$\mathcal{R}=\mathcal{K}+\mathcal{K}\mathcal{R}$ with a short-range kernel $\mathcal{K}$; equations of this type can frequently be solved in the diffusion approximation, and in this case $\mathcal{R}$ is determined from an equation describing stationary diffusion (with absorption or multiplication and a source) in the channel space. The only exception is the case of resonances of the second type, when a long-range interaction of sorts enters into the channel space, due to the proximity to the stationary (or quasistationary) state of the aggregate of channels.

8. CONCLUSION

The analysis presented in this paper is valid only for a system with a very large number of channels, open or closed. The approach indicated above is in essence statistical, although not in the usual sense employed, for example, in nuclear physics. The assumptions (a) and (b) formulated in Sec. 1 are sufficient to make the formulas derived in the preceding sections correct. A necessary condition here is (b), while the first condition can apparently be made much less stringent.

The classification introduced by us for the resonances is not formal but physical. For example, at a given input channel $\bar{\varkappa}$, the cross sections $\sigma_{\varkappa\bar{\varkappa}}$ of different reactions, which are proportional to $|S(\varkappa,\bar{\varkappa})|^2$, will have entirely different dependences on $\varkappa$. This can be easily seen by comparing formulas (36), (44), and (55). We shall not discuss this in detail at present.

One final remark. In all real cases the channel index has several (n) components and can be regarded as vector in n-dimensional space. In particular, two components of the index $\varkappa$ describe the excitation energies of two particles of the channel. As the total excitation energy is gradually increased, we ultimately fall into the region of closed channels. On the other hand, the diffusion laws, as is well known, depend essentially on the dimensionality of the space in which the diffusion takes place. It would therefore be quite incorrect to confine oneself in the approach developed above, only to allowance for the open channels. All the channels, open and closed, must be taken into account in the scheme.

To apply the procedure described in the preceding section to the calculation of concrete systems it is necessary, first, to renumber the channels in correspondence with the requirements of Sec. 1 and, second, to introduce an explicit expression for the kernel K or for the diffusion coefficient. The channels can be numbered by using physical considerations. The form of the kernel K or of the diffusion coefficient should, as a rule, be chosen by starting from the experimental data. Both questions are the subject of a separate investigation.

[1)]This can be done accurately by expanding all the corresponding quantities in powers of $\varkappa-\varkappa'$, but the small corrections that result from this procedure are of no interest.

[2)]The case $G_1<0$ when $\xi\gg1$ is considered in a perfectly analogous manner, but the expansion must now be in powers of ξ^{-1} and not ξ. The formulas of Secs. 4–7 are then slightly modified, but all the physical results remain the same.

[1]A. I. Baz' and M. V. Zhukov, Yad. Fiz. **16**, 60, 958 (1972) [Sov. J. Nucl. Phys. **16**, (1973)].

[2]C. Ngo, B. Tamain, J. Galin, M. Beiner, and R. Lombard, Nucl. Phys. **A240**, 353 (1975).

[3]V. I. Smirnov, Kurs vysshei matematiki (Course of Higher Mathematics), Vol. IV, Gostekhizdat, 1951, Chap. 1, p. 22.

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Bound states of electron-positron pairs in a strong electric field

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It is shown that taking account of the interaction leads to a splitting of electron-positron levels in the field of the nucleus. The levels differ by the number of pairs, each pair consisting of an electron in the K shell and a positron in a quasistationary state. A bound state arises in a small range of Z. The energy spectrum of the positrons, which are emitted upon critical approach of heavy nuclei, should contain several maxima which differ in energy by 10 to 30 keV.

