Long gravitational waves in a closed universe

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An investigation is made of the role of long gravitational waves in the evolution of a homogeneous closed universe (model of type G_3IX in the Bianchi classification). It is shown that the metric of this model can be represented as a sum of a background metric, which describes a nonstationary space of constant positive curvature, and a number of terms that can be interpreted as a collection of gravitational waves of maximal length compatible with the space being closed. This decomposition of the metric is exact and the wave corrections do not have to be small. For this reason, the behavior of the wave terms can be investigated at all stages of their evolution: both in the epoch when the "energy density" and "pressure" of the gravitational waves make a negligibly small contribution to the the dynamics of the background universe and during the epoch when they make a decisive contribution. In particular, it is shown that in the vacuum stage, when the evolution of the background metric is determined by the "energy density" and "pressure" of the gravitational waves, the scale factor of the background universe may pass through the state of a stable regular minimum because the gravitational "pressure" is negative.

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Usually, weak gravitational waves are considered; weak means that they are oscillating corrections to a background metric of space-time. Provided the length of a gravitational wave is small compared with the characteristic radii of curvature of the background space and the period of the wave is short compared with the characteristic time of variation of the background metric, gravitational waves are completely analogous to electromagnetic waves, for example. They propagate with the velocity of light, are transverse, and a collection of gravitational waves behaves like matter with equation of state $p = \epsilon/3$, etc.[1-3] Fundamental differences arise only in regions of space-time where these conditions are not satisfied. For example, one can then have the process of superadiabatic amplification of gravitational waves.[4,5] The existence of these regions is also a necessary condition for the approximation of weak gravitational waves to break down. In other words, in these regions of space-time the relative amplitude of the wave corrections is in general no longer small compared with unity.

The main aim of the present paper is to study the role of wave corrections with allowance for their back reaction on the background metric and also during the stages in the evolution of the background metric when these corrections are no longer small. Concretely, we consider a metric that admits a three-parameter group of motions of the type G_3 IX in the Bianchi classification. We show in Sec. 1 that this metric can be represented as a sum of a "background" metric, which describes a nonstationary space of constant positive curvature, and a collection of terms corresponding to a number of gravitational waves with maximal wavelength compatible with the space's being closed. This decomposition of the metric is exact, and the wave corrections do not have to be small. For this reason, the behavior of the wave terms can be investigated at all stages in the evolution of the universe, and this enables us to elucidate, for a particular example, the properties of "strong" gravitational waves (which are defined in various different ways).

The decomposition of the metric into a background metric and wave corrections is not completely unique. In the linear approximation in the amplitude of the corrections, all definitions of a "gravitational wave" are the same, but already in the second order important differences appear. At the same time, the properties of the "energy density" and the "pressure" of the gravitational waves and the evolution of the background metric depend on the manner in which the wave corrections are separated from the background metric. Below, the background metric is chosen in such a way that the "wave corrections" are eigenfunctions of the Laplacian operator. This choice has certain advantages, but it can also be criticized.

In Sec. 2, we consider weak gravitational waves in the linear and quadratic approximation in the amplitude of the wave corrections. In the linear approximation, the waves give rise to an anisotropy of the deformation of the volume element without perturbing the density or velocity of the matter filling the universe. Perturbations of these quantities appear only in the quadratic approximation as a result of the nonlinear interaction of the waves. Generally speaking, the contribution of the wave terms increases unboundedly as the singularity is approached, i.e., as the epoch in which the waves cease to be high-frequency waves is approached.

There is however a set of initial data, which corresponds to a special choice of the phase of the gravitational waves, for which the wave perturbations are small right up to the singularity. The behavior of the solution in this "quasi-isotropic" regime is considered in Sec. 3. In the general case, the "energy density" of the gravitational waves is predominant near the singularity, and this ensures the existence of the so-called vacuum stage, when the influence of ordinary matter on the behavior of the gravitational field is negligibly small.

Finally, in the limiting case when ordinary matter is completely absent, the evolution of the background metric is determined by the "energy density" and "pressure" of gravitational waves by themselves. We show, in particular, that the background scale factor can pass through a state of a stable regular minimum during the evolution because the gravitational "pressure" is negative (Sec. 4).

The solutions of Einstein's equations for a homogeneous metric of type G_3 IX have been studied by Belinskiĭ, Lifshitz, and Khalatnikov[6] in connection with the problem of singularities, by Misner[7] in his mixmaster program, and also by Bogoyavlenskiĭ, S. Novikov, I. Novikov, Lukash and the authors.[8-11] The decomposition of the metric element employed here may help, on the one hand, in the physical interpretation of these solutions and, on the other, in the development of a theory of strong gravitational waves.[1)]

1. DECOMPOSITION OF THE METRIC ELEMENT

We consider a metric that admits a three-parameter group of motions of the type G_3 IX acting transitively on the hypersurfaces t = const. The metric has the form[14] (the velocity of light is c = 1)

$$ds^2=dt^2-\gamma_{ab}(t)e_i^{(a)}e_k^{(b)}dx^i\,dx^k \quad (a,b,c,d=1,2,3), \tag{1}$$

where the Killing vectors $e_i^{(a)}$ of the reciprocal group of motions are

$$e_i^{(1)}=(\cos x^3, \sin x^3\sin x^1, 0),\ e_i^{(2)}=(-\sin x^3, \cos x^3\sin x^1, 0),\ e_i^{(3)}=(0, \cos x^1, 1).$$

The vectors $e_i^{(a)}$ satisfy the relations $e_{i,k}{}^{(a)}-e_{k,i}{}^{(a)}=C^a_{bc}e_i{}^{(b)}e_k{}^{(c)}$ with the following nonzero structure constants: $C_{23}{}^1=-C_{32}{}^1=C_{31}{}^2=-C_{13}{}^2=C_{12}{}^3=-C_{21}{}^3=1$.

