# Spin waves in quantum crystals

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Quantum theory methods are used to obtain a set of disperison equations for the spin wave spectrum in a magnetically ordered quantum crystal. The dynamical magnetic susceptibility is evaluated. It is assumed that the crystal has gapless Fermi excitations.

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## 1. INTRODUCTION

In preceding papers by Dzyaloshinskiĭ, Levchenkov, and the present author[1,2] we considered the spectrum of spinless boson excitations in a Fermi-type quantum crystal, assuming that the ground state was non-magnetic.

Recently experimental data have been published[3] indicating that at a temperature of ~1 mK solid $^3$He changes into a magnetic phase. In this connection a theoretical study of the properties of magnetically ordered quantum crystals has become timely. The present paper is devoted to a consideration of the spin wave spectrum of a magnetic quantum crystal. We shall be interested in the long-wavelength and low-frequency parts of the spectrum. Since we shall show below that in that case its structure can be expressed in terms of the general characteristics of the system, we shall not make any definite assumptions about the concrete form of the magnetic ordering. We shall merely assume that it is the result of an exchange interaction.

When there are no interactions leading to the nonconservation of the total spin, the existence of low-frequency spin wave branches follows from Goldstone's theorem[4] on systems with broken symmetry. This conclusion remains valid also in the case when there are acting in the crystal also forces on a magnetic nature, provided they are weak compared to the exchange forces. The inclusion of magnetic interactions leads to the formation of gaps in the spin wave Goldstone mode spectrum.

As in[1,2], we shall assume that in the crystal considered there exist gapless Fermi excitations. In that situation there can, apart from the Goldstone modes, exist in the crystal also zero-sound-type spin waves. Of course, interactions lead to a mixing up of the two kinds of spin waves. We shall assume that an external magnetic field acts upon the system, which is such that the energy of the interaction with it $\epsilon_H$ is appreciably smaller than the characteristic magnitude of the exchange energy $\epsilon_0$ ($\epsilon_H \ll \epsilon_0$). At the same time, while assuming that the condition that the characteristic magnetic anisotropy energy $\epsilon_a$ be small ($\epsilon_a \ll \epsilon_0$) is satisfied, we shall not introduce any restrictions on the ratio of the quantities $\epsilon_H$ and $\epsilon_a$.

In the next section we shall briefly consider the properties of the one-particle Green function of a magnetic crystal. In the third section we find a set of dispersion equations for the spin wave spectrum. The spectrum determined by it consists of a group of Goldstone modes and connected with their interaction a family of zero-sound type spin waves. The maximum number of Goldstone modes equals three in an antiferromagnetic and two in a ferromagnetic. For frequencies large compared to the characteristic frequencies of the magnetic interactions ($\omega \gg \epsilon_H, \epsilon_a$) all three branches in an antiferromagnetic have a linear spectrum, while in a ferromagnetic the longitudinal mode has a linear and the transverse mode a quadratic spectrum. In the fourth section we calculate the dynamic susceptibility of a magnetically ordered crystal.

In the Appendix we prove a relation, used in the text, for the variation of the thermodynamic potential of the crystal with respect of its angular orientation in the spin subspace.

The results will be obtained by quantum field theoretical methods for the zero-temperature case. To fix the ideas we shall assume that the crystal consists of spin-½ particles.

## 2. GREEN FUNCTION

One of the basic quantities characterizing the properties of the crystal is the single-particle Green function[5]

$$G_{\alpha\beta}(x, x') = -i\langle T(\hat{\psi}_\alpha(x)\hat{\psi}_\beta^+(x'))\rangle, \qquad (2.1)$$

$\psi_\alpha(x)$ and $\hat{\psi}_\beta^+(x')$ are Heisenberg particle annihilation and creation operators, T is the time-ordering operator, $\langle \ldots \rangle$ indicates averaging over the ground state at T = 0, and $x = \{\mathbf{r}, t\}$ is the space-time coordinate.

Similarly to the case of a nonmagnetic crystal[2] the function $G_{\alpha\beta}(x, x')$ can be expanded in terms of a complete orthonormal set of functions $\psi_{np}(x, \alpha) = \varphi_{np}^\alpha(\mathbf{r}) e^{ipx}$:

$$G_{\alpha\beta}(x, x') = \int \frac{d^4p}{(2\pi)^4} G_{nm}(p)\psi_{np}^*(x, \alpha)\psi_{mp}(x', \beta). \qquad (2.2)$$

Here and henceforth we use the convention of summation over repeated indices, $p = \{\mathbf{p}, \epsilon\}$ is the quasimomentum four-vector, $px = \mathbf{p}\cdot\mathbf{r} - \epsilon t$, $d^4p = d\mathbf{p}\, d\epsilon$, the integration over $\epsilon$ is to infinite limits and that over the quasimomentum is restricted to the first Brillouin zone. The functions $\varphi_{np}^\alpha(\mathbf{r})$ are periodic relative to the crystal lattice. We shall assume that they are chosen such that the matrix $G_{nm}(p)$ is diagonal at $\epsilon = 0$.

When there are gapless Fermi excitations present, as we assume, in the limit as $\epsilon \to 0$ and for values of the quasimomentum $\mathbf{p}$ lying in the vicinity of the Fermi surface the matrix $\hat{G}(p)$ has a form with a pole:

$$\hat{G}(p) = \hat{a}(\mathbf{p})[\varepsilon - \hat{\varepsilon}(\mathbf{p}) + i\delta\,\mathrm{sign}\,\varepsilon]^{-1} + \hat{\tilde{G}}(p), \qquad (2.3)$$

$\delta \to +0$. Here $\hat{\epsilon}(\mathbf{p})$ is the energy of the Fermi quasiparticles, reckoned from the chemical potential, $\hat{\tilde{G}}(p)$ is a function which is finite as $\epsilon \to 0$. The renormalization constant $\hat{a}(\mathbf{p})$ as a matrix in the $\psi_{np}(x, \alpha)$ function space is nonvanishing only for those values of the index n which correspond to bands with a Fermi surface.

We shall assume in what follows for the sake of simplicity that there are only two such bands, which change into one another under time reversal. When there is no magnetic ordering and there are no magnetic forces, such a pair of bands corresponds to two possible quasiparticle spin projections on a chosen direction and is degenerate. In a magnetic crystal the degeneracy is in general lifted. In the case when the corresponding energy splitting is small compared with the width of the whole band it is expedient to consider them together. The values of the matrix elements $\hat{a}(\mathbf{p})$ for the two bands will then be assumed to be the same within the limits of accuracy of Eq. (2.3).