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As is well known, the Dirac equation in the field of a point charge loses meaning at $Z>Z_c=137$. In actual fact the ground state energy is of the form ($\hbar=m=c=1$)

$$\varepsilon_0=[1-(Ze^2)^2]^{1/2}$$

and becomes imaginary for $Z>137$. Allowance for the finite size of the nucleus[1–3] removes this difficulty. However, at a value $Z\approx170$ the energy of the lowest state reaches the value $\varepsilon_0=-1$ and the total energy of a pair vanishes, that is, the vacuum becomes unstable with respect to the creation of electron-positron pairs. Thus, at $Z=Z_c$ the Dirac equation loses the meaning of

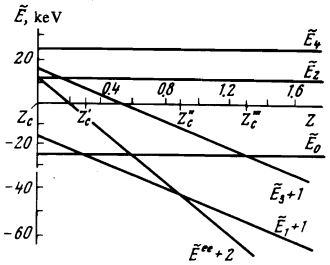
an equation for a single particle. If the K shell is not occupied, two pairs can be created; if there is one electron in the K shell, then according to the Pauli exclusion principle only the creation of a single pair is possible; and, finally, in the case of a filled shell the vacuum remains stable in spite of the appearance of a level with $\varepsilon_0 = -1$. Analogous phenomena for the Klein–Gordon equation (a particle having spin zero) were investigated in detail earlier,[4] and it was shown that the difficulty is removed upon taking account of the interaction between the particles.

In order to be definite we shall talk about π^+ and π^- mesons. A bound state of $\pi^+ - \pi^-$ pairs appears for a potential well depth less than the critical depth, $U < U_c$, but for $U = U_c$ the energy of the bound pair vanishes and the vacuum becomes unstable. It is necessary to solve not the single-particle problem, but rather the problem concerning the field of π^+, π^- mesons with their interactions taken into consideration. In the "dangerous" states such a number of $\pi^+ - \pi^-$ pairs is accumulated that further pair production becomes energetically unfavorable due to their interaction (it is assumed that the interaction is repulsive: $H' = \lambda\varphi^4$, $\lambda > 0$). As a result the effective potential acting on a single pion, which consists of the potential of the external field and the potential of the remaining pions, does not pass through the critical value, which guarantees stability of the vacuum.

In the case of Fermi particles the situation is quite different. As a consequence of the Pauli exclusion principle a large number of particles cannot be accumulated, and their influence only slightly modifies the external field. As we shall see below, the interaction between the particles turns out to have a significant influence on the position of the levels only near the critical value of Z, since degeneracy exists for $Z = Z_c$. In fact, in the case of an unoccupied K shell (a channel with charge $Q = 0$) the energy of the three possible states is the same without taking the interaction into account: 1) the bare nucleus, 2) the nucleus with a single pair, and 3) the nucleus with two pairs.

For the case of a single electron in the K shell (the channel $Q = -e$), the energy of the two states is the same: 1) one electron and 2) electron + pair. The state with two electrons ($Q = -2e$) is not degenerate. As Zel'dovich and Popov showed,[5] the vacuum becomes restructured at $Z > Z_c$—the ground state corresponds to the state with charge $-2e$. For $Z - Z_c \ll Z_c$ this charge is distributed in space with a density similar to the charge distribution in the K shell for $Z = Z_c - 0$, i.e., the charge is localized near the nucleus. The transition to this state is accomplished as the result of the creation of one or two electron-positron pairs, the positrons withdraw to infinity but the electrons are distributed near the nucleus, forming a new vacuum state.

These results were obtained without taking the interaction between electrons and positrons into consideration. The goal of the present article is an investigation of this problem in field theory with allowance for the interactions. The main simplification, permitting us to solve the problem, consists in the fact that it is sufficient to treat only the production of pairs consisting of an electron in the K shell and a low energy positron. All remaining states are separated in energy from the ground state by an amount $\sim mc^2$, and taking them into consideration gives small corrections ($\sim 1/137$). In addition, a simple form is utilized for the wave function of the slow positrons for $Z > Z_c$, which allows us to reduce the integral equation for the determination of the system's energy to a simple algebraic equation.



