Thus the kinetic problems reduce to the calculation of a single function of the angles, which satisfies the equation

$$\int_0^\infty \hbar ckP\{N_0\}\frac{k^2dk}{(2\pi)^3}=\left[\frac{\partial\mathscr{E}}{\partial t}\right]_{\text{coll}}. \quad (2.36)$$

3. PROPERTIES OF THE TRANSVERSE RELAXATION OPERATOR. LAW OF ENERGY CONSERVATION. THE H THEOREM

We shall verify that the collision operator vanishes when T = const and $v = \varkappa\cdot V/c$, where V is a constant vector (this corresponds to the equilibrium distribution). Here the relation (2.18) is satisfied identically, as can be verified with the help of the identity $(l^2+2)\varkappa = 0$. We can demonstrate that (2.33) vanishes by means of the identities

$$l_i\varkappa_i=[l_i,\varkappa_i]=0, \quad (l_il_j+\delta_{ij})\varkappa_n=\delta_{ni}\varkappa_j. \quad (3.1)$$

The energy conservation law is a trivial consequence of the identity

$$\int l_i\psi(\varkappa)d\Omega=0, \quad (3.2)$$

where ψ is an arbitrary function of the direction $\varkappa$ and the integration is carried out over the entire solid angle.

We now calculate the entropy production as a consequence of the collisions, by means of the formula

$$\left[\frac{\partial S}{\partial t}\right]_{\text{coll}}=\int J\{N\}\ln\frac{1+N}{N}\frac{d^3k}{(2\pi)^3}=\int\left[\frac{\partial\mathscr{E}}{\partial t}\right]_{\text{coll}}\frac{d\Omega}{\Theta}. \quad (3.3)$$

Substituting (2.33) here and twice integrating by parts with account of (3.2), we obtain

$$\left[\frac{\partial S}{\partial t}\right]_{\text{coll}}=\int d\Omega M\cdot\left[2\left(l_jl_i\frac{1}{\Theta}\right)\left(l_il_j\frac{1}{\Theta}\right)-\left(l^2\frac{1}{\Theta}\right)^2\right]. \quad (3.4)$$

This quantity can be rewritten in the form of the following integral, the positive definiteness of which is obvious:

$$\left[\frac{\partial S}{\partial t}\right]_{\text{coll}}=2\int\left[\left(l_il_j-\frac{1}{2}\delta_{ij}^{\perp}l^2\right)\frac{1}{\Theta}\right]\left[\left(l_jl_i-\frac{1}{2}\delta_{ij}^{\perp}l^2\right)\frac{1}{\Theta}\right]Md\Omega. \quad (3.5)$$

4. LINEAR APPROXIMATION

In the linear approximation, we assume that

$$\Theta=T(1+Z), \quad (4.1)$$

where T is the temperature of He II at equilibrium and $|Z| \ll 1$. In the expression (2.33) it is not necessary in this case to differentiate M, and the expression takes the form

$$\left[\frac{\partial\mathscr{E}}{\partial t}\right]_{\text{coll}}=-\frac{\mathscr{E}}{\tau_\perp}l^2(l^2+2)Z, \quad (4.2)$$

where $\mathscr{E}(x)$ is the energy density (2.22) and

$$\frac{1}{\tau_\perp}=\frac{M}{\mathscr{E}T}. \quad (4.3)$$

In this case, when the phonon dispersion law has the simplest form (1.1), Eq. (4.3) transforms into

$$\frac{1}{\tau_\perp}=\frac{135F(1+u)^2\gamma^2}{2(2\pi)^5\rho\hbar^8c^9}T^9. \quad (4.4)$$

It must be noted that the quantity Z does not describe the changes of the average temperature of the phonon system and of the drift velocity; therefore it should satisfy the conditions

$$\int Zd\Omega=0, \quad \int Z\varkappa d\Omega=0. \quad (4.5)$$

The expression for $[\partial S/\partial t]_{\text{coll}}$ takes the form

$$\left[\frac{\partial S}{\partial t}\right]_{\text{coll}}=\frac{2\mathscr{E}}{T\tau_\perp}\int\left[\left(l_il_j-\frac{1}{2}\delta_{ij}^{\perp}\right)Z\right]\left[\left(l_jl_i-\frac{1}{2}\delta_{ij}^{\perp}\right)Z\right]d\Omega \quad (4.6)$$

in linear approximation. This quantity, with accuracy to a constant factor, coincides with the variational functional of the work of Benin,[9] which thus represents the production of entropy as a consequence of the phonon-phonon collisions.