The line element

$$dl^2=\gamma_{ab}e_i^{(a)}e_k^{(b)}dx^i\,dx^k \tag{2}$$

with the special choice $\gamma_{ab}=R^2(t)\eta_{ab}/4$ (where η_{ab} is the unit tensor) describes a three-dimensional sphere of variable radius. A three-dimensional sphere of unit radius corresponds to the special case R = 1.

It is easy to see that the metric element (1) can be rewritten in the form

$$ds^2=ds_0{}^2-\delta_{ab}G_{ik}^{(ab)}dx^i\,dx^k. \tag{3}$$

Here, $ds_0{}^2$ is the isotropic line element

$$ds_0{}^2=dt^2-{}^1/_4R^2\eta_{ab}e_i^{(a)}e_k^{(b)}dx^i\,dx^k \tag{4}$$

with scale factor $R^2/4=\gamma_{ab}\eta^{ab}/3\equiv\gamma/3$, and $G_{ik}^{(ab)}$ stand for the six tensors

$$G_{ik}^{(ab)}\equiv{}^1/_2(e_i^{(a)}e_k^{(b)}+e_i^{(b)}e_k^{(a)})-{}^1/_3\eta^{ab}\eta_{cd}e_i^{(c)}e_k^{(d)}; \tag{5}$$

the "amplitudes" δ_{ab} are related to γ_{ab} by

$$\delta_{ab}-{}^1/_3\eta_{ab}\delta=\gamma_{ab}-{}^1/_3\eta_{ab}\gamma \quad (\delta\equiv\delta_{ab}\eta^{ab}).$$

Only five of the tensors (5) are linearly independent since $\eta_{ab}G_{ik}^{(ab)}\equiv 0$. Without changing the form of the decomposition (3), one can always achieve that $\delta=0$, which we shall assume in what follows.

The decisive fact for understanding the metric (1) is that the functions (5) are tensor eigenfunctions of the Laplacian operator on the three-dimensional sphere of unit radius. In other words, each of the functions $G_{ik}^{(ab)}$ satisfies the equations

$$G_{ik;l;}^{(ab)l}=-(n^2-3)G_{ik}^{(ab)},\quad G_i^{(ab)k}{}_{;k}=0,\quad G_i^{(ab)i}=0 \tag{6}$$

for one and the same n, namely n = 3. [In Eqs. (6) all operations of raising and lowering indices and covariant differentiation are performed by means of the metric (2) with $\gamma_{ab}=\eta_{ab}/4$]. The number n = 3 is the smallest[15] of the eigenvalues for the tensor functions, and one can therefore say that the functions $G_{ik}^{(ab)}$ describe the lowest mode of characteristic tensor oscillations of the three-dimensional sphere.

In the linear theory of small perturbations developed by Lifshitz,[15] tensor eigenfunctions correspond to gravitational waves. In the case we are considering, the decomposition of the metric into an isotropic background and a set of tensor harmonics is not approximate but exact. This enables us to treat the correction terms to the isotropic metric in (3) as gravitational waves irrespective of the magnitude of the amplitudes δ_{ab}. Of course, the decomposition of the metric into a background and gravitational waves is not completely unique. For example, one could separate out an isotropic background by introducing a scale factor proportional to the cube root of the volume element, and regarding all that remains as gravitational waves. This decomposition is the same as (3) in the linear approximation in the small perturbations of the isotropic metric.

The differences between the definitions of the scale factor, and, therefore, the wave parts appear already in the quadratic approximation, and they lead to different quantities for the "energy density" and "pressure" of the gravitational waves. We shall use the decomposition (3) because we feel that it corresponds better to the intuitive notion of free gravitational waves as characteristic oscillations of the background geometry independently of the amplitude of the oscillations.

It is interesting that the metric (1) can also be regarded as a combination of vector harmonics of the Laplacian operator on the three-dimensional sphere of unit radius. Indeed, each of the vectors $e_i^{(a)}$ satisfies the equations

$$e_{i;\ k}^{(a)\ ;k}=-(n^2-2)e_i^{(a)},\quad e_{i;}^{(a)i}=0$$

for the smallest eigenvalue, namely, n = 2, for vector functions. According to Lifshitz,[15] from a vector spherical function S_i one can construct a tensor S_{ik}, by means of which the perturbation of the metric can be expressed in accordance with the rule $S_{ik}=S_{i;k}+S_{k;i}$. In the special case n = 2 this tensor is identically zero.[15] A nonzero tensor can be constructed by using a quadratic combination of eigenvectors. Such a combination is $(e_i{}^{(a)}e_k{}^{(b)}+e_k{}^{(a)}e_i{}^{(b)})/2$, which has the form of the polarization tensor for a circularly polarized wave, and it occurs in the metric (3).

Finally, the line element (1) can be represented in the form of an axisymmetric background metric admitting a four-parameter group of motions and a set of vector and tensor harmonics of the Laplacian operator constructed by means of the background metric. If the direction defined by the third frame vector is taken as distinguished direction, the decomposition of the metric takes the form

$$ds^2=ds_c{}^2-\delta_{pq}G_{ik}^{(pq)}dx^i\,dx^k-\sigma_{p3}S_{ik}^{(p3)}dx^i\,dx^k \quad (p,q,r,s=1,2), \tag{7}$$

where the background metric is $[\gamma_{11}(t)=\gamma_{22}(t)=a^2/4,\ \gamma_{33}(t)=c^2/4]$

$$ds_c{}^2=dt^2-dl^2=dt^2-{}^1/_4a^2\eta_{pq}e_i^{(p)}e_k^{(q)}dx^i\,dx^k-{}^1/_4c^2e_i^{(3)}e_k^{(3)}dx^i\,dx^k. \tag{8}$$