The physical significance of the results consists in the following. At $Z > Z_c$ a long-lived quasistationary state of the positron appears, described by a wave function which is similar to the Ψ function of a K electron. As a result of the interaction the degenerate states indicated above are intermixed, where a pair corresponds to an electron in the K shell and a positron in the quasistationary state.

In the case of an unoccupied K shell, three levels arise having a splitting that does not depend on $Z - Z_c$ (to first order in e^2). These levels describe the system consisting of 0, 1, and 2 pairs. In the case of a K shell containing a single electron, two levels appear having a splitting of the same order of magnitude—see the figure. As is clear from the figure, upon allowance for the interaction the ground state of the system up to $Z_c' - Z_c \approx 0.31$ corresponds to zero charge, it corresponds to charge $-1e$ in the interval $Z_c'' > Z > Z_c'$, and finally it corresponds to charge $-2e$ for $Z > Z_c''$ ($Z'' - Z_c \approx 0.87$). All states become quasistationary for $Z > Z_c''$. In accordance with these results, during the collision of two heavy nuclei there is an emission of positrons with an energy spectrum which has several sharp maxima corresponding to transitions between the indicated split states.

It seems to us that problems of similar type may arise in connection with investigations of the levels of an impurity atom in a dielectric or semiconductor when the level sinks below the occupation level of the band. Possible generalizations are left to the reader—only the problem of bound pairs in the Coulomb field near the critical charge will be investigated below.

THE ELECTRON-POSITRON INTERACTION

In what follows we shall only be interested in the states of electrons and positrons in the absence of quanta; therefore, one can use the Lagrangian of the system of electrons and quanta, averaged over the ground state of the electromagnetic field:

$$\mathcal{L}' = \frac{1}{2}\int j_\mu(x) D_{\mu\nu}(x-x') j_\nu(x')\, dx\, dx'; \qquad (1)$$

here $j_\nu(x)$ denotes the current operator:

$$j_\nu(x)=N(\bar{\hat{\Psi}}(x)\gamma_\nu\hat{\Psi}(x)),$$

where N is the symbol for the normal product.

To lowest order in e^2 the propagator $D_{\mu\nu}$ satisfies the equation

$$\Box D_{\mu\nu}(x)=4\pi\delta_{\mu\nu}\delta(x). \tag{2}$$

For our purposes it is most convenient of all to determine $D_{\mu\nu}(x)$ in the coordinate representation with respect to r and in the Fourier representation with respect to time:

$$\Delta D_{\mu\nu}(\mathbf{r},\ \omega)+\omega^2 D_{\mu\nu}(\mathbf{r},\ \omega)=4\pi\delta(\mathbf{r})\delta_{\mu\nu},$$

hence

$$D_{\mu\nu}=-r^{-1}e^{\pm i\omega r}\delta_{\mu\nu}. \tag{3}$$

Evaluation of (3) in the momentum representation gives

$$D_{\mu\nu}(k^2)=(4\pi/k^2)\delta_{\mu\nu},$$

by which our choice of the gauge of $D_{\mu\nu}$ is determined. The normalization of $D_{\mu\nu}$ is taken such that for two slow electrons the interaction goes over into the Coulomb interaction: $V=e^2/|\mathbf{r}-\mathbf{r}'|$.

The Lagrangian (1) corresponds to the following term in the Hamiltonian:

$$H'=-\frac{1}{2}\int j_\mu(\mathbf{r})D_{\mu\nu}(\mathbf{r}-\mathbf{r}',\tau)e^{-iH_0\tau}j_\nu(\mathbf{r}')e^{iH_0\tau}d\mathbf{r}\,d\mathbf{r}'\,d\tau. \tag{4}$$

The maximum element of the interaction operator between two arbitrary states s and s' has the form

$$H'_{ss'}=\frac{1}{2}\int (j_\nu)_{ss_1}(j_\nu)_{s_1s'}\frac{\exp(-iE_{s's_1}|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{r}\,d\mathbf{r}', \tag{5}$$

where summation over s_1 is to be understood and $E_{ss_1}=E_s-E_{s_1}$.

