Scattering by magnetohydrodynamic oscillations and the power-law spectra of relativistic electrons produced in a turbulent plasma

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The quasistationary energy distribution of relativistic electrons interacting with radiation in a turbulent plasma is investigated with diffusion taken into account. It is shown that the diffusion in a magnetic field is determined by scattering by magnetohydrodynamic plasma oscillations and the particle spectrum is given by a power law $\propto \epsilon^{-\gamma}$. An equation is derived for the spectral exponent γ, which depends on the parameters of the plasma turbulence and on the characteristic size of the inhomogeneity of the distribution of fast electrons. It is found that all solutions of this equation lie in the range $2<\gamma<3$.

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1. It was noted previously[1], that in a turbulent plasma the interaction between the fast electrons, radiation, and pulsations occurring during the energy redistribution in spontaneous and induced scattering leads to a quasistationary power-law energy spectrum of the electrons $f_\epsilon \sim \epsilon^{-\gamma}$, where ϵ is the electron energy. This power law is the solution of the completely self-consistent problem of the spectra f_ϵ of the electrons and I_ω of the radiation, with the condition that the turbulence energy density W be constant. Up to certain frequencies ω_L the radiation is formed as a result of the balance of processes of emission and absorption (reabsorption) on relativistic electrons; in a tenuous plasma the radiation spectrum has a universal character $I_\omega \sim \omega^{5/2}$ (previously known for the case of induced reabsorption of synchrotron radiation[2]).

The maximal frequency ω_L is naturally determined by the size L of the system and by the reabsorption coefficient $\gamma_\omega(c/\gamma_\omega L \sim L)$ and the maximal energy has the order of magnitude $(\omega_L/\omega_k)^{1/2}mc^2$, where ω_k is the frequency of the plasma mode of oscillations. In[3] we have proved the universal character of the electron spectrum, taking into account the radiation processes in a magnetic field. Various turbulence modes have been investigated, in particular anisotropic modes[4]; in all cases the only value of the spectrum exponent was $\gamma = 3$.

The present paper is devoted to an investigation of the formation of power-law spectra for the relativistic electrons interacting with reabsorbed radiation ($I_\omega \sim \omega^{5/2}$) in a turbulent plasma, taking into account the finite size of the region occupied by plasma turbulence. In such a system the distribution of the electrons is not uniform and the outflow of particles on account of spatial diffusion is a source of additional losses which have not been taken into account previously. The source of fast particles may be the turbulent plasma where the strong Langmuir turbulence[5] may give rise to self-injection of relativistic electrons.

Plasma turbulence leads to both particle acceleration and energy loss of individual particles. As we show in the present paper, the relativistic electrons turn out to have a power-law quasistationary spectrum. The spectrum exponent γ depends on the characteristic size of the inhomogeneity and is situated in the interval $2 < \gamma < 3$.

2. We shall consider the inhomogeneity of the electron distribution along the z-axis, f_ϵ, the axis being oriented along the uniform magnetic field **H**. The equation obeyed by f_ϵ is:

$$\frac{\partial f_\varepsilon}{\partial t}+\mu v\frac{\partial f_\varepsilon}{\partial z}=\frac{\partial}{\partial\varepsilon}\left(\varepsilon^2 D_\varepsilon\frac{\partial}{\partial\varepsilon}\frac{f_\varepsilon}{\varepsilon^2}+A_\varepsilon f_\varepsilon\right)+\frac{\partial}{\partial\mu}D_\mu\frac{\partial f_\varepsilon}{\partial\mu}, \qquad (1)$$

where μ is the cosine of the angle between the electron velocity vector **v** and the direction of **H**. The coefficient D_ϵ and A_ϵ describe the diffusion of the electrons in energy space and are determined by scattering processes of plasma waves on relativistic electrons, with conversion into high-frequency electromagnetic radiation ($\omega \gg \omega_k$). These processes lead to an insignificant angular diffusion which may be neglected if the distribution function does not change noticeably over the characteristic angle $\Delta\theta \simeq mc^2/\epsilon$.[3]