The functions $G_{ik}^{(pq)}$ and $S_{ik}^{(p3)}$ are determined by means of the Laplacian operator constructed with the metric dl^2 in (8); the same metric is used for all operations of covariant differentiation and raising and lowering of indices. The tensor eigenfunctions $G_{ik}^{(pq)}$ can be expressed by means of the relations

$$G_{ik}^{(pq)}={}^1/_2(e_i^{(p)}e_k^{(q)}+e_i^{(q)}e_k^{(p)})-{}^1/_2\eta^{pq}\eta_{rs}e_i^{(r)}e_k^{(s)},\quad \eta_{pq}G_{ik}^{(pq)}\equiv 0$$

and they satisfy the equations

$$G_{ik;l;}^{(pq)l}=-\nu G_{ik}^{(pq)},\quad G_{ik;}^{(pq)k}=0,\quad G_i^{(pq)i}=0;$$

$$\nu=4(4/c^2-4/a^2+3c^2/2a^4).$$

From the vector eigenfunctions $e_i^{(p)}$, which satisfy the equations

$$e_{i;k;}^{(p)k} = -\mu e_i^{(p)}, \quad e_{i;}^{(p)i} = 0;$$

$$\mu = 4(1/c^2 - 1/a^2 + c^2/2a^4),$$

one can construct components of "solenoidal" tensors in accordance with the usual rule[15]:

$$S_{ik}^{(p3)} = e_i^{(p)}{}_{;k} + e_k^{(p)}{}_{;i},$$

which leads to the concrete expressions

$$S_{ik}^{(13)} = (1 - c^2/a^2)(e_i^{(2)} e_k^{(3)} + e_i^{(3)} e_k^{(2)}), \quad S_{ik}^{(23)} = -(1 - c^2/a^2)(e_i^{(1)} e_k^{(3)} + e_i^{(3)} e_k^{(1)}).$$

The scale factors $(1/4)a^2$ and $(1/4)c^2$ of the background metric and the amplitudes δ_{pq} and σ_{p3} can be expressed in terms of the components $\gamma_{ab}(t)$ of the original metric in accordance with the equations

$$^1/_4 a^2 = {}^1/_2 \gamma_{pq} \eta^{pq}, \quad {}^1/_4 c^2 = \gamma_{33},$$
$$\delta_{pq} - {}^1/_2 \eta_{pq} (\delta_{rs} \eta^{rs}) = \gamma_{pq} - {}^1/_2 \eta_{pq} (\gamma_{rs} \eta^{rs}),$$
$$\gamma_{13} = \sigma_{23}(1 - c^2/a^2), \quad \gamma_{23} = -\sigma_{13}(1 - c^2/a^2).$$

The physical advantage of the decomposition (3) over (7) is that, as follows from Einstein's equations in vacuum, the evolution of the background in (3) is entirely determined by the existence of sources in the form of the wave terms (there is no solution of the equations of gravitation for the background metric in the absence of sources!). At the same time, the background metric in (7) can also exist as a solution of the vacuum Einstein equations in the absence of sources, i.e., for $\delta_{pq} \equiv 0$ and $\sigma_{p3} \equiv 0$.

Note that other homogeneous metrics admit a decomposition into a background and a set of eigenfunctions of the Laplacian operator constructed from the background metric.[16] The background metric is distinguished by the condition that it admits the larger group G_4 (or G_6) acting on the hypersurfaces t = const, the given group G_3 being a subgroup of this group (for the types of metric for which the larger group exists). A decomposition of this kind and an interpretation on this basis of homogeneous type G_3 IX metrics were considered by Lukash.[17]

2. WEAK GRAVITATIONAL WAVES IN A MATTER-FILLED UNIVERSE

Using the representation (3), let us consider weak gravitational waves in a matter-filled universe. For simplicity, we assume that the equation of state of the matter in the universe is $p = \epsilon/3$. In the limiting case $\delta_{ab} \equiv 0$, we obtain from Einstein's equations the standard solution for a closed universe:

$$R_F = R_0 \sin\eta, \quad \varepsilon_F = 3R_0^2/R_F^4, \quad R_0 = \text{const}, \tag{9}$$

where the time η is related to t by $dt = R_F d\eta$. The components of the four-velocity of the matter are $u_0 = 1$, $u_i = p_a e_i^{(a)} = 0$.

To treat the general case of the metric (1), we introduce the notation

$$\gamma_{ab} = {}^1/_4 R_F^2 (\eta_{ab} + \varkappa_{ab}), \quad \varepsilon = \varepsilon_F (1+q), \tag{10}$$
$$u_0 = 1 + v_0, \quad p_a = v_a.$$

Here, κ_{ab}, q, v_0, v_a are functions of the time. If they are equal to zero, we come back to the Friedmann solution (9). By means of (10), we readily find R^2 and δ_{ab} in the decomposition (3):

$$R^2 = R_F^2 (1 + {}^1/_3 \varkappa), \quad \delta_{ab} = {}^1/_4 R_F^2 (\varkappa_{ab} - {}^1/_3 \eta_{ab} \varkappa),$$

where $\kappa = \eta^{ab} \kappa_{ab}$.

We are interested in small corrections to the solution (9) in the linear and quadratic approximation in the small amplitude of the perturbations, and we therefore assume that the corrections can be represented in the form

$$\varkappa_{ab} = \varkappa_{ab(1)} + \varkappa_{ab(2)}, \quad q = q_{(1)} + q_{(2)},$$
$$v_0 = v_{0(1)} + v_{0(2)}, \quad v_a = v_{a(1)} + v_{a(2)}.$$

Einstein's equations for arbitrary metric (1) have the form

$$\dot{d} + d_{ab} d^{ab} = {}^1/_3 \varepsilon (1 - 4u_0^2), \tag{11}$$

$$d_a^b C_{bc}^a = -{}^4/_3 \varepsilon u_0 p_c, \tag{12}$$

$$\dot{d}_{ab} + d d_{ab} - 2 d_{ac} d_b^c - P_{ab} = {}^1/_3 \varepsilon (\gamma_{ab} - 4 p_a p_b), \tag{13}$$

where the point denotes d/dt; $d_{ab} = \dot{\gamma}_{ab}/2$, $d = d_{ab}\gamma^{ab}$; P_{ab} are the components of the three-dimensional Ricci tensor.[2)] The Einstein gravitational constant is assumed equal to unity.