The matrix elements of the operator j_ν correspond to the creation or the annihilation of a pair. We shall see below that in this problem only positron states with small momenta $k\ll 1$ are important, while the electron vanishes or is created in the K shell. For $Z\approx Z_c$ the energy of such pairs is close to zero, $E_{ss_1}\ll 1$. Meanwhile the distances $|\mathbf{r}-\mathbf{r}'|$ are determined by the dimensions of the K shell and $|\mathbf{r}-\mathbf{r}'|\sim 1$. Therefore, the factor $\exp(iE_{ss_1}|\mathbf{r}-\mathbf{r}'|)$ can be omitted in expression (4). As a result the interaction H' takes the form

$$H'=\frac{e^2}{2}\int\frac{N(\hat{\Psi}^+(\mathbf{r})\alpha_\nu\hat{\Psi}(\mathbf{r}))N(\hat{\Psi}^+(\mathbf{r}')\alpha_\nu\hat{\Psi}(\mathbf{r}'))}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{r}\,d\mathbf{r}', \tag{6}$$

where $\alpha_\nu=\gamma_0\gamma_\nu$ are the Dirac matrices.

To find the matrix elements of H', in Eq. (6) one should substitute the wave functions of the initial and final states instead of the quantum operators $\hat{\Psi}$ (see the analogous calculations in[6]).

THE POSITRON WAVE FUNCTION

The wave functions of low energy positrons are needed for the subsequent calculations. One can show[7] that for $Z>Z_c$ a quasistationary level appears for a positron energy $\varepsilon_0=1+\beta(\zeta-\zeta_c)$, $\zeta=Ze$; the level width γ is determined by penetration of the Coulomb barrier. Expressions for β and γ are given below. In this connection the wave function of a positron of small energy $\varepsilon-1=k^2/2\ll 1$ may be written in the form

$$\Psi_\varepsilon(\mathbf{r})=\Delta^{1/2}(\varepsilon)\bar{\Psi}_0(\mathbf{r}),\quad r\ll 1/k, \tag{7}$$

where $\bar{\Psi}_0(r)$ is obtained from the wave function of a K electron (for $Z=Z_c$) by the operation of charge conjugation, and $\Delta(\varepsilon)$ is a function of the energy having the form of a Breit-Wigner resonance:

$$\Delta(\varepsilon)=\gamma/\pi[(\varepsilon-\varepsilon_0)^2+\gamma^2]. \tag{8}$$

The wave function (7) corresponds to $j=1/2$. Thanks to the resonance factor (8) these states give a major contribution to the matrix elements evaluated below. Relations (7) and (8) correspond to normalization to a δ-function with respect to the energy:

$$\int\Psi_\varepsilon(\mathbf{r})\Psi_{\varepsilon'}(\mathbf{r})d\mathbf{r}=\delta(\varepsilon-\varepsilon').$$

In this connection

$$\int_{-\infty}^{\infty}\Delta(\varepsilon)d\varepsilon=1.$$

Let us cite certain relationships[6] in order to clarify these assertions and also for subsequent calculations. In a spherical field the solution of the Dirac equation may be represented in the form

$$\Psi_{jlM\varepsilon}=\begin{Bmatrix} f_\varepsilon(r)\Omega_{jlM}(\mathbf{n}) \\ g_\varepsilon(r)\Omega_{jl'M}(\mathbf{n}) \end{Bmatrix}, \tag{9}$$

where Ω_{jlM} is a spherical spinor; j and M denote the angular momentum and its component ($j=l+1/2$), l is the orbital angular momentum, and $l'+l=2j$. The relationship