## 3. SPIN WAVE SPECTRUM

The spin wave spectrum, like the other branches of Bose excitations of Fermi systems, is determined by the poles of the two-particle vertex part $\hat{\Gamma}(p_1, p_2; k)$ with respect to the quasimomentum transfer $k = \{\mathbf{k}, \omega\}$. The function $(\hat{\Gamma}(p_1, p_2; k))_{nmlq}$ is the result of expanding the vertex part $\Gamma_{\alpha\beta\gamma\delta}(x_1, x_2; x_3, x_4)$, defined in the usual way[5], in terms of the set of functions $\psi_{np}(x, \alpha)$. The variables $p_1 + k$, $p_2$; $p_1$, $p_2 + k$ of the function $\hat{\Gamma}(p_1, p_2; k)$ correspond to the variables $x_1, x_2; x_3, x_4$ of the function $\Gamma_{\alpha\beta\gamma\delta}(x_1, x_2; x_3, x_4)$, respectively. In the case $\mathbf{k} \to 0$ in which we are interested there are no Umklapp processes and the law of conservation of quasimomentum has the same form as for momentum in free space. For the sake of simplicity we shall omit in what follows the band indices and the signs of summation over them.

The existence of low-frequency poles of $\hat{\Gamma}$ caused by magnetic ordering follows from the following simple consideration: We perform an infinitesimal rotation of the system of reference through an angle $\delta\varphi$ in the spin subspace. The particle annihilation and creation operators then transform according to the formulae

$$\hat{\psi}_\alpha'(x) = \hat{\psi}_\alpha(x) + i\delta\varphi s_{\alpha\gamma}\hat{\psi}_\gamma(x),$$
$$\hat{\psi}_\beta'^{+}(x) = \hat{\psi}_\beta^{+}(x) - i\hat{\psi}_\delta^{+}(x) s_{\delta\beta}\delta\varphi, \tag{3.1}$$

$\mathbf{s}_{\alpha\gamma}$ is a spin matrix. This transformation entails a change in the Green function $G_{\alpha\beta}(x, x')$ and in the irreducible self-energy part $\Sigma_{\alpha\beta}(x, x')$. Since the function $\hat{\Sigma}$ can be obtained by using block diagrams, we can regard it as a functional of the exact Green function and the bare interaction in the system.[5] Therefore, any infinitesimal change in $\hat{\Sigma}$ can be written in the form

$$\delta\Sigma_{\alpha\beta}(x, x') = \int dx_1\, dx_1' \frac{\delta\Sigma_{\alpha\beta}(x, x')}{\delta G_{\gamma\delta}(x_1, x_1')} \delta G_{\gamma\delta}(x_1, x_1') + (\delta\Sigma_{\alpha\beta}(x, x'))_G. \tag{3.2}$$

The second term indicates here the variation of $\hat{\Sigma}$ when the quantity $\hat{G}$ is fixed.

Since the functional derivative of $\hat{\Sigma}$ with respect to $\hat{G}$ is determined by the irreducible vertex part $\hat{\Gamma}^{(1)}$[5], it follows that Eq. (3.2) becomes, after changing to the representation with the $\psi_{np}(x, \alpha)$ as base

$$\delta\hat{\Sigma}(p, k) = -i \int \frac{d^4p'}{(2\pi)^4} \hat{\Gamma}^{(1)}(p, p'; k) \delta\hat{G}(p', k) + (\delta\hat{\Sigma}(p, k))_G. \tag{3.3}$$

The variables $p + k$ and $p$ of the functions $\delta\hat{G}(p, k)$ and $\delta\hat{\Sigma}(p, k)$ correspond to the arguments $x$ and $x'$ of the function $\delta G_{\alpha\beta}(x, x')$ and $\delta\Sigma_{\alpha\beta}(x, x')$. The function $\hat{\Gamma}^{(1)}$ is connected with the total vertex part through the relation[5]

$$\hat{\Gamma}(p, p'; k) = \hat{\Gamma}^{(1)}(p, p'; k) - i \int \frac{d^4p_1}{(2\pi)^4} \hat{\Gamma}^{(1)}(p, p_1; k) \hat{G}(p_1 + k) \hat{G}(p_1) \hat{\Gamma}(p_1, p'; k). \tag{3.4}$$

We assume now that there is no external magnetic field acting on the crystal and that in the crystal itself there are no magnetic interactions. In that case its Hamiltonian is invariant under uniform rotations in the spin subspace and the quantity $(\delta\hat{\Sigma}(p, 0)/\delta\varphi_i)_G$ vanishes. Using this fact and also using the connection between $\delta\hat{\Sigma}$ and $\delta\hat{G}$ which follows from the Dyson equation[5]

$$\hat{G}^{-1} = \hat{G}_0^{-1} - \hat{\Sigma}, \tag{3.5}$$

and the fact that the Green function of the non-interacting system $\hat{G}_0$ remains unchanged under a rotation over a constant angle we are, on the basis of Eq. (3.3), led to the conclusion that when $\omega$, $\mathbf{k} = 0$ the equation corresponding to (3.4), which is homogeneous in $\hat{\Gamma}$, has a nontrivial solution proportional to $\delta\hat{G}^{-1}(p, 0)/\delta\varphi_i$. This fact indicates that as $\omega$, $\mathbf{k} \to 0$ the function $\hat{\Gamma}(p, p'; k)$ has a pole with residue $\sim \delta\hat{G}^{-1}(p, 0)/\delta\varphi_i$. Indeed, in view of the vector nature of the residue the function $\hat{\Gamma}(p, p'; k)$ has, in general, three poles.

We note that the singularities we have found are not of the zero-sound type, since the latter are absent in the limiting cases $\omega = 0$, $\mathbf{k} \to 0$ and $\mathbf{k} = 0$, $\omega \to 0$. When there is no magnetic ordering $G_{\alpha\beta}(x, x') \propto \delta_{\alpha\beta}$ and hence $\delta\hat{G}^{-1}(p, 0)/\delta\varphi_i = 0$ and the above conclusion about the poles of $\hat{\Gamma}$ loses its validity. The singularities we found for $\hat{\Gamma}$ are thus the result of magnetic ordering when the interaction is of an exchange nature.

It is perfectly obvious that the switching on of a magnetic field and of magnetic interactions does not change the conclusion about the low-frequency poles of $\hat{\Gamma}$, provided that $\epsilon_H$, $\epsilon_a \ll \epsilon_0$. We write the function $\hat{\Gamma}$ in the following form:

$$\hat{\Gamma}(p, p'; k) = \hat{\tilde{\Gamma}}(p, p'; k) + \hat{g}_i(p, k) D_{ij}(k) \hat{g}_j^{(+)}(p, k). \tag{3.6}$$

The function $D_{ij}(k)$ contains explicitly the singularity of the vertex $\hat{\Gamma}$ caused by the magnetic ordering. The quantities $\hat{\tilde{\Gamma}}$, $\hat{g}_i$, and $\hat{g}_j^{(+)}$ do not have such a singularity. The quantity $D_{ij}(k)$ can be called the spin wave Green function and the functions $\hat{g}_i(p, k)$ and $\hat{g}_j^{(+)}(p', k)$ correspond to the amplitudes for the emission and absorption of spin waves by the Fermi quasi-particles.