We also write down the solution of one of the equations of hydrodynamics that is a consequence of the system (11)–(13):

$$\varepsilon^{3/4} u_0 \sqrt{|\gamma|} = \text{const} = k. \tag{14}$$

The constant k has the physical meaning of the entropy of the matter in unit comoving volume.

Introducing a small perturbation of the Friedmann solution, one will in general also introduce a small change in the constant k. However, we are interested in the behavior of an anisotropic model that has the same value of the entropy density as the isotropic model, i.e., we shall require that the constant k keep its Friedmann value.

Substituting (10) into Eqs. (11)–(13), we can find the corrections to the Friedmann solution (9). It must be borne in mind that some of the solutions obtained may be "fictitious," i.e., they may not correspond to a real variation of the gravitational field but merely to small transformations of the coordinates.[15,19] We are interested in perturbations that leave the coordinate system synchronous and the metric homogeneous. One can show[3)] that these conditions are satisfied by the class of small transformations

$$x^{\alpha'} = x^\alpha + \eta^\alpha, \quad \eta^\alpha = \{\eta^0, k^c \xi_{(c)}^i + m^c e_{(c)}^i\},$$

where ξ^i are Killing vectors of the given group and η^0, k^c, m^c are arbitrary constants. Concretely, for type G_3 IX fields fictitious perturbations of the isotropic metric are due solely to a shift in time and have the form $\delta g_{ik} = \dot{\gamma}_{ab} \eta_0 e_i^{(a)} e_k^{(b)}$, i.e.

$$\delta\gamma_{11} = \delta\gamma_{22} = \delta\gamma_{33} = (R^2/4)^{\cdot} \eta_0, \quad \delta\gamma_{12} = \delta\gamma_{13} = \delta\gamma_{23} = 0.$$

Therefore, this perturbation mode can be eliminated by a small transformation of t.

The solution of the system (11)–(13) in the linear approximation can be written in the form

$$\varkappa_{ab(1)} = \frac{1}{\sin\eta}(\alpha_{ab}\cos 3\eta + \beta_{ab}\sin 3\eta) + \frac{C_1 \cos\eta}{\sin^2\eta}\eta_{ab},$$

where the constants α_{ab} and β_{ab} are related by the conditions

$$\eta^{ab}\alpha_{ab} = 0, \quad \eta^{ab}\beta_{ab} = 0. \tag{15}$$

The term $C_1 \cos \eta \eta_{ab}/\sin^2 \eta$ is a fictitious perturbation of the metric $4R^{-2}\delta\gamma_{ab}$ (a = b) and can be eliminated by a transformation of the time coordinate. Therefore, the solution can be written in the form

$$\frac{4}{R_F^2}\delta_{ab(1)}=\varkappa_{ab(1)}=\frac{1}{\sin\eta}(\alpha_{ab}\cos 3\eta+\beta_{ab}\sin 3\eta), \tag{16}$$

where α_{ab} and β_{ab} are related by the conditions (15). Hence $\kappa_{(1)} = 0$, and therefore in (14) we also have $q_{(1)} = 0$.

Thus, in the linear approximation only the metric is perturbed, and the corrections, as one would expect, correspond to gravitational waves of the maximal possible wavelength $\lambda = 2\pi R/n$, where n = 3. The waves deform space, causing anisotropy in the expansion, but the volume of each element of space does not change. The matter remains at rest, distributed uniformly in space, and there is the same law of variation of the density with time as in the unperturbed solution. The mode $\beta_{ab} \cos 3\eta/\sin\eta$ corresponds to perturbations of the metric that remain small during the whole of the evolution. The mode $\alpha_{ab} \cos 3\eta/\sin\eta$ gives rise to corrections that diverge near the singularities ($\eta \to 0$, $\eta \to \pi$).

The waves are standing, circularly polarized waves, as can be seen from the coordinate expression and from the invariant definition[20]: The Weyl tensor with allowance for the wave terms always belongs to the Petrov type I classification (the Weyl tensor of the background metric is of type 0).

Not all of the constants in the solution (16) are "physically" arbitrary since they can be changed by a transformation of the coordinates that does not affect the form of the metric. This transformation reduces to an orthogonal transformation of the matrix γ_{ab}.[10] Without changing the Friedmann part of the matrix γ_{ab}, it mixes up the corrections and contains three arbitrary parameters (three Eulerian angles). In particular, using it one can reduce γ_{ab} to diagonal form at any fixed time or make equal to zero three of the coefficients α_{ab} ($a \neq b$) or β_{ab} ($a \neq b$). Of course, α_{ab} (a = b) and β_{ab} (a = b) then change. Ir the particular case when α_{ab} and β_{ab} satisfy the conditions

$$K_c\equiv 3C^b_{dc}\,\beta_{ab}\alpha^{ad}=0, \tag{17}$$

where $\alpha^{ad} = \alpha_{bc}\eta^{ab}\eta^{dc}$, the matrix γ_{ab} can be reduced to diagonal form for all times at once. As follows from Eqs. (12), this case is realized when there is no motion of the matter, i.e., in the approximation under consideration when $v_{a(2)} = 0$.

In the quadratic approximation, there are perturbations of the density and the velocity of the matter, which can be described as the back reaction of the gravitational waves on the background metric. Contracting Eq. (13) with γ_{ab} and subtracting (11), we obtain an equation that contains ϵ on the right-hand side. We write this equation in the form

$$\frac{3}{R^4}\left(\left(\frac{dR}{d\theta}\right)^2+R^2\right)=\varepsilon+\varepsilon_g. \tag{18}$$

Here, $d/d\theta = R d/dt$, and ϵ_g is the energy density of the gravitational waves and it consists of the terms that remain after the left hand side of Eq. (18) has been separated.