$$\Omega_{jl'M}=-(\sigma\mathbf{n})\Omega_{jlM}, \tag{10}$$

where $\mathbf{n}=\mathbf{r}/r$, is valid for the angular functions Ω_{jlM} and $\Omega_{jl'M}$. We have the following equations for the radial functions $F=rf_\varepsilon$ and $G=rg_\varepsilon$ corresponding to an energy ε:

$$\frac{dG}{dr}+\frac{\varkappa}{r}G-(\varepsilon+1-V)F=0,\quad \frac{dF}{dr}-\frac{\varkappa}{r}F+(\varepsilon-1-V)G=0, \tag{11}$$

where $\varkappa=\mp(j+1/2)$; the sign of $\varkappa$ is chosen in accordance with the sign in the relation $j=l\pm 1/2$. The sign of $\varkappa$ together with the quantities j, M, and ε is an integral of the motion.

At $Z\lesssim Z_c$ the wave function of a K electron corresponds to the values

$$j=1/2,\quad \varkappa=-1,\quad \varepsilon=-1+\beta(\zeta_c-\zeta),$$

where β coincides with the quantity introduced above for the description of the position of the positron quasistationary level for $Z>Z_c$. Upon the replacements $F\rightleftarrows G$ and $\varkappa\to-\varkappa$ (charge conjugation) the system (11) describes a positron with energy $-\varepsilon$ moving in the field $-V$. Thus, for $Z=Z_c$ the system (11) simultaneously describes an electron in the K shell with energy $\varepsilon=-1$ and a positron with energy $\varepsilon=1$. Upon an increase of Z, a formal solution of the system (11) exists for $\varepsilon=\varepsilon_0+i\gamma$, where $\varepsilon_0=-1-\beta(\zeta-\zeta_c)$, which also indicates the

energence of a quasistationary level in the positron spectrum.

We note that the wave function of a positron with energy $\varepsilon \approx 1$ (or of an electron with energy $\varepsilon \approx -1$) is expressed in terms of the solution of the Schrödinger equation with energy $(\varepsilon^2-1)/2$ and an effective potential $U(r)$.[8] In the Coulomb case of interest to us, when $U(r)=\zeta/r$ (for $r>R$, where R denotes the nuclear radius), the effective potential has the form

$$U(r)=\frac{\zeta}{r}-\frac{\zeta^2-\varkappa^2+1/4}{2r^2}. \tag{12}$$

At $r<R$ this expression is cut off by the influence of the nucleus, and the problem retains physical meaning in spite of the "collapse" to the center, the condition for which is given by $\zeta^2-\varkappa^2>0$ (see, for example,[9]). For $Z>Z_c(\varkappa^2=1,\ \zeta_c^2>1)$ the effective potential corresponds to an attraction at small distances ($r \lesssim 1$), which changes into Coulomb repulsion for $r \gg 1$.

The width γ of the quasistationary level can be calculated as the reciprocal lifetime of a positron in the potential barrier (12). The following estimate is obtained[8]:

$$\gamma(k)=\gamma_0 \exp\{-2\pi[\zeta/k-(\zeta^2-1)^{1/2}]\},\quad \gamma_0 \sim 1. \tag{13}$$

A numerical calculation gives the following result[8] for the value of β:

$$\beta=-(d\varepsilon/d\zeta)_{\zeta=\zeta_c}=8.2. \tag{14}$$

For what follows it is essential that $\gamma(k) \ll \varepsilon_0-1=k^2/2$ Under this condition the width γ does not appear in the final expressions.

THE EQUATION FOR THE DETERMINATION OF THE SYSTEM ENERGY AT $Z>Z_c$

Let us obtain the equation for the energies and eigenstates of the system with the interaction (6) taken into consideration. Let us start with the case when there is a single electron in the "K shell" in the absence of the interaction. The words "K shell" are enclosed in quotation marks since there is no appropriate solution of the Dirac equation at $Z>Z_c$. However, at $Z-Z_c \ll Z_c$ the state in which the electron is found near the nucleus can, by virtue of continuity, be approximately described by the wave function of the K shell at $Z=Z_c-0$.