The angular diffusion (related to the anisotropy of the distribution function) is determined by the resonant interaction of electrons with plasma oscillations, representing the effect of the quasilinear approximation. The magnetohydrodynamic (MHD) waves are distinguished for processes of resonant interaction with relativistic particles. Thus, the effectiveness of electron scattering by whistlers is c/v_A times smaller than for MHD waves, and the effectiveness of scattering by Langmuir waves is $(m_i/m_e)^{1/2}c/v_A$ times smaller[6]. For such an interaction the angular diffusion occurs approximately $(c/v_A)^2$ times faster than the diffusion in energy, fact which is taken into account in Eq. (1), where the second term of the right-hand side describes the angular diffusion with a diffusion coefficient D_μ ($v = Hc/(4\pi nm)^{1/2}$ is the Alfvén velocity).

It is well known that in a turbulent plasma the magnetohydrodynamic waves (Alfvén waves and fast magnetosonic waves) can propagate along the external magnetic field and that the spectral density of the turbulence energy of MHD oscillations behaves like $W_\omega \sim 1/\omega$[7]. The diffusion coefficient D_μ of relativistic electrons interacting with turbulent pulsations can in this case be represented in the form

$$D_\mu=\frac{\pi^2e^2v_Ac}{\varepsilon^2}\frac{1-\mu^2}{|\mu|}W_\omega|_{\omega=eHv_A/|\mu|\varepsilon};\quad c\gg v_A\gg v_s=(T/m_i)^{1/2},$$

where $\int W_\omega d\omega = W = W^A + W^M$ is the energy density of the MHD oscillations, W^A is the energy density of Alfvén oscillations and W^M is the energy density of the magnetosonic oscillations.

We shall be interested in a sufficiently low inhomogeneity. This means that all quantities change little over distances of the order of the mean free path. If the characteristic size of the inhomogeneity of the distribution of the fast electrons is L we assume that

$$v/D_\mu \ll L, \qquad (2)$$

or, in other words,

$$L \gg \frac{\varepsilon}{mc^2}\frac{H^2}{W}\frac{c}{\omega_{He}}, \quad \omega_{He}=\frac{eH}{mc}. \quad (3)$$

At the same time the process of establishment of a power-law distribution $f_\epsilon \sim \epsilon^{-\gamma}$ as a result of a redistribution of energy among the electrons and radiation assumes that

$$c/\gamma_\omega < L, \quad (4)$$

where γ_ω is the reabsorption coefficient. The first terms of the right-hand side of Eq. (1) may be considered small if

$$\left(\frac{\varepsilon}{mc^2}\right)^2 \ll \frac{\omega_{He}}{\omega_{pe}}\frac{nT}{H^2}\left(\frac{c}{v_{Te}}\right)^5 N_d, \quad N_d=n\left(\frac{v_{Te}}{\omega_{pe}}\right)^3. \quad (5)$$

In this case the fastest process is angular diffusion and the diffusion in energy space can be taken into account in the next approximation. Taking this into account one can easily derive from (1) the equation for spatial diffusion. Indeed, the assumption (2) means that the diffusion approximation is applicable and that it is possible to solve the equation for f_ϵ by separating a small μ-dependent correction to the isotropic distribution function, $f_\epsilon = f_\epsilon^{(0)} + f_\epsilon^{(1)}(\mu)$. Then, retaining the leading terms in (1) we obtain for the quasistationary distribution the equation

$$\mu v\frac{\partial f_\varepsilon^{(0)}}{\partial z}=\frac{\partial}{\partial\mu}D_\mu\frac{\partial f_\varepsilon^{(1)}}{\partial\mu}. \quad (6)$$

Integrating this equation, substituting $f_\epsilon^{(1)}(\mu)$ into (1) and averaging over the angles, we obtain the required equation:

$$\frac{\partial f_\varepsilon}{\partial t}=\frac{\partial}{\partial\varepsilon}\left(\varepsilon^2\bar{D}_\varepsilon\frac{\partial}{\partial\varepsilon}\frac{f_\varepsilon}{\varepsilon^2}+\bar{A}_\varepsilon f_\varepsilon\right)+D_{zz}\frac{\partial^2 f_\varepsilon}{\partial z^2}. \quad (7)$$