In the first nonvanishing approximation, ϵ_g contains quadratic combinations of the small quantities κ_{ab}. After simple calculations, we can find[4)]

$$\varepsilon_g=\frac{1}{24}\varepsilon_F\sin^2\eta\left[4\chi^2+\mu^2+4\frac{R_F'}{R_F}(\chi^2)'\right], \tag{19}$$

where

$$\chi^2=\varkappa_{11}^2+\varkappa_{22}^2+\varkappa_{33}^2+2\varkappa_{12}^2+2\varkappa_{13}^2+2\varkappa_{23}^2,$$

$$\mu^2=\varkappa_{11}'^2+\varkappa_{22}'^2+\varkappa_{33}'^2+2\varkappa_{12}'^2+2\varkappa_{13}'^2+2\varkappa_{23}'^2.$$

For a slowly varying metric of the background, when one can ignore the last term in (19), ϵ_g is an essentially positive quantity.

The gravitational influence of ϵ_g means that ϵ and R differ from their Friedmann values: $\epsilon = \epsilon_F(1 + q_{(2)})$, $R = R_F(1 + \kappa_{(2)}/6)$. In order to find $\kappa_{(2)}$ and $q_{(2)}$ explicitly, we use Eq. (18), the second approximation of Eq. (14):

$$q_{(2)}+{}^2/_3\varkappa_{(2)}-{}^1/_3\chi^2=0$$

and the approximate equation $d\eta/d\theta = 1 + \kappa_{(2)}/6$. For $\kappa_{(2)}$ we readily obtain the equation

$$\operatorname{ctg}\eta\,\varkappa_{(2)}'+\frac{1}{\sin^2\eta}(1+\cos^2\eta)\,\varkappa_{(2)}=\frac{2+\sin^2\eta}{2\sin^2\eta}\chi^2+\frac{1}{8}\mu^2+\frac{1}{2}\operatorname{ctg}\eta\,(\chi^2)'.$$

Using the result of integration of this inhomogeneous equation, whose right-hand side is a combination of the known functions $\kappa_{ab(1)}$, we write down the final expression for $q_{(2)}$:

$$\begin{aligned} q_{(2)}=&C_{1(2)}\frac{\cos\eta}{\sin^2\eta}-\frac{1}{6}\left(l^2+\frac{1}{2}p^2\right)\frac{\cos\eta}{\sin^2\eta}\ln\left|\operatorname{tg}\frac{\eta}{2}\right| \\ &-\left(\frac{3}{4}p^2+\frac{2}{3}r^2-\frac{8}{3}m^2+\frac{11}{2}l^2\right)\frac{1}{\sin^2\eta}+\frac{8}{3}n^2\operatorname{ctg}\eta \\ &-\frac{1}{3}(p^2-r^2+2l^2-2m^2)\frac{\cos\eta}{\sin^2\eta}\left(-\frac{1}{5}\cos 5\eta+\frac{1}{3}\cos 3\eta\right) \\ &-\frac{1}{3}(s^2-2n^2)\frac{\cos\eta}{\sin^2\eta}\left(-\frac{1}{5}\sin 5\eta+\frac{1}{3}\sin 3\eta\right). \end{aligned}$$

Here, we have introduced the notation

$$p^2=\sum_a\alpha_{aa}^2,\quad r^2=\sum_a\beta_{aa}^2,\quad s^2=2\sum_a\alpha_{aa}\beta_{aa},$$

$$l^2=\alpha_{12}^2+\alpha_{13}^2+\alpha_{23}^2,\quad m^2=\beta_{12}^2+\beta_{13}^2+\beta_{23}^2,\quad n^2=2(\alpha_{12}\beta_{12}+\alpha_{13}\beta_{13}+\alpha_{23}\beta_{23}).$$

The constant $C_{1(2)}$ has appeared as a result of integration of the homogeneous equation for $\kappa_{(2)}$ and is associated with a small (of second order) transformation of the time coordinate.

The perturbation of the velocity can be found from Eq. (12): $v_{c(2)}/R_F = K_c/8$, where K_c is the constant quantity defined in (17). This velocity is solenoidal since the divergence of the three-velocity reduces to the divergence $e^i_{(a);i}$ and is identically zero for the metric (1).

Finally, we give the result of integration of the equation for $\kappa_{ab(2)}$. For simplicity, we assume that α_{ab} and β_{ab} with indices $a \neq b$ are equal to zero. Then

$$\begin{aligned} \varkappa_{aa(2)}=&\frac{1}{3}\varkappa_{(2)}+\frac{1}{2}\left(\varkappa^2_{aa(1)}-\frac{1}{3}\chi^2\right) \\ &+\frac{40}{9}\frac{1}{\sin\eta}\left[(3\alpha_{aa}^2-p^2)\Phi_1+(3\beta_{aa}^2-r^2)\Phi_2+\left(3\alpha_{aa}\beta_{aa}-\frac{1}{2}s^2\right)\Phi_3\right] \\ &+\frac{1}{\sin\eta}(\alpha_{aa(2)}\cos 3\eta+\beta_{aa(2)}\sin 3\eta), \end{aligned} \tag{20}$$

where

$$\Phi_1={}^1/_2\eta\cos 3\eta-\sin 3\eta\ln\sin\eta+{}^1/_{48}(-9\sin 5\eta+52\sin 3\eta-30\sin\eta),$$

$$\Phi_2={}^1/_2\eta\cos 3\eta+{}^1/_{48}(9\sin 5\eta-4\sin 3\eta-18\sin\eta),$$

$$\Phi_3=-\eta\sin 3\eta-{}^1/_{24}(-9\cos 5\eta+4\cos 3\eta+18\cos\eta),$$

$\alpha_{aa(2)}$ and $\beta_{aa(2)}$ are arbitrary second-order constants that satisfy the conditions $\eta^{ab}\alpha_{ab(2)} = 0$ and $\eta^{ab}\beta_{ab(2)} = 0$.