The equation describing the system has the form

$$\begin{aligned}(E-E_1^0)a&=\int h_{1\varepsilon}\, b_\varepsilon\, d\varepsilon + h_{11}a,\\ (E-E_\varepsilon^0)\, b_\varepsilon&=h_{\varepsilon 1}a+\int h_{\varepsilon\varepsilon'} b_{\varepsilon'} d\varepsilon'.\end{aligned} \tag{15}$$

Here a is the state with a single electron in the "K shell", b_ε is the state with two electrons in the "K shell" and with a single positron of energy ε, h_{11} denotes the average value of the perturbation H' in the state a, and $h_{1\varepsilon}$ and $h_{\varepsilon\varepsilon'}$ denote the corresponding matrix elements of the operator H'. It is assumed that the part corresponding to renormalization of the electron's mass and the radiative correction to the K-electron's energy has been subtracted from h_{11}. It is not difficult to verify that allowance for the radiative corrections (the Lamb shift and the correction to the Coulomb interaction) reduces to a small change of the critical charge. Denoting the corresponding shift of the K-electron energy by $e^2\nu$, we obtain

$$\tilde{Z}_c=Z_c+\nu/\beta.$$

Using the estimate ($\nu \approx 0.8$) obtained in[10] for the value of ν, we find $\tilde{Z}_c-Z_c \sim 0.8/8.2 \approx 0.1$. This small shift in the value of Z_c does not seem to have any influence on the phenomena investigated below.

The region of small energies ε is essential in Eqs. (15), since only in this case can the smallness of the matrix elements ($\sim e^2$) in the expressions for b_ε be compensated by the small difference $E-E_\varepsilon^0$. Transitions into states containing more than two pairs cannot be taken into consideration since the energy of these states differs markedly from the energy of the state a. Positron states with quantum numbers different from the values indicated above ($j=1/2$, $\varkappa=1$) also should not be taken into consideration since here the region of small energies is not amplified by the factor $\Delta(\varepsilon)$ (formula (8)). The quantities E_1^0 and E_ε^0 may be written in the following form:

$$E_1^0=-1-\beta(\zeta-\zeta_c),\quad E_\varepsilon^0=-2-2\beta(\zeta-\zeta_c)+\varepsilon, \tag{16}$$

where β is given by expression (14).

One can easily obtain an expression for E in the form of a perturbation-theory series. Thus, by discarding the second term on the right hand side of the second equation in (15) and substituting into the first equation, we find

$$E \approx E_1^0+h_{11}+\int \frac{|h_{1\varepsilon}|^2}{E-E_\varepsilon^0}\, d\varepsilon. \tag{17}$$

We shall obtain an integral equation for E without using the perturbation theory series. For this purpose we write the second equation of (15) in the form

$$(E-E_\varepsilon^0)\, b_\varepsilon=A_{\varepsilon 1}a. \tag{15'}$$

From Eqs. (15') and (15) we obtain the following integral equation for the determination of the energy:

$$E=E_1^0+h_{11}+\int \frac{h_{1\varepsilon'}A_{\varepsilon' 1}}{E-E_{\varepsilon'}^0}\, d\varepsilon', \tag{18}$$

and also the following equation for the determination of $A_{\varepsilon 1}$:

$$A_{\varepsilon 1}=h_{\varepsilon 1}+\int h_{\varepsilon\varepsilon'}\frac{A_{\varepsilon' 1}}{E-E_{\varepsilon'}^0}\, d\varepsilon'. \tag{19}$$

Equations (18) and (19) reduce to simple algebraic equations if the properties of the positron functions are utilized.