Here $D_{ZZ} \sim H^2\epsilon/Wm\omega_{He}$ is the spatial diffusion coefficient, $\bar{D}_\epsilon$ and $\bar{A}_\epsilon$ represent respectively the coefficients D_ϵ and A_ϵ averaged with respect to μ. It can easily be seen from Eq. (6) that the inequality $f_\epsilon^{(1)} \ll f_\epsilon^{(0)}$ used in the derivation of (7) is in fact equivalent to $v/D_\mu \ll L$, i.e., it reduces to the condition (2) already considered before. One should also note that in the equation for f_ϵ we do not take into account the particle acceleration during resonant interaction with MHD waves. However, the effectiveness of such acceleration is proportional to:

$$\frac{D_\varepsilon^R}{\varepsilon^2} \propto \frac{mc^2}{\varepsilon}\left(\frac{v_A}{c}\right)^2\frac{W}{H^2}\omega_{He}, \quad (8)$$

and one may neglect this effect if

$$\left(\frac{\varepsilon}{mc^2}\right)^2 \gg \frac{\omega_{He}}{\omega_{pe}}\frac{nT}{H^2}\left(\frac{c}{v_{Te}}\right)^5\left(\frac{v_A}{c}\right)^2 N_d.$$

In the sequel we also consider the following condition to be valid:

$$D_\varepsilon^R/\varepsilon^2 \ll D_{zz}/L^2. \quad (9)$$

3. We can now analyze the character of the quasistationary energy spectrum of the relativistic electrons, determined by the equation (7). Of importance is, first of all, the fact that $D_{ZZ} \sim \epsilon$. Therefore, taking into account the energy dependence of $\bar{D}_\epsilon$ and $\bar{A}_\epsilon$ ($\bar{D}_\epsilon \sim \epsilon^3$, $\bar{A}_\epsilon \sim \epsilon^2$,[3]) it is easy to see that the spectrum is given by a power law: $f_\epsilon \sim \epsilon^{-\gamma}$ where the spectrum exponent γ does not depend on ϵ. It is known that the first term in the right-hand side of (7), a term which describes the formation of the power-law spectrum under the conditions of homogeneous spatial distribution of the electrons, leads to an equation for γ which has the solution $\gamma = 3$[3,4] (and also $\gamma = 2$, for nonvanishing flux in the energy space). It is easy to show what changes in γ are produced by taking the inhomogeneity into account. The term $D_{ZZ}\partial^2 f_\epsilon/\partial z^2 \simeq -D_{ZZ}f_\epsilon/L^2$ in Eq. (7) depicts the loss of particles on account of diffusion. This effect signifies a violation of the balance between the acceleration of particles $(d\langle\epsilon\rangle/dt)_{D_\epsilon} > 0$ and the energy losses $(d\langle\epsilon\rangle/dt)_{A_\epsilon} < 0$ and the shift of this balance in the direction of increased losses. Then the spectrum must become harder, i.e., be characterized by exponents $\gamma < 3$ in order for the equilibrium to reestablish itself. This becomes completely clear if one takes into account that the acceleration mechanism under consideration leads to such a change of the average electron energy for which $(d\langle\epsilon\rangle/dt)_{D_\epsilon} \sim \langle\epsilon^2\rangle$.