The energy density ϵ_g of the gravitational waves, which causes the scale factor to deviate from R_F, leads as a result to a perturbation of the matter density $q_{(2)}$. This perturbation does not depend on the coordinates. The density of matter is the same at all points of space, but its law of variation with the time differs from the Friedmann law. The corrections ϵ_g/ϵ_F and $q_{(2)}$ increase as the singularities are approached. The term with the greatest divergence in $q_{(2)}$ has the asymptotic behavior $\eta^{-2} \ln \eta$ as $\eta \to 0$ and is caused by the divergent mode $\alpha_{ab} \cos 3\eta/\sin \eta$ in the perturbations of the metric. For a special choice of the initial data, $\alpha_{ab} = 0$, which corresponds to a particular choice of the phase of the gravitational waves, and ϵ_g/ϵ_F and all the corrections to the Friedmann solution remain small during the complete evolution right up to the singularity. Therefore, for $\alpha_{ab} = 0$ the divergent terms in $q_{(2)}$ have the dependence $4/15\,(31/3\,m^2 - 3r^2)/\eta^2$ as $\eta \to 0$. The term $C_{1(2)} \cos \eta/\sin^2 \eta$ has the same asymptotic dependence on the time. Choosing the constant $C_{1(2)}$ appropriately, one can eliminate the divergence in $q_{(2)}$. The same is true of $\chi_{aa(2)}$ in (20) subject to the additional condition $\alpha_{aa(2)} = 0$.

Similarly, one can eliminate the divergent terms in all the following approximations. This is proved in Sec. 3 by the direct construction of a class of quasi-isotropic solutions that differ little from the Friedmann solution as $\eta \to 0$.

3. QUASI-ISOTROPIC SOLUTION

Note that the elimination of the divergent terms as $\eta \to 0$ requires the vanishing of the coefficients α_{ab} in all approximations, and this is equivalent to the assumption that there are no (solenoidal) velocities of the matter. In this case, as we have noted above, the matrix γ_{ab} can be reduced to diagonal form for all times at once, i.e., if the matrix is reduced to diagonal form at a particular time, it remains diagonal subsequently by virtue of Einstein's equations. Suppose

$$\gamma_{11}={}^1/_4 R_F{}^2 e^{2a}, \quad \gamma_{22}={}^1/_4 R_F{}^2 e^{2b}, \quad \gamma_{33}={}^1/_4 R_F{}^2 e^{2c}, \quad \varepsilon=\varepsilon_F e^{\varsigma}, \quad \gamma_{12}=\gamma_{13}=\gamma_{23}=0.$$

We write the functions a, b, c, q in the form of series in even powers of η (because the perturbation of the metric and the density in the neighborhood of $\eta = 0$ found in the previous section are expanded in even powers of η):

$$a=\sum_{n=0}^{\infty} a_n \eta^{2n}, \quad b=\sum_{n=0}^{\infty} b_n \eta^{2n}, \quad c=\sum_{n=0}^{\infty} c_n \eta^{2n}, \quad q=\sum_{n=0}^{\infty} q_n \eta^{2n}. \tag{21}$$

Substituting the expansions (21) into Eqs. (11) and (13) and comparing terms with equal powers of η, we obtain a system of equations that relate the unknown coefficients. In the principal approximation ($\sim\eta^{-2}$),

$$q_0=0, \tag{22}$$

from which we see that $(\epsilon - \epsilon_F)/\epsilon_F \sim \eta^2$ as $\eta \to 0$.

With allowance for (22), Eq. (14) leads to the relations

$$a_0+b_0+c_0=\text{const}=A_0, \quad q_n+{}^4/_3(c_n+b_n+c_n)=0 \quad (n=1,\ 2,\ 3,\ldots).$$

The constant A_0 must become zero if we require that the entropy density preserve its Friedmann value.

Expressing q_n in terms of a_n, b_n, c_n and substituting this relation into the expansions of the equations (13), we obtain a system of three algebraic equations for determining a_n, b_n, c_n in each approximation in η^{2n}. Each n-th term of the expansion can be expressed in terms of a combination of the preceding terms by means of the equations

$$\begin{aligned}
&4(n^2+n+{}^1/_3)a_n+(2n+{}^4/_3)b_n+(2n+{}^4/_3)c_n\\
&=F_n{}^1(a_{n-1},\ldots,a_0;\quad b_{n-1},\ldots,b_0;\quad c_{n-1},\ldots,c_0),\\
&(2n+{}^4/_3)a_n+4(n^2+n+{}^1/_3)b_n+(2n+{}^4/_3)c_n\\
&=F_n{}^2(a_{n-1},\ldots,a_0;\quad b_{n-1},\ldots,b_0;\quad c_{n-1},\ldots,c_0),\\
&(2n+{}^4/_3)a_n+(2n+{}^4/_3)b_n+4(n^2+n+{}^1/_3)c_n\\
&=F_n{}^3(a_{n-1},\ldots,a_0;\quad b_{n-1},\ldots,b_0;\quad c_{n-1},\ldots,c_0).
\end{aligned} \tag{23}$$

The functions F_n can be reduced ultimately to combinations of the initial data a_0, b_0, c_0. The determinant of the system (23) is nonzero for all n = 1, 2, ... Thus, there always exists a nontrivial solution of this system provided that not all a_0, b_0, c_0 are equal to zero.

There are recursion relations that enable one to express the coefficients a_n, b_n, c_n in terms of the foregoing coefficients. We shall merely give as an example the explicit form of the first terms of the expansions:

$$\begin{aligned}
q_1&={}^1/_3(e^{-2a_0}+e^{-2b_0}+e^{-2c_0})-{}^1/_6(e^{2(2a_0-A_0)}+e^{2(2b_0-A_0)}+e^{2(2c_0-A_0)})-{}^1/_2,\\
a_1&={}^1/_8-{}^1/_{72}(38e^{-2a_0}-10e^{-2b_0}-10e^{-2c_0}+29e^{2(2a_0-A_0)}-19e^{2(2b_0-A_0)}-19e^{2(2c_0-A_0)}).
\end{aligned}$$

The coefficients b_1 and c_1 are obtained from a_1 by cyclic permutation of a_0, b_0, c_0. If the initial data satisfy the restrictions $|a_0|$, $|b_0|$, $|c_0| \ll 1$, we obtain a solution that is nearly Friedmann as $\eta \to 0$.