Turning to the definition of the functions $D_{ij}(k)$, $\hat{g}_i(p, k)$, $\hat{g}_j^{(+)}(p', k)$ we perform an infinitesimal rotation of the system of reference in the spin subspace through an angle that depends on the coordinates and the time as $\delta\varphi\, e^{ikx}$ ($kx = \mathbf{kr} - \omega t$). The corresponding changes in the functions $\hat{G}$ and $\hat{\Sigma}$ are connected by Eqs. (3.3) and (3.5). Eliminating from them the quantity $\delta\hat{\Sigma}(p, k)$, we get for $\delta\hat{G}^{-1}(p, k)/\delta\varphi_i$ the equation

$$\frac{\delta\hat{G}^{-1}(p, k)}{\delta\varphi_i} = \hat{\mathscr{E}}_i(p, k) - i \int \frac{d^4p'}{(2\pi)^4} \hat{\Gamma}^{(1)}(p, p'; k) \hat{G}(p' + k) \frac{\delta\hat{G}^{-1}(p', k)}{\delta\varphi_i} \hat{G}(p'). \tag{3.7}$$

The function $\mathscr{E}_i(p, k)$ is defined by the equation

$$\hat{\mathscr{E}}_i(p, k) = \frac{\delta\hat{G}_0^{-1}(p, k)}{\delta\varphi_i} - \left(\frac{\delta\hat{\Sigma}(p, k)}{\delta\varphi_i}\right)_G. \tag{3.8}$$

We eliminate the function $\hat{\Gamma}^{(1)}$ from (3.7) by using Eq. (3.4) and for the function $\hat{\Gamma}$ we substitute its expression (3.6):

$$\frac{\delta G^{-1}(p, k)}{\delta\varphi_i} = \hat{\mathscr{E}}_i(p, k) - i \int \frac{d^4p'}{(2\pi)^4} \hat{\tilde{\Gamma}}(p, p'; k) \hat{G}(p' + k) \hat{\mathscr{E}}_i(p', k) \hat{G}(p')$$
$$- i\hat{g}_l(p, k) D_{lm}(k) \operatorname{Sp} \int \frac{d^4p'}{(2\pi)^4} \hat{g}_m^{(+)}(p', k) \hat{G}(p' + k) \hat{\mathscr{E}}_i(p', k) \hat{G}(p'). \tag{3.9}$$

The trace symbol here and henceforth indicates a trace over the band indexes which number the basis functions $\psi_{np}(x, \alpha)$.

In the sense of Eq. (3.6), the function $\hat{g}_i$, $\hat{g}_j^{(2)}$, and $D_{ij}$ are determined, accurate to the transformation

$$\hat{g}_i(p,k)\to\hat{g}_l(p,k)f_{li}(k),\qquad \hat{g}_j^{(+)}(p',k)\to f_{jm}(-k)\hat{g}_m^{(+)}(p'\ k),$$
$$D_{ij}(k)\to f_{il}^{-1}(k)D_{lm}(k)f_{mj}^{-1}(-k),$$

in which $f_{ij}(k)$ is arbitrary. We fix its choice by the condition

$$-iD_{lm}(k)\,\mathrm{Sp}\int\frac{d^4p'}{(2\pi)^4}\hat{g}_m^{(+)}(p',k)\hat{G}(p'+k)\hat{\mathscr{E}}_i(p',k)\hat{G}(p')=\delta_{li}. \tag{3.10}$$

Substituting this equation into (3.9) we get the following expression for $\hat{g}_i(p, k)$:

$$\hat{g}_i(p,k)=\frac{\delta\hat{G}^{-1}(p,k)}{\delta\varphi_i}-\hat{\mathscr{E}}_i(p,k)$$
$$+i\int\frac{d^4p'}{(2\pi)^4}\hat{\Gamma}(p,p';k)\hat{G}(p'+k)\hat{\mathscr{E}}_i(p',k)\hat{G}(p'). \tag{3.11}$$

By virtue of the obvious symmetry of the function $\hat{\Gamma}$ in the coordinate representation

$$\Gamma_{\alpha\beta\gamma\delta}(x_1,\ x_2;\ x_3,\ x_4)=\Gamma_{\beta\alpha\delta\gamma}(x_2,\ x_1;\ x_4,\ x_3)$$

the functions $\hat{g}_i$ and $\hat{g}_j^{(+)}$ are connected through the relation

$$\hat{g}_i^{(+)}(p,k)=\hat{g}_i(p+k,-k). \tag{3.12}$$

Using this and Eqs. (3.11) and (3.8) we get from (3.10)

$$D_{ij}^{-1}(k)=-i\,\mathrm{Sp}\int\frac{d^4p}{(2\pi)^4}\frac{\delta\hat{G}(p+k,-k)}{\delta\varphi_i}\frac{\delta\hat{G}_0^{-1}(p,k)}{\delta\varphi_j}$$
$$+i\,\mathrm{Sp}\int\frac{d^4p}{(2\pi)^4}\frac{\delta\hat{G}(p+k,-k)}{\delta\varphi_i}\left(\frac{\delta\hat{\Sigma}(p,k)}{\delta\varphi_j}\right)_G+i\,\mathrm{Sp}\int\frac{d^4p}{(2\pi)^4}\hat{G}(p)\hat{\mathscr{E}}_i(p+k,-k)$$
$$\times G(p+k)\left[\mathscr{E}_j(p,k)-i\int\frac{d^4p'}{(2\pi)^4}\hat{\Gamma}(p,p';k)\hat{G}(p'+k)\hat{\mathscr{E}}_j(p',k)\hat{G}(p')\right]. \tag{3.13}$$

We now transform the right-hand side of the obtained equation. We explain below that in the limit $\omega\neq 0$, $k\to 0$ the function $D_{ij}^{-1}(k)$ can be expressed in terms of the equilibrium characteristics of the crystal. As to that part of it which vanishes with $\mathbf{k}$, we can establish for it the analytical structure and give an expression in terms of phenomenological quantities, Using the relations

$$\frac{\delta\hat{G}(p,k)}{\delta\varphi_i}=i[\hat{s}_i(\mathbf{p},\mathbf{k})\hat{G}(p)-\hat{G}(p+k)\hat{s}_i(\mathbf{p},\mathbf{k})], \tag{3.14}$$

$$\frac{\delta\hat{G}_0^{-1}(p,k)}{\delta\varphi_j}=-i\omega\hat{s}_j(\mathbf{p},\mathbf{k})+o(\mathbf{k}); \tag{3.15}$$

$$(\hat{s}_i(\mathbf{p},\mathbf{k}))_{nm}=\int_{v_c}d\mathbf{r}\,\varphi^{\alpha*}_{n\mathbf{p}+\mathbf{k}}(\mathbf{r})s_{\alpha\beta}{}^i\varphi_{m\mathbf{p}}{}^\beta(\mathbf{r})$$

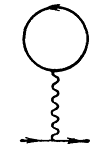
($v_c$ is the volume of the elementary lattice cell), which follow from Eq. (1) and from the expression for the Green function $\hat{G}_0$ of the non-interacting system when there is no magnetic field (the interaction in it is assumed to be completely taken into account by the self-energy part $\hat{\Sigma}$), the first term on the right-hand side of (3.13) (we denote it by $Y_{ij}^{(1)}(k)$) is transformed to the form

$$Y_{ij}^{(1)}(k)=ie_{ijl}\omega s_l+o(\mathbf{k}^2), \tag{3.16}$$

where $e_{ijl}$ is the completely antisymmetric unit tensor. Here

$$s_i=-i\lim_{\delta\to+0}\mathrm{Sp}\int\frac{d^4p}{(2\pi)^4}e^{i\epsilon\delta}\hat{s}_i(\mathbf{p},0)\hat{G}(p) \tag{3.17}$$

is the macroscopically averaged magnitude of the spin



moment per unit volume of the crystal. We shall assume that if the crystal is ferromagnetic its sample takes the form of the ellipsoid so that the macroscopic density distribution of the spontaneous spin moment over it is uniform.