To illustrate the method, we calculate the first-viscosity coefficient η of He II. In the calculation of η, we must assume that $v_S = 0$ and (see the book of Khalatnikov,[2] p. 134)

$$P\{N_0\}=-\frac{1}{2}V_{ij}\left(\varkappa_i\varkappa_j-\frac{1}{3}\delta_{ij}\right)\frac{\hbar ck}{T}n_0'\left(\frac{\hbar ck}{T}\right), \quad (4.7)$$

where

$$V_{ij}=\frac{\partial v_{ni}}{\partial x_j}+\frac{\partial v_{nj}}{\partial x_i}-\frac{2}{3}\delta_{ij}\operatorname{div}\mathbf{v}_n,$$

and n_0' is the derivative of the Planck function with respect to the argument.

Equation (2.37) takes in this case the form

$$2\left(\varkappa_i\varkappa_j-\frac{1}{3}\delta_{ij}\right)V_{ij}=-\frac{1}{\tau_\perp}l^2(l^2+2)Z. \quad (4.8)$$

The solution of this equation, with account of the conditions (4.5), is

$$Z=-{}^1/_{12}V_{ij}(\varkappa_i\varkappa_j-{}^1/_3\delta_{ij})\tau_\perp. \quad (4.9)$$

The dissipative part of the momentum flux density tensor in our case is given by

$$\Pi_{ij}=\frac{\pi}{30}\frac{T^4}{(\hbar c)^3}\int\varkappa_i\varkappa_jZd\Omega=-\eta V_{ij}, \quad (4.10)$$

whence

$$\eta=\frac{\pi^2}{675}\frac{T^4}{(\hbar c)^3}\tau_\perp. \quad (4.11)$$

This expression is identical with that obtained in[9].5)

5. THE ROLE OF MULTIPHONON PROCESSES

The temperature dependence of the viscosity coefficient (4.11) in the case $\xi(k) = \gamma k^2$ is identical with the dependence of the coefficient η calculated by Landau and Khalatnikov[1,2] and is due to the contribution of four-phonon processes of the type $2 \rightleftarrows 2$ (two phonons, colliding, are transformed into two other phonons). According to[1,2], we have

$$\eta_{\text{LKh}}=\frac{2^{13}(2\pi)^9}{4050\cdot 13!}\left(\frac{\hbar c}{T}\right)^5\frac{\rho^2c^2}{(1+u)^4\hbar}. \quad (5.1)$$

The ratio of this quantity to the quantity (4.11) is 0.12 for $\gamma = 4 \times 10^{-17}$ cm². If we are given a larger value of γ (as follows from[3-5,8,11]), then this ratio falls off like γ^{-2}. Thus, in spite of the same temperature dependence, it seems that three-phonon processes are noticeably more effective than the four-phonon processes.

Let explain qualitatively how it is that four-phonon processes, which occur more rarely than three-phonon ones, give the same temperature dependence of the viscosity. For this purpose, we must estimate the contributions from multiphonon processes to the collision operator. Such an estimate has been made by us with the use of the Keldysh diagram technique. In the present work we cannot set forth the details of the corresponding calculations, and limit ourselves to the results.

We shall distinguish between processes by the number of real phonons participating in them. Analysis of the conservation laws for the processes $2 \to 2$, $2 \rightleftarrows 3$, and so on, shows that the phonons that take part in them can propagate at large angles with respect to one another. It turns out that for large-angle scattering processes, the contribution from each higher order is less than the contribution of the preceding one in, roughly speaking, the ratio Γ/ω, where Γ is the attenuation coefficient of the phonon, to which the principal contribution is made by the three-phonon interaction, such that[6] $\Gamma \sim \tau_{\parallel}^{-1}$, and ω is the characteristic frequency of the phonon, usually of order $T/\hbar$. The quantity $\tau_{\parallel}^{-1}$ for thermal phonons is proportional to T^5; therefore, $1/\omega\tau_{\parallel} \sim (T/T_0)^4$, where $T_0 \sim (\rho\hbar^3 c^5)^{1/4}$, i.e., of the order of several degrees.