The energy density of the gravitational waves diverges as the singularity is approached, although the relative contribution ϵ_g/ϵ_F tends to zero as η^2 in the limit $\eta \to 0$.

The region of applicability of the formal solution we have constructed is determined by the radius of convergence of the series (21). It is possible that there exist solutions which differ little from the Friedmann solution during the whole of the evolution between the singularities $\eta = 0$ and $\eta = \pi$. In this case, the dynamical properties of the universe at all stages of the evolution are determined by the "ordinary" matter, and not by gravitational waves.

Note that the class of quasi-isotropic solutions is distinguished by initial data that have measure zero in the set of all initial data for type G_3 IX solutions. Nevertheless, according to the interesting arguments of Bogoyavlenskiĭ and Novikov[8] quasi-isotropic solutions are in a certain sense typical if one considers the evolution of solutions away from a singularity and not toward it. The quasi-isotropic solution is particularly important in connection with allowance for a new physical effect—creation of particles in an anisotropic gravitational field.[21,22]

4. GRAVITATIONAL WAVES IN VACUUM

In the general case, the gravitational-wave corrections and the energy density of the gravitational waves diverge near the singularity, ensuring the existence of the so-called vacuum stage,[19] when the gravitational influence of the ordinary matter is unimportant. We shall consider the limiting case when there is no ordinary matter at all, i.e., we consider the solution of the Einstein equations for the metric (1) in vacuum.

The decomposition (3) leads to the following statement of the problem. An harmonic of an external wave field (the gravitational field in the given case) is put into the isotropic universe (4) and the problem is to elucidate its behavior and find the gravitational influence it has on the evolution of the scale factor R(t).

This formulation of the problem in the case of other fields (scalar, electromagnetic) presupposes the construction of a gravitational "source" in the form of the energy–momentum tensor of the corresponding field and, possibly, averaging of this source over the spatial coordinates in order to obtain quantities that depend only on the time, which is assumed by the symmetry properties of the metric (4). The equations of gravitation then establish a connection between the Einstein tensor calculated in accordance with the metric (4) and the energy-momentum tensor of the corresponding field. The absence of sources is compatible only with the metric of a flat universe, R = const.

In the gravitational case which we now consider, the concept of an energy-momentum tensor is absent, and we are therefore forced to construct a source by writing down the Einstein equations for the metric (1) in vacuum and decomposing them into two parts. On the left, we shall have the Einstein tensor for the metric (4); on the right, all the remaining terms, which appear as a source.[5)] The homogeneity of the metric (1) ensures that the effective energy and pressure densities depend solely on the time.

As we have already noted above, in the vacuum case the matrix γ_{ab} can be made diagonal. We introduce the notation

$$\gamma_{11}={}^1/_4R^2e^{2\alpha},\quad \gamma_{22}={}^1/_4R^2e^{2\beta},\quad \gamma_{33}={}^1/_4R^2e^{2\gamma}. \tag{24}$$

Then Einstein's equations can be written in the form

$$\frac{3}{R^2}(\dot{R}^2+1)=-\Big\{(\dot\alpha\dot\beta+\dot\alpha\dot\gamma+\dot\beta\dot\gamma)+\frac{1}{R^2}e^{-2(\alpha+\beta+\gamma)}\cdot \tag{25}$$
$$\times[2(e^{2\alpha+2\beta}+e^{2\alpha+2\gamma}+e^{2\beta+2\gamma})-(e^{4\alpha}+e^{4\beta}+e^{4\gamma})-3e^{2(\alpha+\beta+\gamma)}]+2\frac{\dot R}{R}(\alpha+\beta+\gamma)^{\cdot}\Big\}\equiv\varepsilon_g,$$

$$\frac{\ddot R}{R}+2\frac{\dot R^2}{R^2}+\frac{2}{R^2}=-\Big\{\ddot\alpha+\frac{\dot R}{R}(4\alpha+\beta+\gamma)^{\cdot}+\dot\alpha(\alpha+\beta+\gamma)^{\cdot}$$
$$+\frac{2}{R^2}e^{-2(\alpha+\beta+\gamma)}[e^{4\alpha}-(e^{2\beta}-e^{2\gamma})^2-e^{2(\alpha+\beta+\gamma)}]\Big\}\equiv\frac{1}{2}(\varepsilon_g-p_g{}^{(1)}). \tag{26}$$

Two further equations can be obtained from the last equation by cyclic permutation of the functions α, β, γ. These equations determine $(\epsilon_g - p_g^{(2)})/2$ and $(\epsilon_g - p_g^{(3)})/2$. By virtue of Einstein's equations, we must have $p_g^{(1)} = p_g^{(2)} = p_g^{(3)} \equiv p_g$. A consequence of the above system is the equation

$$6\frac{\ddot R}{R}=-2\Big\{(\alpha+\beta+\gamma)^{\cdot\cdot}+2\frac{\dot R}{R}(\alpha+\beta+\gamma)^{\cdot}-(\dot\alpha^2+\dot\beta^2+\dot\gamma^2)\Big\}\equiv-(\varepsilon_g+3p_g). \tag{27}$$

If the function R is to be identical to the scale factor of the metric (4), α, β, and γ must be related by

$$e^{2\alpha}+e^{2\beta}+e^{2\gamma}=3. \tag{28}$$

If Einstein's equations are satisfied, then ϵ_g is equal to the essentially positive quantity on the left-hand side of (25), and therefore $\epsilon_g > 0$. As regards the pressure p_g, it can in general take values such that $\epsilon_g + 3p_g < 0$. This inequality is a necessary condition for the existence of a regular minimum of the function R(t), i.e., of an instant of time t_0 at which simultaneously

$$R>0,\quad \dot R=0,\quad \ddot R>0. \tag{29}$$

We shall show that a stable regular minimum at some time t_0 is possible.