When evaluating the second term on the right-hand side of Eq. (3.13), we must bear in mind that the exchange contribution corresponding to it is proportional to $\mathbf{k}^2$ by virtue of which we can, within the required accuracy, put $\omega = 0$. As far as the contribution from the magnetic forces is concerned, owing to the long-range nature of the magnetic dipole interaction, it has a non-analytical structure in $\mathbf{k}$.

The only block diagram for $\hat{\Sigma}$ in which the long-range magnetic forces are important is shown in the figure. The solid line depicts the G-function and the wavy line the potential of the magnetic dipole interaction

$$V_{\alpha\beta\gamma\delta}(x-x_1)=(2\mu_0)^2s_{\alpha\gamma}{}^i\frac{\partial^2}{\partial r_i\,\partial r_{1j}}\frac{\delta(t-t_1)}{|\mathbf{r}-\mathbf{r}_1|}s_{\beta\delta}{}^j,$$

$\mu_0$ is the magnetic moment of an isolated atom of the crystal. Using the analytical expression for the above-mentioned diagram we can write the function $\Sigma_{\alpha\beta}(x, x')$ in the form

$$\Sigma_{\alpha\beta}(x,x')=-2\mu_0\mathbf{s}_{\alpha\beta}\mathbf{H}(\mathbf{r})\delta(x-x')+\Sigma_{\alpha\beta}'(x,x'), \tag{3.18}$$

$$H_i(\mathbf{r})=H_i^e-2\mu_0\frac{\partial}{\partial r_i}\int d^4x_1\frac{\partial}{\partial r_{1j}}\frac{\delta(t-t_1)}{|\mathbf{r}-\mathbf{r}_1|}(-is_{\gamma\delta}{}^j\langle G_{\delta\gamma}(x_1,x_1+0)\rangle). \tag{3.19}$$

Here $H_i(\mathbf{r})$ is the strength of the macroscopically averaged effective magnetic field, $H_i^e$ the external field strength; $\langle G_{\delta\gamma}(x_1, x_1 + 0)\rangle$ is the macroscopically averaged expression for the function $G_{\delta\gamma}(x_1, x_2)$ for coincident values of the arguments, $\mathbf{r}_2 = \mathbf{r}_1$, $t_2 = t_1 + 0$. The quantity $\Sigma'_{\alpha\beta}(x, x')$ is determined by the set of block diagrams for $\Sigma$ in which the long-wavelength part of the magnetic interaction is unimportant. In particular, together with all other diagrams for $\Sigma$, $\hat{\Sigma}'$ includes the short-wavelength contribution of the diagram, a contribution obtained by replacing the function $G_{\delta\gamma}(x_1, x_1 + 0)$ by the quantity $G_{\delta\gamma}(x_1, x_1 + 0) - \langle G_{\delta\gamma}(x_1, x_1 + 0)\rangle$.

Substituting Eqs. (3.18) and (3.19) into the second term on the right-hand side of (3.13) (we denote it by $Y_{ij}^{(2)}(k)$) and using (3.14) and the cyclic property of the trace we get

$$Y_{ij}^{(2)}(k)=i\frac{\delta}{\delta\varphi^s}\left[\mathrm{Sp}\int\frac{d^4p}{(2\pi)^4}\Sigma_m(p)\frac{\delta\hat{G}(p)}{\delta\varphi_i}\right]_H$$
$$+16\pi\mu_0^2e_{ilm}s_me_{jl'm'}s_{m'}\frac{k_lk_{l'}}{\mathbf{k}^2}+o(\mathbf{k}^2). \tag{3.20}$$

Here $\hat{\Sigma}_m(p)$ is the contribution to the function $\hat{\Sigma}(p)$ caused by the magnetic forces; $\delta[\dots]_{\mathbf{H}}/\delta\varphi_j^s$ denotes the derivative for fixed vector $\mathbf{H}$, determined by Eq. (3.19), taking into account only the dependence on the angle $\varphi$ which follows through the function $\hat{G}$,

$$\delta\hat{G}(p)/\delta\varphi_i\equiv\delta\hat{G}(p,0)/\delta\varphi_i.$$

Substituting in (3.20) Eq. (A.1) which is proved in the Appendix

$$\frac{\delta\Omega}{\delta\varphi_i}=-i\,\mathrm{Sp}\int\frac{d^4p}{(2\pi)^4}\hat{\Sigma}_m(p)\frac{\delta\hat{G}(p)}{\delta\varphi_i} \tag{3.21}$$

($\Omega$ is the thermodynamic potential per unit volume of the system), we get

$$Y_{ij}^{(2)}(k)=\left(\frac{\delta^2\Omega}{\delta\varphi_j^s\,\delta\varphi_i^s}\right)_H+16\pi\mu_0^2 e_{ilm}s_m e_{jl'm'}s_{m'}\frac{k_l k_{l'}}{\mathbf{k}^2}+o(\mathbf{k}^2). \quad (3.22)$$

In deriving this formula we neglected the effects of the retardation of the magnetic interaction. We thereby neglected the coupling between the spin and the electromagnetic waves.

We note that the effective field $\mathbf{H}$, defined by Eq. (3.19) is the same as the magnetic field strength in macroscopic electrodynamics.[6]

In contrast to $Y_{ij}^{(1)}(k)$ and $Y_{ij}^{(2)}(k)$, the third term on the right-hand side of (3.13) contains poles caused by the phonon excitations and Fermi-liquid singularities. One can take the phonon poles into account using a consideration similar to one used earlier.[2] It is well known[7] that the coupling between spin waves and elastic waves is determined by the ratio of the magnetic dipole interaction energy ($\sim\epsilon_a$) to the elastic energy ($\epsilon_{el}$). In the case of solid $^3$He the ratio $\epsilon_a/\epsilon_{el}\sim 10^{-8}$ and the magnon-phonon interaction is anomalously small, so that we shall neglect it. Because of this we replace the quantity $\hat{\Gamma}$ which occurs in the right-hand side of (3.13) by the function $\bar{\Gamma}$ in which apart from the poles contained in the D-function also the phonon poles have been eliminated.