On the other hand, the three-phonon collisions take place at small angles. As a result, as we have seen, $\tau_{\perp}^{-1}$ is proportional to $(\gamma k^2)^2 \sim T^4$. Thus the relative rarity of four-phonon processes $2 \to 2$ is compensated by the fact that they take place at large angles, so that the temperature dependence of the two contributions to the viscosity turns out to be the same. If the character of the disturbance is such that it covers a small range of angle $\Delta\theta$ in k space,[7] then the role of the three-phonon processes turns out to be more effective, not numerically but literally—in the ratio $1/(\Delta\theta)^4$.

In the case of a decay spectrum, three-phonon processes $3 \rightleftarrows 1$ are also allowed. The conservation laws allow the participation in such processes only of phonons whose wave vectors are almost collinear. The corresponding amplitude of the transition at small angles has a singularity.[1] Similar singularities arise at amplitudes of the multiphonon processes calculated in the higher approximations of perturbation theory. These singularities lead to the appearance of quadrically diverging integrals. The summation of these divergences in the Keldysh technique leads to the replacement of the Green's function of the zeroth approximation by Green's functions with an self-energy part due to three-phonon processes.

The remaining integrals converge. The most important contribution is made by processes for which the wave vectors of the participating phonons are almost collinear. The transition probabilities in this range of angles in each successive order acquire the parameter[8] $1/\tau_{\parallel}\Delta\omega$ ($\Delta\omega = ck\xi$). (The transition probabilities describing the scattering at large angles contains the smaller parameter $1/\omega\tau_{\parallel}$).

We estimate the contribution to the phonon relaxation at large angles of processes of the type $3 \rightleftarrows 1$. The relaxation at large angles due to such processes is also described by an operator of the superdiffusion type, and the corresponding reciprocal relaxation time is described by multiplication of the probability of the process by ξ^2. Therefore, the ratio of the inverse relaxation times due to the processes $2 \rightleftarrows 1$ and $3 \rightleftarrows 1$ turns out to be of the order of $1/\tau_{\parallel}\Delta\omega \ll 1$ (for the simplest dispersion law, $1/\tau_{\parallel}\omega \sim T^2/T_0^2$).

Thus the contribution of four-phonon processes turns out to be numerically small, in comparison with the three-phonon processes (for the processes $2 \to 2$) or parametrically small (for the processes $3 \rightleftarrows 1$).

We now discuss the contribution of five-phonon processes. Just as in the four-phonon case, they are of two types. (1) The processes $1 \rightleftarrows 4$, which are allowed only in a narrow range of angles between the participating phonons. The contribution of these processes is small in comparison with the processes $1 \rightleftarrows 3$, in accord with the parameter $1/\tau_{\parallel}\Delta\omega$. (2) The processes $2 \rightleftarrows 3$; these are allowed over a large range of angles. Their contribution is small in comparison with the contribution of processes $2 \to 2$ in the ratio $1/\omega\tau_{\parallel}$.

The smallness of the contributions of processes of still higher order can be traced out in similar fashion.

In conclusion, we express our deep gratitude to I. M. Khalatnikov, who made a number of extremely valuable critical comments. We are very grateful to A. F. Andreev for a very interesting discussion.

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[1] The distribution function (1.3) is given in a set of coordinates moving with the superfluid velocity v_s, while the drift velocity V in (1.3) is none other than the difference $v_n - v_s$, where v_n is the normal velocity.

[2] Landau and Khalatnikov [1] first used the pressure-dependent temperature for the description of phonon relaxation processes in He II. They also first studied the transverse relation of phonons in He II due to four-phonon processes, and calculated the viscosity coefficient resulting from them. A comparison of the relative roles of three- and four-phonon processes in kinetic phenomena in He II at low temperatures will be given in Sec. 5 of this paper.

[3] We are very grateful to Benin for sending a preprint of his paper.

[4] It differs from the quantum mechanical angular-momentum operator by the quantity $i/\hbar$, so that $[l_i, l_h] = -e_{ikj}l_j$, $[l_i, k_m] = -e_{imj}k_j$.

[5] The time $\tau_\perp$ that we introduced is connected with the relaxation time of the second spherical harmonic, introduced in [9], by the relation $\tau_\perp = \tau_2/6$.

[6] In subsequent discussions, the attenuation coefficient of the phonons due to three-phonon processes is used as a graphic measure of the nonlinear multiphonon interactions in He II. Actually, this means that for estimates we can express the multiphonon amplitude in order of magnitude in terms of the three-phonon interactions.

[7] For example, in the case of high-frequency sound.

[8] The existence of such a parameter was first noted by Zakharov [14] in an analysis of acoustic turbulence on the basis of classical theory.

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Translated by R. T. Beyer
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