Introducing the parameter L we can obtain an equation for the spectrum exponent γ from Eq. (7) without solving the latter. This equation has the form

$$(\gamma-2)\left[\sum_{\sigma=1,2}\overline{R_3{}^\sigma\frac{R_{\gamma-1}^\sigma}{R_\gamma{}^\sigma}}-\sum_{\sigma=1,2}\overline{R_2{}^\sigma}\right]=\delta. \quad (10)$$

where

$$\delta=24\ln\frac{k_{max}}{k_{min}}N_d\left(\frac{c}{v_{Te}}\right)^3\frac{1}{1+\alpha}\frac{\omega_{He}}{\omega_{pe}}\left(\frac{W^A}{nmc^2}\right)^{-2}\frac{(c/L)^2}{\omega_{pe}{}^2}, \quad \alpha=\frac{W^M}{W^A}.$$

Now it can be seen, in particular, that for $\delta = 0$ one of the solutions is $\gamma = 3$ (or $\gamma = 2$). The equation involves averages with respect to μ containing the functions R_γ^σ which determine the dependence on γ of the coefficients D_ϵ and A_ϵ[3] ($\sigma = 1, 2$ denotes summation over the two polarizations of transverse waves). We list these functions here only for the concrete case of magnetohydrodynamic turbulence, in order to show how the parameters which determine the value of the spectrum exponent calculated for a turbulent plasma enter in Eq. (10). We have

$$\begin{aligned}R_\gamma{}^1&=\left(\frac{k_{max}|\mu|}{\omega_{pe}}\right)^{(\gamma-2)/2}(1-\beta^{(\gamma-2)/2})a_\gamma{}^1(\alpha,p^A,\mu)\\&\quad+\frac{9}{2}(1-\mu^2)\frac{H^2}{8\pi W^A}(2\zeta)^{(\gamma-2)/2}b_\gamma{}^1,\\R_\gamma{}^2&=\alpha\left(\frac{k_{max}|\mu|}{\omega_{pe}}\right)^{(\gamma-2)/2}(1-\beta^{(\gamma-2)/2})a_\gamma{}^2(\alpha,p^M,\mu)\\&\quad+\frac{9}{2}(1-\mu^2)\frac{H^2}{8\pi W^A}\cdot(2\zeta)^{(\gamma-2)/2}b_\gamma{}^2,\\\zeta&=\frac{3\omega_{He}\sqrt{1-\mu^2}}{4\omega_{pe}}, \quad \omega_{pe}{}^2=\frac{4\pi ne^2}{m}, \quad \beta=\frac{k_{min}}{k_{max}},\end{aligned} \quad (11)$$

where k_{min} and k_{max} determine the region of the wave numbers of turbulent pulsations. The quantities a_γ^σ and b_γ^σ depending on γ have the form:

$$a_\gamma{}^1=\frac{6}{(\gamma-2)\ln\beta^{-1}}\left[\frac{(\gamma+4)(\gamma+6)+4(3+\alpha\mu^2)}{(\gamma+2)(\gamma+4)(\gamma+6)}\pm\frac{4p^A}{(\gamma+2)(\gamma+4)}\right],$$

$$a_\gamma{}^2=\frac{6}{(\gamma-2)\ln\beta^{-1}}\left[\frac{(\gamma+4)(\gamma+6)\mu^2+4(\alpha^{-1}+3\mu^2)}{(\gamma+2)(\gamma+4)(\gamma+6)}\mp\frac{4p^M\mu^2}{(\gamma+2)(\gamma+4)}\right],$$

$$b_\gamma^{1,2}=\frac{\sqrt{3}}{4\pi}\Gamma\left(\frac{\gamma}{4}+\frac{1}{6}\right)\Gamma\left(\frac{\gamma}{4}+\frac{5}{6}\right)\left[\frac{\gamma}{4}+\frac{5}{6}\mp\frac{\gamma+2}{4}\right]\frac{1}{\gamma+2}$$

We note that in the case of anisotropic turbulence under consideration a_γ^σ depend on a parameter which determines the relation between the energy density W_+ of the turbulent pulsations propagating along the direction of the magnetic field and the density W_- of those propagating against the magnetic field ($W_+ + W_- = W$):

$$p^{A,M}=(W_+^{A,M}-W_-^{A,M})/(W_+^{A,M}+W_-^{A,M}). \quad (12)$$

An idea about the character of the solutions of (10) can be obtained from the γ-dependence of the left-hand side of this equation. This ratio is represented in the figure for various values of the parameters. As can already be seen from the general expression, this term in the equation vanishes for $\gamma = 2$ and $\gamma = 3$. Thus, the possible values of the spectrum exponent which are solutions of Eq. (10) are situated in the region $2 < \gamma < 3$.

