$$U_{\rm th}^2=\frac{8\pi^3}{\varepsilon_\perp}\frac{\sigma_\parallel+\sigma_\perp}{\sigma-\sigma_\perp}\left(\frac{3}{2}K_{22}^*K_{33}^*\right)^{1/2}\frac{n}{2}. \qquad (4)$$

This relationship describes the jumps of $U_{\rm th}$ resulting from variation of n (with the exception of the $B\to A$ transition corresponding to complete untwisting of a CLQ into an NLQ). The dependence of $U_{\rm th}$ on L within one region can be explained simply by assuming that K_{22}^* and K_{33}^* vary with L. Thus, the oscillatory nature of the dependences of $U_{\rm th}$ and λ on L can be explained qualitatively.

Experimental studies of the Fréedericksz transition and untwisting of cholesteric helices is an electric field gave the values of $K_{33}=1.0\times10^{-6}$ dyn and $K_{22}=0.3\times10^{-6}$ dyn. Figures 1–3 include the dependences of $U_{\rm th}$ and λ on L calculated on the basis of Eqs. (1) and (2) for the equilibrium value of the pitch. We can see that the Helfrich–Hurault theory (derived for $L\gg P_0$) describes only qualitatively the average dependence of $U_{\rm th}$ and λ on L for $L\sim P_0$.

The authors are grateful to M. I. Barnik for measurements of the anisotropy of the electrical conductivity and permittivity components, and for a discussion of the results.

[1)] Mixture A consists of two parts of p-n-butyl-p'-methoxy-azoxybenzene and one part of p-n-butyl-p'-heptanoyloxy-azoxybenzene.

[1] W. Helfrich, Appl. Phys. Lett. 17, 531 (1970).
[2] W. Helfrich, J. Chem. Phys. 55, 839 (1971).
[3] J. P. Hurault, J. Chem. Phys. 59, 2068 (1973).
[4] H. Arnould-Netillard and F. Rondelez, Mol. Cryst. Liq. Cryst. 26, 11 (1974).
[5] H. Hervet, J. P. Hurault, and F. Rondelez, Phys. Rev. A8, 3055 (1973).
[6] T. J. Scheffer, Phys. Rev. Lett. 28, 593 (1972).
[7] M. I. Barnik, L. M. Blinov, M. F. Grebenkin, S. A. Pikin, and V. G. Chigrinov, Phys. Lett. A51, 175 (1975).
[8] M. I. Barnik, L. M. Blinov, M. F. Grebenkiu, S. A. Pikin, and V. G. Chigrinov, Zh. Eksp. Teor. Fiz. 69, 1080 (1975) [Sov. Phys.-JETP 42, 550 (1975)].
[9] L. M. Blinov, V. A. Kizel, V. G. Rumyantsev, and V. V. Titov, J. Phys. (Paris) 36, Colloq. C1-69 (1975).

Translated by A. Tybulewicz

Nonlinear Raman interaction between first and second sounds in liquid helium II

N.I. Pushkina and V. K. Rudenko

Nuclear Physics Institute of the Moscow State University
(Submitted June 25, 1975)
Zh. Eksp. Teor. Fiz. 70, 191–195 (January 1976)

The nonlinear Raman interaction between first and second sounds in liquid helium II, i.e., the parametric excitation of second sound by first sound in a resonator, is considered. An expression is obtained for the threshold first-sound intensity. The intensities of the stationary waves are found. The deviation from exact synchronism of the interacting waves and the difference between the frequencies of these waves and the natural frequencies of the resonator are taken into account. The stability of the obtained solutions is investigated.