The quantity $\hat{\bar{\Gamma}}(p, p'; k)$ as function of the quasimomentum transfer k contains only the Fermi-liquid singularities. It is well known[8] that they arise thanks to the fact that the poles of the single-particle Green functions with arguments p and p + k approach each other. We can separate them off by analogy with the case of a nonmagnetic crystal by formally splitting the product $\hat{G}(p+k)\hat{G}(p)$ into singular and regular terms which has the form, using Eq. (2.3)

$$\hat{G}(p+k)\hat{G}(p)=2\pi i\hat{a}^2(\mathbf{p})\delta(\varepsilon)\delta(\hat{\varepsilon}(\mathbf{p}))\hat{\Phi}(\mathbf{p},k)+\overline{\hat{G}(p+k)\hat{G}(p)}. \quad (3.23)$$

As $k\to 0$ the term $\overline{\hat{G}(p+k)\hat{G}(p)}$ remains regular. The function $\hat{\Phi}(p, k)$ has the form

$$(\hat{\Phi}(\mathbf{p},k))_{nmn'm'}=\frac{\varepsilon_n(\mathbf{p})-\varepsilon_m(\mathbf{p})+\mathbf{v}_n\mathbf{k}}{\omega-\varepsilon_n(\mathbf{p})+\varepsilon_m(\mathbf{p})-\mathbf{v}_n\mathbf{k}}\delta_{nn'}\delta_{mm'}. \quad (3.24)$$

Here $\mathbf{v}=\partial\hat{\varepsilon}(\mathbf{p})/\partial\mathbf{p}$ is the velocity of the quasiparticles on the Fermi surface. Because of the meaning of the Fermi-liquid singularities Eq. (3.24) for the off-diagonal matrix elements of the function $\hat{\Phi}(\mathbf{p}, k)$ is valid only when the quasiparticle energy splitting is small compared to the band width. In that case we must assume the velocity $\mathbf{v}(\mathbf{p})$ to be the same for both bands and neglect the energy splitting in the argument of the $\delta$-function in Eq. (3.23). If, however, the splitting is not small we must put the off-diagonal matrix elements equal to zero.

For the further development it is convenient to introduce the function $\mathscr{T}$:

$$\hat{\bar{\mathscr{T}}}(\mathbf{p}_1,\mathbf{p}_2;k)=\hat{a}(\mathbf{p}_1)\hat{\bar{\Gamma}}(p_1,p_2;k)\hat{a}(\mathbf{p}_2)|_{\varepsilon=\varepsilon'=0}. \quad (3.25)$$

Proceeding in the same way as in the corresponding derivation in[2] and using Eqs. (3.4) and (3.6) and the foregoing definition of the function $\hat{\bar{\Gamma}}$, we get an equation for $\hat{\bar{\mathscr{T}}}(p_1, p_2; k)$:

$$\hat{\bar{\mathscr{T}}}(\mathbf{p}_1,\mathbf{p}_2;k)=\hat{f}(\mathbf{p}_1,\mathbf{p}_2)+\int dS'\hat{f}(\mathbf{p}_1,\mathbf{p}')\hat{\Phi}(\mathbf{p}',k)\hat{\bar{\mathscr{T}}}(\mathbf{p}',\mathbf{p}_2;k). \quad (3.26)$$

We have introduced here the notation

$$\int dS=\int\frac{d\mathbf{p}}{(2\pi)^3}\delta(\varepsilon(\mathbf{p})).$$

The quantity $\hat{f}(\mathbf{p},\mathbf{p}')$ has the meaning of the Landau function which describes the interaction between the quasiparticles. It follows from Eq. (3.26), if we use (3.24) that the function $f(\mathbf{p},\mathbf{p}')$ is connected with the limiting value $\hat{\bar{\mathscr{T}}}^k(\mathbf{p},\mathbf{p}')$ of the function $\hat{\bar{\mathscr{T}}}(\mathbf{p},\mathbf{p}';k)$ for $\omega=0$, $\mathbf{k}\to 0$ through the equation

$$\hat{f}(\mathbf{p},\mathbf{p}')=\hat{\bar{\mathscr{T}}}^k(\mathbf{p},\mathbf{p}')+\int dS_1\hat{f}(\mathbf{p},\mathbf{p}_1)\hat{\bar{\mathscr{T}}}^k(\mathbf{p}_1,\mathbf{p}').$$

Turning to the evaluation of the third term on the right-hand side in Eq. (3.13) (we denote it by $Y_{ij}^{(3)}(k)$) we note that according to the definition (3.8) the diagrams for $\mathscr{E}_i(p, k)$ do not contain intersections such as $\hat{G}(p+k)\hat{G}(p)$ so that this quantity has no Fermi-liquid singularities. Up to terms caused by the magnetic anisotropy we can write the function $\mathscr{E}_i(p, k)$, using (3.8), (3.15), and (3.18) in the form

$$\hat{\mathscr{E}}_i(p,k)=-\hat{s}_l(p)F_{li}(\omega)-\hat{b}_{il}(p)k_l,$$
$$F_{ij}(\omega)=\omega\delta_{ij}-ie_{ijl}2\mu_0H_l,\quad \hat{s}_l(\mathbf{p})=\hat{s}_l(\mathbf{p},0). \quad (3.27)$$

We further introduce the following quantities:

$$\hat{x}_i(\mathbf{p},\mathbf{k})=-\hat{a}(\mathbf{p})\left[\hat{s}_i(p)-i\int\frac{d^4p'}{(2\pi)^4}\hat{\Gamma}(p,p',k)\hat{G}(p'+k)\hat{s}_i(p')\hat{G}(p')\right]\Bigg|, \quad (3.28)$$

$$\hat{c}_{il}(\mathbf{p},k)=\hat{a}(\mathbf{p})\left[\hat{b}_{il}(p)-i\int\frac{d^4p'}{(2\pi)^4}\hat{\Gamma}(p,p';k)\hat{G}(p'+k)\hat{b}_{il}(p')\hat{G}(p')\right]\Bigg|_{\varepsilon=0} \quad (3.29)$$

$$I_{ij}(k)=i\,\mathrm{Sp}\int\frac{d^4p}{(2\pi)^4}\hat{G}(p)\hat{s}_i(\mathbf{p})\hat{G}(p+k)$$
$$\times\left[\hat{s}_j(\mathbf{p})-i\int\frac{d^4p'}{(2\pi)^4}\hat{\Gamma}(p,p';k)\hat{G}(p'+k)\hat{s}_j(\mathbf{p}')\hat{G}(p')\right]. \quad (3.30)$$