4. Thus, we have obtained the result that in a turbulent plasma the exponent of the power-law spectrum of relativistic electrons can vary in a definite range. For astrophysical applications it is particularly important that these values correspond to h the region of most probable values obtained in the investigation of cosmic sources of radiation as well as from the spectrum of primary electrons near the Earth. The concrete value of γ depends on the parameter δ which, as was seen, is determined by the characteristic size L of the inhomogeneity in the distribution of fast electrons.

The determination of this parameter requires a more detailed investigation. Thus, L may be determined from the intensity of sources of fast electrons such as regions with strong plasma turbulence which effectively accelerate electrons in the whole energy range.

However, already a general analysis, taking into account the peculiarities of the electron acceleration mechanisms considered here imposes definite restrictions on the possible values of the parameter L. The radiation reabsorption coefficient on relativistic electrons γ_ω has the representation[3]

$$\gamma_\omega \sim \frac{1}{m}\left(\frac{\varepsilon_*}{mc^2}\right)^{\gamma-1}\left(\frac{\omega_{pe}}{\omega}\right)^{(\gamma+4)/2}\cdot \omega_{pe}R_\gamma n_*\frac{W}{nme^2},$$

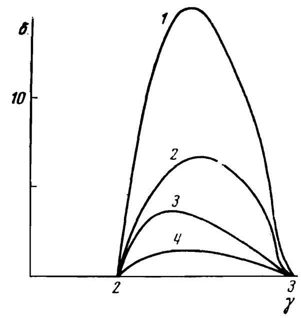
where n_* and ϵ_* are the density and minimal energy of the fast electrons in the region of the power-law spectrum[8]. Therefore, making use of the relation $\omega \sim k_{max}c(\epsilon/mc^2)^2$ between the frequency ω of the radiation and the energy ϵ of the emitted relativistic electrons in a turbulent plasma, and of the conditions (4), (9) together with (11), we obtain

$$\frac{c}{\omega_{pe}}\left(\frac{\varepsilon}{mc^2}\right)^5\left(\frac{k_{max}c}{\omega_{pe}}\right)^3\frac{n}{n_*}\frac{nmc^2}{W^A} < L \ll \frac{c}{\omega_{He}}\frac{\varepsilon}{mc^2}\frac{H^2}{W^A}\frac{c}{v_A}. \quad (13)$$

We note, in conclusion, that of the two values of the spectrum exponent which are possible for a given δ (cf. the Figure) the stable one will be that for which a deviation, e.g., in the direction of larger γ than the equilibrium value, corresponds to an increase of the intensity of particle acceleration compared with the intensity of energy losses

$$\left(\frac{d\langle\varepsilon\rangle}{dt}\right)_{D_\varepsilon} > \left|\left(\frac{d\langle\varepsilon\rangle}{dt}\right)_{A_\varepsilon + D_{zz}}\right|,$$

and vice versa. However, the proof of the stability of the power-law spectrum $f_\epsilon \sim \epsilon^{-\gamma}$ under arbitrary perturbations remains a problem for further investigations.



The dependence of δ on γ for different values of the parameters: 1–$H^2/8\pi W^A = 10^2$, $\omega_{He}/\omega_{pe} = 10^{-7}$; 2–$H^2/8\pi W^A = 10^2$, $\omega_{He}/\omega_{pe} = 10^{-6}$; 3–$H^2/8\pi W^A = 10$, $\omega_{He}/\omega_{pe} = 10^{-7}$; 4–$H^2/8\pi W^A = 10$, $\omega_{He}/\omega_{pe} = 10^{-6}$. For all the curves $\alpha = 0.1$, $p^A = p^M = -1$, $k_{max}/\omega_{pe} = 10^{-3}$, $\beta = 0.1$.

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