PACS numbers: 67.40.Tr

It is well known that liquid helium II begins to manifest nonlinear properties when sufficiently intense first- and second-sound waves propagate in it. Thus, Osborne[1] observed the formation of second-sound shock waves, the theory of which is given in Khalatnikov's paper,[2] where first-sound shock waves are also considered. The formation of shock waves is, in essence, a self-action of waves, and is not connected with nonlinear mixing of waves of different nature. Another type of nonlinear interaction—the so-called nonlinear Raman interaction connected with the nonlinear intermixing of first- and second-sound waves—is also possible. In the present paper we consider one of the examples of the latter interaction, namely, the parametric excitation of second-sound waves at the expense of the exciting first-sound waves. This process is, in particular, of interest from the point of view of the generation of high-frequency second sound. Such an interaction was considered in[3] for the case of traveling waves. However, since the threshold first-sound intensity starting from which second-sound wave excitation becomes possible essentially depends on the attenuation of the waves,[3] a more favorable case for the experimental observation of the process in question is the case of waves in a resonator, since losses in a resonator with a sufficiently high Q are less than in traveling waves. In view of this, in the present paper we consider the parametric excitation of second sound by first sound for the case of waves in a resonator.

In solving the problem, we proceed from the nonlinear equations of the hydrodynamics of a superfluid liquid,[4] transformed into a form suitable for our problem.[3] These equations are coupled, nonlinear wave equations for pressure and temperature. Since we can assume the nonlinearity of the medium to be sufficiently slight and the losses to be sufficiently small, we shall use the method of slowly varying amplitudes (see, for example, [5]) to solve the basic system of equations. We shall seek the solution in the form of a sum of three standing waves[1)]: the exciting first-sound wave

$$\tfrac{1}{4}\{P(t)\exp i(k_0x-\omega_0t)+P(t)\exp[-i(k_0x+\omega_0t)]+\text{c.c.}\} \tag{1a}$$

and two temperature waves (second-sound waves).

$$\tfrac{1}{4}\{T_{1,2}(t)\exp i(k_{1,2}x-\omega_{1,2}t)+T_{1,2}(t)\exp[-i(k_{1,2}x+\omega_{1,2}t)]+\text{c.c.}\}, \tag{1b}$$

where the complex amplitudes P, T_1, and T_2 are slowly varying functions of the time. Substituting (1a) and (1b) into the system of equations (1) from[3], we obtain the following system of the so-called truncated equations for the amplitudes $T_1(t)$, $T_2(t)$, and $P(t)$:

$$\begin{aligned}\frac{dT_1}{dt}+\frac{1}{2}(\delta-i\Delta\omega)T_1&=\frac{e^{-i\Delta kl}-1}{-i\Delta kl}iB_1PT_2^{*},\\ \frac{dT_2}{dt}+\frac{1}{2}(\delta-i\Delta\omega)T_2&=\frac{e^{-i\Delta kl}-1}{-i\Delta kl}iB_2PT_1^{*},\\ \frac{dP}{dt}+\frac{1}{2}\alpha P&=\frac{e^{i\Delta kl}-1}{i\Delta kl}iB_0T_1T_2+\lambda.\end{aligned} \tag{2}$$

Here δ and α are the decay coefficients for the waves

$$B_{1,2}=\frac{\omega_{1,2}A}{8\rho_s},\qquad B_0=\frac{S^2\rho^2\omega_0}{8\rho_n}\left(\frac{c_1}{c_2}\right)^2A,$$

ρ_n and ρ_s are the normal and superfluid densities, ρ is the density of the medium, S is the entropy, c_1 and c_2 are the velocities of first and second sounds, and

$$A=\left(1+2\frac{\rho_s}{\rho}\right)\frac{1}{c_1^2}-\frac{\rho}{\rho_n}\frac{\partial\rho_n}{\partial p}.$$

In contrast to the case of running waves,[3] here we take into account the deviation of the frequencies ω_1 and ω_2 of the temperature waves from the eigenfrequencies ω_{10} and ω_{20} of the resonator and the deviation of the wave numbers from synchronism:

$$\omega_1-\omega_{10}=\Delta\omega_1,\qquad \omega_2-\omega_{20}=\Delta\omega_2, \tag{3a}$$

$$\mathbf{k}_0-\mathbf{k}_1-\mathbf{k}_2=\Delta\mathbf{k}. \tag{3b}$$

It is not difficult to show that the frequencies satisfy the relations

$$\omega_0-\omega_1-\omega_2=0,\qquad \omega_0-\omega_{10}-\omega_{20}=\Delta\omega,$$