Writing down the positron functions, entering into the matrix elements, in the form (7) and using the narrowness of the resonance, we obtain

$$E=E_1^0+h_{11}+h_{13}A_{31}/(E-E_3^0), \tag{18'}$$

$$A_{31}=h_{31}+h_{33}A_{31}/(E-E_3^0). \tag{19'}$$

Here the state 3 is the same state as if a third particle—a positron with its spin component coinciding with the

spin component of one of the electrons—were found in the K shell in addition to the two electrons. The energy E_3^0 corresponds to the maximum of the resonance curve $\Delta(\varepsilon)$, i.e., it corresponds to $\varepsilon = 1+\beta(\zeta-\zeta_c)$. As a result, according to Eq. (16) we find

$$E_3^0 = -1-\beta(\zeta-\zeta_c).$$

Determining A_{13} from expression (19′) and substituting into (18′), we obtain

$$E = E_1 + |h_{13}|^2/(E-E_3), \tag{20}$$

where $E_1 = E_1^0 + h_{11}$ and $E_3 = E_3^0 + h_{33}$. The new energy levels, replacing E_1 and E_3, are obtained as the roots of the quadratic equation (20):

$$\tilde{E}_{1,3} = \frac{E_1+E_3}{2} \mp \left[\frac{(E_1-E_3)^2}{4} + |h_{13}|^2\right]^{1/2}. \tag{21}$$

Let us introduce the notation

$$(h_{11}-h_{33})^2/4 + |h_{13}|^2 = h_0^2, \quad (h_{11}+h_{33})/2 = \bar{h}.$$

Then from Eqs. (16) we find

$$\tilde{E}_{1,3} = -1-\beta(\zeta-\zeta_c)+\bar{h} \pm h_0.$$

Upon a reduction of Z to values smaller than Z_c, the energy of a pair becomes positive, i.e., $E_3^0 > E_1^0$, and from Eq. (21) it follows that the minus sign corresponds to the state without any pairs. Therefore, it is natural to denote the energy corresponding to the minus sign by $\tilde{E}_1$, and that corresponding to the plus sign by $\tilde{E}_3$.

Expression (21) can also be obtained directly from the the following considerations without integration over the positron energy. A long-lived positron state exists, which is described by the wave function $\tilde{\Psi}_0(\mathbf{r})$. This state is represented by a wave packet composed of the functions $\Psi_\varepsilon(\mathbf{r})$:

$$\tilde{\Psi}_0(\mathbf{r}) = \int C_\varepsilon \Psi_\varepsilon(\mathbf{r})\, d\varepsilon,$$

where $C_\varepsilon = \sqrt{\Delta(\varepsilon)}$. The probability of finding the state $\Psi_0(\mathbf{r})$ during the time t is given by the square of the integral

$$J(t) = \int_0^\infty |C_\varepsilon|^2 e^{-i\varepsilon t}\, d\varepsilon = e^{-i\varepsilon_0 t} e^{-\gamma t}, \quad t \gg 1/\varepsilon_0.$$

For $t \gg 1/\varepsilon_0$ the quantity $J(t)$ is determined by the pole of expression (8).

Thus, the attenuation of the wave packet is determined by the quantity γ, i.e., γ is the width of the quasistationary level. Since the important energy differences in (18′) and (19′) are of order $h_{nm} \sim e^2$, the condition for neglect of attenuation is given by $\gamma \ll e^2$ for $Z \sim Z_c$. This condition is satisfied with great accuracy. Thus, the positron state can be treated as stationary, and the system of equations is written in the form

$$(E-E_1)a = h_{12}b, \quad (E-E_3)b = h_{31}a,$$

which also leads to expression (21).

Using this consideration, it is not difficult to treat the case of an unoccupied K shell. In this case the system of equations connects three states: a_0—the nucleus without any pairs, b_0—the nucleus with a single pair, and c_0—the nucleus with two pairs.