Using (3.23), (3.25), and (3.26) and the substitution $\hat{\Gamma}\to\hat{\bar{\Gamma}}$ we then get the following equation for the third term on the right-hand side of Eq. (3.13):

$$Y_{ij}^{(3)}(k)=F_{il}(\omega)I_{lm}F_{mj}(\omega)-\mathscr{P}_{ij}(k)+o(\mathbf{k}^2), \quad (3.31)$$

$$\mathscr{P}_{ij}(k)=-\mathrm{Sp}\int dS(F_{il}(\omega)\hat{x}_l(\mathbf{p})-\hat{c}_{il}(\mathbf{p})k_l)$$
$$\times\hat{\Phi}(\mathbf{p},k)(\hat{x}_m(\mathbf{p},k)F_{mj}(\omega)-\hat{c}_{jm}(\mathbf{p},k)k_m). \quad (3.32)$$

As in Eqs. (3.16) and (3.22) the term $o(\mathbf{k}^2)$ is analytical in nature. The functions $\hat{x}_i(\mathbf{p})$, $c_{il}(\mathbf{p})$, and the tensor $I_{ij}$ are connected with the quantities $\hat{x}_i(\mathbf{p}, k)$, $\hat{c}_{il}(\mathbf{p}, k)$, and $I_{ij}(k)$ through the equations

$$\hat{x}_i(\mathbf{p},k)=\hat{x}_i(\mathbf{p})+\int dS'\hat{f}(\mathbf{p},\mathbf{p}')\hat{\Phi}(\mathbf{p}',k)\hat{x}_i(\mathbf{p},k), \quad (3.33)$$

$$\hat{c}_{il}(\mathbf{p},k)=\hat{c}_{il}(\mathbf{p})+\int dS'\hat{f}(\mathbf{p},\mathbf{p}')\hat{\Phi}(\mathbf{p}',k)c_{il}(\mathbf{p}',k), \quad (3.34)$$

$$I_{ij}(k)=I_{ij}-\mathrm{Sp}\int dS\hat{x}_i(\mathbf{p})\hat{\Phi}(\mathbf{p},k)\hat{x}_j(\mathbf{p},k). \quad (3.35)$$

Indeed, it is convenient to assume that the functions $\hat{x}_i(\mathbf{p})$, $\hat{c}_{il}(\mathbf{p})$, and the tensor $I_{ij}$ are given phenomenologically. Equations (3.33) to (3.35) will then serve to define the quantities $\hat{x}_i(\mathbf{p}, k)$, $\hat{c}_{il}(\mathbf{p}, k)$, and $I_{ij}(k)$.

Substituting now Eqs. (3.16), (3.22), (3.31), and (3.32) into Eq. (3.13) we get the final expression for the spin wave Green function:

$$D_{ij}^{-1}(k)=\bar{D}_{ij}^{-1}(k)-\mathscr{P}_{ij}(k), \quad (3.36)$$

$$\bar{D}_{ij}^{-1}(k)=i\omega e_{ijl}s_l+F_{il}(\omega)I_{lm}F_{mj}(\omega)$$
$$-\beta_{ij}(\mathbf{H})+16\pi\mu_0^2 e_{ilm}s_m e_{jl'm'}s_{m'}k_lk_{l'}/\mathbf{k}^2-\gamma_{lm}^{ij}k_lk_m. \quad (3.37)$$

The quantity $\beta_{ij}(\mathbf{H})$ is defined by the equation

$$\beta_{ij}(\mathbf{H})=\left(\frac{\delta^2\Omega}{\delta\varphi_i^s\delta\varphi_j^s}\right)_H. \quad (3.38)$$

Since at equilibrium

$$\delta\Omega/\delta\varphi_j=0,$$

the tensor $\beta_{ij}(H)$ is symmetric, $\beta_{ij} = \beta_{ji}$.

The last term in Eq. (3.37) is the sum of the analytical contributions $o(k^2)$ in the expressions for $Y_{ij}^{(1)}(k)$, $Y_{ij}^{(2)}(k)$, and $Y_{ij}^{(3)}(k)$. The tensor $\gamma_{lm}^{ij}$ has the symmetry property $\gamma_{lm}^{ij} = \gamma_{ml}^{ji}$.

The spin wave spectrum is determined by the condition that the set of homogeneous equations

$$D_{ij}^{-1}(k)\,\delta\varphi_j=0 \quad (3.39)$$

has a non-trivial solution. As the function $\mathscr{P}_{ij}(k)$ in (3.36) according to (3.32) to (3.34) has poles of a Fermi-liquid origin, the set (3.39) determines the spectrum of a family of two interacting groups of spin excitations. On the one hand, this includes the group of Goldstone modes defined by the set

$$\bar{D}_{ij}^{-1}(k)\,\delta\varphi_j=0, \quad (3.40)$$

which would occur in a magnetically ordered crystal when there are no Fermi excitations. On the other hand, this includes the spin waves which would occur in the subsystem of Fermi quasiparticles when there is no magnetic ordering. By virtue of the linear connection between $\mathscr{P}_{ij}(k)$ and $\hat{\bar{\Gamma}}(p, p'; k)$ which follows from (3.37) and (3.18) the poles of $\mathscr{P}_{ij}(k)$ and, hence, the spin wave spectrum in the subsystem of the Fermi quasiparticles are determined by Eq. (3.26).

The maximum number of Goldstone modes described by the set (3.40) is equal to three. These modes correspond to the usual spin waves of a solid state kind. The properties of their spectrum in the region of frequencies large compared to the frequencies determined by the energies of the interaction with the magnetic field and of the magnetic anisotropy ($\omega \gg \epsilon_H, \epsilon_a$; $\epsilon_H = 2\mu_0 H$) depend on whether the crystal possesses a spontaneous spin moment $s^0$. In an antiferromagnetic ($s^0 = 0$) for frequencies $\omega \gg \epsilon_H$, $(\epsilon_a\epsilon_0)^{1/2}$ ($\epsilon_0$ is characteristic for the magnitude of the exchange energy) all three branches have a linear spectrum. If, however, the crystal is a strong ferromagnetic ($s^0 \propto N$, N is the number of atoms per unit volume) the presence of a term in (3.37) which is linear in the frequency and proportional to $s^0$ leads to a lowering of the number of low-frequency branches to two, one of which (transverse with respect to the direction of $\mathbf{s}^0$) has a quadratic spectrum for $\omega \gg \epsilon_H, \epsilon_a$, while the second (longitudinal) is linear when $\omega \gg \epsilon_H$, $(\epsilon_a\epsilon_0)^{1/2}$. The lowering of the number of branches is connected with the fact that when $s^0 \neq 0$ the frequency of one of them becomes of the order of the exchange energy and because of this such a branch, if it exists at all, goes beyond the framework of our considerations. The necessary condition for the existence of a longitudinal branch in a ferromagnetic is that the longitudinal component of the tensor $I_{ij}$, which we shall show in the next section to be connected with the longitudinal susceptibility, is non-vanishing. As $\mathbf{k} \to 0$ the modes described by the set (3.40) have a gap determined by the anisotropy and the magnetic field.