By virtue of the constraint (28), we can assume that the model is described by three independent functions: R, α, β. At the initial time, we must specify the values of the functions themselves and their derivatives, and they must satisfy the relation (25), i.e., only five quantities remain independent. If we specify (29) at the initial time, not more than two parameters remain independent. Suppose these are the initial values α_0 and β_0. The nature of the connections between the initial data may be such that one cannot find any α_0 and β_0 compatible with (29); a regular minimum is then absent. If the set of α_0 and β_0 contains a region (which does not reduce to a point or line) of values compatible with (29), a regular minimum is possible, and it is stable.

Using (25), (28), and three equations of the type (26), we can reduce the question of the existence of a regular minimum to the question of the existence of solutions for $\dot\alpha$ and $\dot\beta$ in the following system of equations:

$$\dot\alpha^2e^{2\alpha}+\dot\beta^2e^{2\beta}+\dot\alpha\dot\beta[2(e^{2\alpha}+e^{2\beta})-3]+\frac{1}{R^2}e^{-2(\alpha+\beta)}A_1=0,$$
$$\dot\alpha^2e^{2\alpha}(3-e^{2\beta})+\dot\beta^2e^{2\beta}(3-e^{2\alpha})+2\dot\alpha\dot\beta e^{2(\alpha+\beta)}-\frac{1}{R^2}e^{-2(\alpha+\beta)}A_2 \tag{30}$$
$$=\frac{3}{2}\frac{\ddot R}{R}(3-e^{2\alpha}-e^{2\beta})>0;$$

where

$$A_1=9+4(e^{4\alpha}+e^{4\beta}+e^{2\alpha+2\beta}-3e^{2\alpha}-3e^{2\beta}),$$
$$A_2=3A_1+12e^{2(\alpha+\beta)}(3-e^{2\alpha}-e^{2\beta}).$$

A fairly cumbersome investigation shows that for every $\ddot{R}/R \geq 0$ there exists an entire region of α and β values for which the system (30) can be solved. This region contains α and β values satisfying the inequalities

$$A_1>e^{2\alpha}(3-2e^{2\alpha})+2e^{2\beta}(3-e^{2\beta}),\quad A_1\geqslant0,$$

but the region is not restricted to these values. The restrictions are satisfied, for example, by α and β satisfying $e^{2\alpha} \ll 1$, $e^{2\beta} \ll 1$. Thus, the model allows the existence of a stable regular minimum of the scale factor R(t).

Note that if the function R(t) were chosen proportional to the cube root of the volume element constructed in accordance with (24), i.e., subject to the condition $\alpha + \beta + \gamma = 0$, then, as readily follows from (27), a regular minimum of this scale factor would be impossible.

The solutions of Einstein's equations in vacuum for a type G_3 IX metric have been studied in detail in [6,11]. The evolution of the solutions in the direction of the singularity is made up of successive periods (called eras) during which two of the three functions γ_{11}, γ_{22}, γ_{33} oscillate, and the third decreases monotonically. On the transition to a new era, two other functions begin to oscillate, and one of those that oscillated decreases. Each era is described by a sequence of successive "Kasner" epochs, and the transition from one epoch to the next is made in accordance with a definite rule.

Using the ready solution and the representation (25)–(28), we can calculate R^2, ϵ_g, p_g and find the corresponding equation of state. It is curious that the critical times in the evolution of the solution such as, for example, the era transitions coincide with the critical times in the changing equation of state. A typical element of the solution is the Kasner epoch during which with high accuracy

$$\gamma_{11}=t^{2p_1},\quad \gamma_{22}=t^{2p_2},\quad \gamma_{33}=t^{2p_3}, \tag{31}$$

where

$$p_1=-\frac{u}{1+u+u^2},\quad p_2=\frac{1+u}{1+u+u^2},\quad p_3=\frac{u(1+u)}{1+u+u^2};$$

the constant u can lie in the range $1 \leq u < \infty$.

The solution (31) is obtained by neglecting terms without derivatives on the left-hand sides of Eqs. (25) and (26). The transition to a new Kasner epoch is accompanied by a reduction of the parameter u by unity. At the end of each Kasner epoch (which corresponds to $t \to 0$) we have approximately $R^2/4 \approx t^{2p_1}/3$ and, calculating ϵ_g and p_g, we find

$$\varepsilon_g \approx 3p_1^2 t^{-2},\quad p_g = -p_1(3p_1-2)t^{-2}.$$

Thus, we always have $\epsilon_g \geq 0$ and $p_g \leq 0$. As $u \to \infty$, the exponents p_1, p_2, p_3 tend to the set (0, 0, 1) characteristic of a flat universe, and ϵ_g and p_g vanish in this approximation. The value of u decreases on the transition to new Kasner epochs, and as a result the equation of state relating p_g to ϵ_g also changes:

$$p_g/\varepsilon_g = -{}^2/_3(u+1+1/u)-1.$$

An era ends when u = 1, which corresponds to attainment of the critical equation of state $p_g + 3\epsilon_g = 0$.

In conclusion, we note that the monotonic nature of the Taub solution[14] (two of the three functions γ_{11}, γ_{22}, γ_{33} are equal), as opposed to the oscillatory solutions in the general case of a type G_3 IX metric, can be understood on the basis of the fact that in the Taub universe the number of independent states of the gravitational waves is one less.[6)] As a result, there is a distinguished direction along which the waves do not prevent the monotonic collapse of the volume element. This can be seen particularly clearly by separating a background metric that admits the group G_4. In this case, the Taub metric does not contain gravitational waves at all, whereas the general metric of type G3 IX admits a standing gravitational wave along the distinguished direction.

[1)]The main results of this paper were reported at the Third Soviet Gravitational Conference [12] and at the 64-th Symposium of the International Astronomical Union [13].

[2)]For the explicit form of P_{ab} see, for example, [18].

[3)]We are indebted to M. Demyanskiĭ for assistance in the analysis of this question.

[4)]The prime denotes differentiation with respect to the time η.

[5)]Decompositions of this kind have frequently been studied, the first time by Einstein himself. Usually, they are considered in a scheme of successive approximations, although attemps have also been made to do this in an exact theory [23] (see also [25]).

[6)]We recall Wheeler's [24] characterization of the Taub metric as a universe with a standing gravitational wave.

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