where $\Delta\omega_1=\Delta\omega_2=\Delta\omega/2$. For the collinear case under consideration by us the equality (3b) has the form

$$k_1-k_0-k_2=\Delta k.$$

The parameter λ in (2) describes the effect of the sound source (the piezoelectric plate) concentrated in a plane perpendicular to the resonator axis. We have $\lambda=c_1^2\rho u_0/l$, where u_0 is the amplitude of the velocity of the oscillations of the piezoelectric crystal. In the presence of a sound source on the right-hand side of the basic nonlinear wave equation for the pressure ((1) from [3]), there arises the additional term

$$c_1^2\rho\frac{\partial u}{\partial t}\delta(x)$$

(see, for example, [6]), where u is the velocity of the oscillations of the piezoelectric crystal ($\delta(x)$ is the δ-function); when we go over to the truncated equations (2), this term gets transformed into the parameter λ.

We shall be interested in the steady states of the system. Let us find them from the Eqs. (2), setting $dT_{1,2}/dt=dP/dt=0$. We obtain that two different steady-state regimes are possible in the system. The first of them is realized at a value of λ less than some threshold value λ_{thr}; in this case only pressure, and not temperature, waves are excited: $P=2\lambda/\alpha$; $T_1=T_2=0$. The second steady-state regime is possible when $\lambda>\lambda_{\text{thr}}$; in this case besides the pressure wave there also exist stationary second-sound oscillations ($T_1\neq 0$, $T_2\neq 0$). The quantity λ_{thr} and the corresponding threshold pressure P_{thr} then turn out to be equal to:

$$\begin{aligned}\lambda_{\text{thr}}&=\frac{1}{2}\alpha P_{\text{thr}},\qquad |P_{\text{thr}}|^2=\frac{\delta^2(1+\tilde{\Delta}^2)}{4B_1B_2F(\Delta kl)},\\ \tilde{\Delta}&=\Delta\omega/\delta,\qquad F(\Delta kl)=\frac{\sin^2(\Delta kl/2)}{(\Delta kl/2)^2}.\end{aligned} \tag{4}$$

Notice that an analogous situation in nonlinear optics, but without allowance for the detunings $\Delta\omega$ and Δk, is described in[7].

The minimum threshold value is naturally realized in the case when $\Delta\omega=\Delta k=0$:

$$|P_{\text{thr}}|^2_{\min}=\delta^2/4B_1B_2.$$

Let us numerically estimate the threshold intensity J_{thr} of the sound at $T=1.5\,^\circ\text{K}$. Let us assume that

$$\begin{gathered}\omega_0=2\cdot 2\pi\cdot 10^5\ \text{sec}^{-1},\qquad \omega_1\approx\omega_2\approx\omega_0/2,\\ c_1=2.3\cdot 10^4\ \text{cm/sec}\qquad c_2=2\cdot 10^3\ \text{cm/sec}\\ \rho=0{,}14\ \text{g/cm}^3\quad \rho_s=0.9\rho,\quad \rho_n=0.1\rho,\quad Q=10^4\end{gathered}$$

(Q denotes the Q-factor of the resonator).

Let us estimate the quantity $\partial\rho_n/\partial\rho$, using the data on the pressure dependence of the quantities: the velocity c_2[8] and the entropy and specific heat.[9] We obtain

$$(J_{\text{thr}})_{\min}=\frac{|P_{\text{thr}}|^2_{\min}}{4\rho c_1}\approx(10^{-3}-10^{-4})\ \text{W/cm}^2.$$

The first-sound intensity J_{thr} necessary for the excitation of second sound in a resonator is an order-of-magnitude less than the intensity in the case of running waves.[3]