In the case of a strong splitting of the bands the off-diagonal elements of the quantity $\hat{\Phi}(\mathbf{p}, \mathbf{k})$ given in (3.24) are not present and Eq. (3.26) for the function $\hat{\mathscr{T}}$ takes the form which is standard for the Fermi-liquid theory and, hence, describes zero-sound type spin waves. The same situation also arises for $\omega \gg \Delta\epsilon$ when the band splitting $\Delta\epsilon$ is small. The presence of off-diagonal components of $\hat{\Phi}$ when $\Delta\epsilon \neq 0$ in the case $\mathbf{k} \to 0$ leads to the formation of a gap in the sound-like mode spectrum.

Moreover, when $|\mathbf{v}||\mathbf{k}| < \Delta\epsilon$ additional vibrational branches may exist with frequencies which decrease with increasing $|\mathbf{k}|$. Their spectrum cannot be extended into the region $|\mathbf{v}||\mathbf{k}| > \Delta\epsilon$ because of strong Landau damping.

When we include the interaction between the groups of Goldstone and Fermi-liquid vibrational modes which are present in the set (3.39) their above formulated asymptotic properties remain unchanged in the large and small frequency ranges. Their nature in the intermediate frequency range is to a large extent determined by the principle of non-intersection of terms of the same symmetry.[9]

In concluding this section we note that by virtue of the fact that when evaluating the function $\hat{\mathscr{E}}_i(p, k)$ we neglected the contribution of the magnetic anisotropy and in the case of the quantity $Y_{ij}^{(2)}$ we neglected the dependence of that contribution on the frequency and on the magnetic field, the spectrum of the longitudinal Goldstone mode in a ferromagnetic and all three modes in an antiferromagnetic can be evaluated with a relative accuracy $\propto(\epsilon_a/\epsilon_0)^{1/2}$ while the spectrum of the transverse mode in a ferromagnetic is calculated with accuracy up to quantities $\propto \epsilon_a/\epsilon_0$.

## 4. DYNAMICAL SUSCEPTIBILITY

We evaluate the dynamical magnetic susceptibility $\chi_{ij}(k)$ of a crystal describing the linear response of the macroscopically averaged magnetic moment density to an external field of strength $h(x) = he^{ikx}$ ($kx = \mathbf{k}\cdot\mathbf{r} - \omega t$).

The Hamiltonian of the interaction of the crystal with the magnetic field $\mathbf{h}(x)$ has the form

$$\hat{H}_{int}=-\int d\mathbf{r}\hat{M}_i(x)h_i(x), \quad (4.1)$$

$$\hat{M}_i(x)=-2\mu_0\hat{\psi}_\alpha^+(x)s_{\alpha\beta}^i\hat{\psi}_\beta(x), \quad (4.2)$$

$\hat{M}_i(x)$ is the magnetic moment density operator. In accordance with the general rules of the diagram technique, using (4.1) and (4.2) the quantity $\chi_{ij}(k)$ has the form

$$\chi_{ij}(k)=i(2\mu_0)^2\Big\{\mathrm{Sp}\int\frac{d^4p}{(2\pi)^4}\hat{G}(p)\hat{s}_i(\mathbf{p}+\mathbf{k},-\mathbf{k})\hat{G}(p+k)$$
$$\times\Big[\hat{s}_j(\mathbf{p},\mathbf{k})-i\int\frac{d^4p'}{(2\pi)^4}\hat{\Gamma}(p,p';k)\hat{G}(p'+k)\hat{s}_j(\mathbf{p}',\mathbf{k})\hat{G}(p')\Big]\Big\}. \quad (4.3)$$

In the long-wavelength and low-frequency approximation in which we are interested the quantities $\hat{s}_i(\mathbf{p}+\mathbf{k}, -\mathbf{k})$ and $\hat{s}_i(\mathbf{p}, \mathbf{k})$ must be replaced by $\hat{s}_i(\mathbf{p}) \equiv \hat{s}_i(\mathbf{p}, 0)$, and for the function $\hat{\Gamma}(p, p'; k)$ we must substitute its expression (3.6). Neglecting as in the derivation of the spectrum the magnon-phonon interaction we replace the function $\hat{\Gamma}$ by $\hat{\bar{\Gamma}}$ and use Eq. (3.30). As a result we obtain

$$\chi_{ij}(k)=(2\mu_0)^2\{I_{ij}(k)-A_{il}(k)D_{lm}(k)A_{mj}^{(+)}(k)\}, \quad (4.4)$$

$$A_{ij}(k)=-i\,\mathrm{Sp}\int\frac{d^4p}{(2\pi)^4}\hat{G}(p)\hat{s}_i(p)\hat{G}(p+k)\hat{g}_j(p,k),$$
$$A_{ij}^{(+)}(k)=A_{ji}(-k). \quad (4.5)$$

Substituting then (3.11) and (3.27) to (3.30) we will get

$$A_{ij}(k)=e_{ijl}s_l-iI_{il}(k)F_{lj}(\omega)-i\,\mathrm{Sp}\int dS\hat{x}_i(\mathbf{p})\hat{\Phi}(\mathbf{p},k)c_{il}(\mathbf{p},k)k_l. \quad (4.6)$$

According to Eqs. (4.4) to (4.6) the static susceptibility of an anti-ferromagnet in zero field (s = 0, H = 0) is equal to

$$\chi_{ij}{}^k=\chi_{ij}(\omega=0,\ \mathbf{H}=0,\ \mathbf{k}\to 0)=(2\mu_0)^2 I_{ij}{}^k,$$
$$I_{ij}{}^k=I_{ij}(\omega=0,\ \mathbf{k}\to 0). \quad (4.7)$$

The same derivation also refers to the longitudinal susceptibility of a ferromagnet. According to (3.35) and (3.24) the tensor $I_{ij}^k$ is given by the equation

$$I_{ij}{}^k=I_{ij}+\mathrm{Sp}\int dS\,\hat{x}_i(\mathbf{p})\hat{x}_j{}^k(\mathbf{p}),$$
$$\hat{x}_j{}^k(\mathbf{p})=\lim_{\omega=0,\ \mathbf{k}\to 0}\hat{x}_j(\mathbf{p},k). \quad (4.8)$$

It follows from (3.33) and (3.24) that the function $\hat{x}_i^k(p)$ satisfies the equation

$$\hat{x}_i{}^k(\mathbf{p})=\hat{x}_i(\mathbf{p})-\int dS'\hat{f}(\mathbf{p},\mathbf{p}')\hat{x}_i{}^k(\mathbf{p}'). \quad (4.9)$$

We note that as the response of the quantity $\delta\hat{G}^{-1}(p, k)$ to the magnetic field is connected linearly with the vertex part $\hat{\Gamma}(p, p'; k)$, the most singular part of $\delta\hat{G}^{-1}(p, k)$ is by virtue of Eqs. (3.6), (3.11), and (4.5) in the limit $k\to 0$ equal to

$$\frac{\delta\hat{G}^{-1}(p,k)}{\delta\varphi_i}D_{ij}(k)A_{jl}^{(+)}(k)h_l.$$

This means that the product $D_{ij}(k)A_{jl}^{(+)}(k)h_l$ gives the response of the angular orientation of the system in the spin subspace $\delta\varphi_i$ to the magnetic field $he^{ikx}$. Hence it also follows that the spin waves whose spectrum is determined by the poles of the function $D_{ij}(k)$ correspond to oscillations of the crystal orientation in the spin subspace. However, the quantity $(2\mu_0)^2 I_{ij}(k)$ is according to (4.4) the response of the magnetic moment density to the external magnetic field for a fixed orientation $\varphi$:

$$I_{ij}(k)=(2\mu_0)^{-2}\left(\frac{\partial M_i(k)}{\partial h_j}\right)_\varphi. \quad (4.10)$$