Let us now find the steady-state values of the amplitudes of the interacting waves for $\lambda>\lambda_{\text{thr}}$. As λ is increased above the threshold value, the amplitude of the pressure remains constant: $|P|=|P_{\text{thr}}|$, since, as can be seen from the system (2), no steady-state solutions for T_1 and T_2 exist when $|P|>|P_{\text{thr}}|$. Consequently, any increase (as a result of the increase of λ) in the energy entering into the resonator is wholly converted into parametrically excited second-sound waves T_1, T_2. Thus, in the steady-state regime $|P|=|P_{\text{thr}}|$ and the values of $|T_1|$ and $|T_2|$ depend on the value of λ. Let us write them out explicitly, introducing the quantities J_0, J_1, and J_2, which have the meaning of energy dissipation per unit time in each of the three waves ($J_0=E_0\alpha$, $J_{1,2}=E_{1,2}\delta$, where $E_{0,1,2}$ is the energy density in the corresponding wave):

$$J_0=|P_{\text{thr}}|^2\alpha/4\rho c_1^2, \tag{5a}$$

$$J_{1,2}=\rho\rho_sS^2|T_{1,2}|^2\delta/4\rho_nc_2^2. \tag{5b}$$

From (2) we obtain

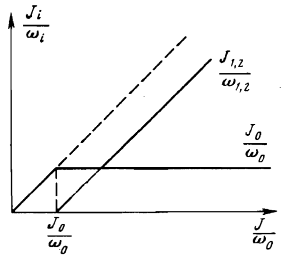

FIG. 1. The wave energy multiplied by the corresponding attenuation factor as a function of the power fed into the resonator. $J_0=E_0\alpha$; $J_{1,2}=E_{1,2}\delta$, where $E_{0,1,2}$ is the energy density in the wave.

$$J_{1,2}=\frac{\omega_{1,2}}{\omega_0}J_0\frac{\lambda}{\lambda_{thr}}\frac{1}{1+\tilde{\Delta}^2}\left(\sqrt{1+\left[1-\left(\frac{\lambda_{thr}}{\lambda}\right)^2\right]\tilde{\Delta}^2}+\frac{\lambda_{thr}}{\lambda}\tilde{\Delta}^2\right)-\frac{\omega_{1,2}}{\omega_0}J_0. \qquad (6)$$

In order to understand the structure of the expression for $J_{1,2}$, let us find the total energy dissipation J per unit time (in all the three waves):

$$J=J_0+J_1+J_2=J_0\frac{\lambda}{\lambda_{thr}}\frac{1}{1+\tilde{\Delta}^2}\left(\sqrt{1+\left[1-\left(\frac{\lambda_{thr}}{\lambda}\right)^2\right]\tilde{\Delta}^2}+\frac{\lambda_{thr}}{\lambda}\tilde{\Delta}^2\right). \qquad (7)$$

It can be seen from this that

$$J_{1,2}=\frac{\omega_{1,2}}{\omega_0}J-\frac{\omega_{1,2}}{\omega_0}J_0. \qquad (8)$$

Consequently, for a given power input J the dependence of the obtainable power outputs J_1 and J_2 on the detunings is determined by the second term in (8) (or (6)), where J_0 is a known function of the detunings (see (4) and (5a)).

Notice that in the expression (7) for J

$$J_0\frac{\lambda}{\lambda_{thr}}=\frac{|P_{thr}|u}{2l},$$

$$\frac{1}{1+\tilde{\Delta}^2}\left(\sqrt{1+\left[1-\left(\frac{\lambda_{thr}}{\lambda}\right)^2\right]\tilde{\Delta}^2}+\frac{\lambda_{thr}}{\lambda}\tilde{\Delta}^2\right)=\cos\varphi, \qquad (9)$$

where φ is the phase shift between P and the velocity u of the piezoelectric crystal ((9) can be obtained from (2)). Consequently, as it should be, the quantity J is equal to the work done per unit time by the piezoelectric crystal against the pressure P for unit volume:

$$J=\frac{|P_{thr}|u}{2l}\cos\varphi=\frac{\overline{Pu}}{l}.$$

The quantities J_0, J_1, and J_2 as functions of the input energy are shown in Fig. 1.

The stability of the above-obtained steady-state solutions against infinitesimal perturbations of the amplitudes and phases was investigated, using the method expounded in[10]. The analysis of the stability equation in the complex perturbation-"frequency" plane yielded the following results (see Fig. 2). There always exists, when the pump slightly exceeds the threshold, a region of unstable motions with an upper boundary determined by the equation

$$\left(\frac{\lambda}{\lambda_{thr}}\right)^2_{cr}=\frac{\sqrt{1+2\tilde{\Delta}^2}+1+3\tilde{\Delta}^2}{2(1+\tilde{\Delta}^2)}.$$

For pump values exceeding the critical value, i.e., for $\lambda/\lambda_{thr}>(\lambda/\lambda_{thr})_{cr}$, the boundaries of the region of stable, steady-state motions may expand or contract, depending on the ratio α/δ of the attenuation constants for the first and second sounds. Figure 3 shows the regions of stable and unstable values of the parameter α/δ. Thus, by increasing the losses with respect to the first sound (or decreasing the losses with respect to the second sound), we expand the region of stable motions in the system. And, conversely, by decreasing the losses with respect to the first sound (or increasing the losses with respect to the second sound), we make the region of admissible stable motions shrink. The boundary of the stable region is determined by the equation

$$\alpha/\delta=\sqrt{1+2\tilde{\Delta}^2}-1.$$

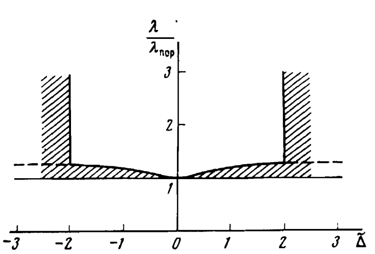

FIG. 2. The regions of stable and unstable (hatched) steady-state motions in the system for $(\alpha/\delta)=2$, as functions of $\tilde{\Delta}$.

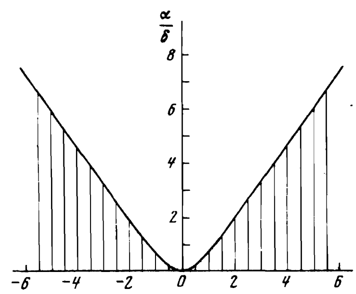

FIG. 3. The regions of stable and unstable (hatched) values of α/δ, as functions of $\tilde{\Delta}$.

The authors are sincerely grateful to R. V. Khokhlov for proposing the problem and supervising the work.

1) We consider the one-dimensional case. The pressure and temperature have antinodes at the resonator boundaries $x=0$ and $x=l$. The wave numbers $k_{0,1,2}$ are equal to: $k_{0,1,2}=\pi n_{0,1,2}/l$ ($n_{0,1,2}$ are whole numbers).

[1] D. V. Osborne, Proc. Phys. Soc. A64, 114 (1951).
[2] I. M. Khalatnikov, Zh. Eksp. Teor. Fiz. 23, 253 (1952).
[3] N. I. Pushkina and R. V. Khokhlov, Pis'ma Zh. Eksp. Teor. Fiz. 19, 672 (1974) [JETP Lett. 19, 348 (1974)].
[4] L. D. Landau and E. M. Lifshitz, Mekhanika sploshnykh sred (Fluid Mechanics), Gostekhizdat, 1953 (Eng. Transl., Pergamon, London, 1959).
[5] S. A. Akhmanov and R. V. Khokhlov, Problemy nelineĭnoĭ optiki (Problems of Nonlinear Optics), 1964 (Eng. Transl., Gordon & Breach, New York, 1972).
[6] P. Morse, Vibrations and Sound, McGraw-Hill, New York, 1948 (Russ. Transl., Gostekhizdat, 1949).
[7] A. Yariv and W. H. Louisell, IEEE J. Quantum Electron. **QE**-2, 418 (1966).
[8] V. P. Peshkov and K. N. Zinov'eva, Zh. Eksp. Teor. Fiz. 18, 438 (1948).
[9] I. Wilks, The Properties of Liquid and Solid Helium, Clarendon Press, Oxford, 1967.
[10] Yu. V. Grigor'ev, V. K. Rudenko, and R. V. Khokhlov, Izv. Vyssh. Uchebn. Zaved. Radiofiz. 9, 932 (1966) [Sov. Radiophys. 9, 545 (1966)].

Translated by A. K. Agyei