One can show similarly that the quantity $\hat{x}_i^k(p)$ can be expressed in terms of the response of the quasiparticle energy $\hat{\epsilon}(p)$ to the static magnetic field for a fixed orientation $\varphi$:

$$\hat{x}_i{}^k(\mathbf{p})=(2\mu_0)^{-1}\left(\frac{\partial\hat{\varepsilon}(\mathbf{p})}{\partial H_i}\right)_\varphi. \quad (4.11)$$

In conclusion we note that as there are at the present moment no data on the properties of solid $^3$He in its magnetic phase it is premature to talk about a comparison of the results with experimental data.

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## APPENDIX

We prove here the relation (3.21) which we used in the text:

$$\frac{\delta\Omega}{\delta\varphi_i}=-i\,\mathrm{Sp}\int\frac{d^4p}{(2\pi)^4}\hat{\Sigma}_m(p)\frac{\delta\hat{G}(p)}{\delta\varphi_i}. \quad (A.1)$$

To do this we use the formula

$$\delta\Omega_m/\delta\hat{G}'(p)=-i\hat{\hat{\Sigma}}_m(p). \quad (A.2)$$

Here $\Omega_m$ is the contribution to the thermodynamic potential per unit volume of the system caused by the magnetic interactions, $\hat{\hat{\Sigma}}_m(p)$ is the contribution of the magnetic forces to the reducible self-energy part, and $\hat{G}'(p)$ is the Green function calculated neglecting the magnetic forces. The quantities $\hat{\hat{\Sigma}}_m(p)$ and $\hat{G}'(p)$ are connected with $\hat{\Sigma}_m(p)$ and $\hat{G}(p)$ through the following equations which follow from their definition:

$$\hat{\hat{\Sigma}}_m(p)=\hat{\Sigma}_m(p)+\hat{\Sigma}_m(p)\hat{G}(p)\hat{\Sigma}_m(p), \quad (A.3)$$
$$\hat{G}(p)=\hat{G}'(p)+\hat{G}'(p)\hat{\Sigma}_m(p)\hat{G}(p). \quad (A.4)$$

Equation (A.2) is a consequence of the diagram expansion of the thermodynamic potential.[5] Indeed, if we take as the unperturbed system the crystal considered without the magnetic forces, the magnetic contribution $\Omega_m$ to the thermodynamic potential of the crystal is determined by all connected diagrams containing magnetic interactions. Each diagram will then correspond to a factor 1/n which in an essential way depends on its order (n is the order of the diagram with respect to the magnetic interaction Hamiltonian). The situation is here analogous to the one met with in the paper by Dzyaloshinskiĭ and Pitaevskiĭ[10] (see also[5]) when they calculated the contribution from the long-wavelength fluctuations of the electromagnetic field to the free energy of a dielectric. As for each n-th order connected diagram there are n topologically equivalent ways to split off the G′ line when varying the quantity $\Omega_m$ with respect to the function $\hat{G}'(p)$ the factor 1/n cancels. The sequence of diagrams which then occurs corresponds to all possible diagrams with one entrance and one exit for the G′-lines without any restrictions as to their reducibility with respect to the internal lines. But such a sequence just corresponds to the function $\hat{\hat{\Sigma}}_m(p)$. The general factor −i is established by considering the lowest approximation diagrams.

It follows from Eq. (A.2) that the change in the thermodynamic potential under an infinitesimal uniform rotation of the spins of the system is given by the formula

$$\frac{\delta\Omega}{\delta\varphi_i}=-i\,\mathrm{Sp}\int\frac{d^4p}{(2\pi)^4}\hat{\hat{\Sigma}}_m(p)\frac{\delta\hat{G}'(p)}{\delta\varphi_i}. \quad (A.5)$$

We used here the fact that thanks to the invariance of the exchange forces under uniform rotations in the spin subspace

$$\frac{\delta\Omega}{\delta\varphi_i}=\frac{\delta\Omega_m}{\delta\varphi_i}.$$

Substituting now $\hat{\hat{\Sigma}}_m(p)$ from (A.3) into (A.5) and also the expression for $\delta\hat{G}'(p)/\delta\varphi_i$ which follows from (3.14) we can use the cyclic properties of the trace, Eq. (A.4), and once again Eq. (3.14). As a result we are led to Eq. (A.1) which we had to prove.

[1] I. E. Dzyaloshinskiĭ, P. S. Kondratenko, and V. S. Levchenkov, Zh. Eksp. Teor. Fiz. 62, 1574 (1972) [Sov. Phys.-JETP 35, 823 (1962)].
[2] I. E. Dzyaloshinskiĭ, P. S. Kondratenko, and V. S. Levchenkov, Zh. Eksp. Teor. Fiz. 62, 2313 (1962) [Sov. Phys.-JETP 35, 1209 (1962)].
[3] W. P. Halperin, C. N. Archie, F. B. Rasmussen, R. A. Buhrman, and R. C. Richardson, Phys. Rev. Lett. 32, 927 (1974).
[4] J. Goldstone, Nuovo Cimento 19, 154 (1961).
[5] A. A. Abrikosov, L. P. Gor'kov, and I. E. Dzyaloshinskiĭ, Metody kvantovoĭ teorii polya v statisticheskoĭ fizike (Quantum Field Theory Methods in Statistical Physics) Fizmatgiz, 1962 [English translation published by Pergamon Press, Oxford, 1965].

[6] L. D. Landau and E. M. Lifshitz, Elektrodinamika sploshnykh sred (Electrodynamics of Continuous Media) Gostekhizdat, 1957 [English translation published by Pergamon Press, Oxford, 1960].
[7] A. I. Akhiezer, V. G. Bar'yakhtar, and S. V. Peletminskiĭ, Spinovye volny (Spin Waves) Fizmatgiz, 1967 [English translation published by North Holland, Amsterdam, 1968].
[8] L. D. Landau, Zh. Eksp. Teor. Fiz. **35**, 97 (1958) [Sov. Phys.-JETP **8**, 70 (1959)].
[9] L. D. Landau and E. M. Lifshitz, Kvantovaya mekhanika (Quantum Mechanics) Fizmatgiz, 1963 [English translation published by Pergamon Press, Oxford, 1965].
[10] I. E. Dzyaloshinskiĭ and L. P. Pitaevskiĭ, Zh. Eksp. Teor. Fiz. **36**, 1797 (1959) [Sov. Phys.-JETP **9**, 1282 (1959)].

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