The possible values of the energy are determined from the condition that the system determinant be equal to zero:

$$\begin{Vmatrix} E-E_0 & -h_{02} & -h_{04} \\ -h_{20} & E-E_2 & -h_{24} \\ -h_{40} & -h_{42} & E-E_4 \end{Vmatrix} = 0. \tag{22}$$

Here $E_0 = E_0^0 + h_{00}$ denotes the energy of the state without any pairs, including the diagonal matrix element of the meteraction; $E_2 = E_2^0 + h_{22}$ and $E_4 = E_4^0 + h_{44}$ are the analogous expressions for the states containing one and two pairs; E_0^0, E_2^0, and E_4^0 are the corresponding expressions without taking the interaction into account. As has already been mentioned, the self-energy parts are excluded from the diagonal matrix elements h_{nn}. The energies $\tilde{E}_1, \tilde{E}_3, \tilde{E}_0, \tilde{E}_2$, and $\tilde{E}_4$, obtained as the result of a calculation of the matrix elements $h_{nm} \equiv e^2 f_{nm}$, are given below. The energy E_0^0 is taken equal to zero. In addition we shall need the energy of the state containing two electrons:

$$\tilde{E}^{ee} = -2-2\beta(\zeta-\zeta_c)+e^2 f^{ee}. \tag{23}$$

CALCULATION OF THE MATRIX ELEMENTS

It is not difficult to verify that all of the matrix elements h_{nm} contain the wave functions of the initial and final states, which differ only by the sign of the spin component. In the intersection (6) the operator for the disappearance of an electron of a negative level having a wave function $\Psi^0_{\pm 1/2}$ corresponds to the positron state. The two electrons in the "K shell" also have the same kind of functions. One should not be confused by the fact that the momentum distribution in the wave packet, which describes the hole, coincides to a high degree of accuracy with the momentum distribution of the electron state. If the positron state exactly coincided with the electron state, the state containing such a pair would not differ from the vacuum. In the present case the electron state is stationary whereas the state of the positron is quasistationary (having a width γ) and the corresponding wave packets differ by a quantity of order γ. In measurements in which a positron is not emitted, the states of an electron and of an electron + a pair only differ by a quantity $\sim\gamma$.

Thus, all four functions of expression (6) in all matrix elements h_{nm} differ only by spin projections and coincide with the wave functions $\Psi_{+1/2}$ of a K electron for $Z \approx Z_c$.

For the calculation of the matrix elements, we represent the operator H' in terms of the operators for the creation and annihilation of electrons and positrons in the states $\Psi_1 \equiv \Psi^0_{1/2}$ and $\Psi_2 \equiv \Psi^0_{-1/2}$. Let us write down the quantized operator $\hat{\Psi}$ in the current operator in the form

$$\hat{\Psi} = a_1\Psi_1 + a_2\Psi_2 + b_1^+\Psi_1^{p^*} + b_2^+\Psi_2^{p^*},$$

where $a_{1,2}$ and $b_{1,2}$ are the electron and positron annihilation operators. Since the positron function $\Psi^*_{1,2} = \Psi^p_{2,1}$, we have

$$\hat{\Psi}=a_1\Psi_1+a_2\Psi_2+b_1^+\Psi_2+b_2^+\Psi_1.$$

After substitution into H' we obtain a sum of terms, each of which contain four operators for the creation or annihilation of particles. As follows from Eq. (6), the coefficient associated with each of these terms may be written in the form

$$h_{\alpha\beta}^{\gamma\delta}=\frac{e^2}{2}\left\{\int\frac{(\Psi_\alpha^*-\Psi_\beta)_r(\Psi_\gamma^*\Psi_\delta)_{r'}\,dr\,d\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}-\sum_{i=1}^{3}\int\frac{(\Psi_\alpha^*\alpha_i\Psi_\beta)_r(\Psi_\gamma^*\alpha_i\Psi_\delta)_{r'}\,d\mathbf{r}\,dr'}{|\mathbf{r}-\mathbf{r}'|}\right\}\equiv\mathcal{K}^0_{\alpha\beta\gamma\delta}-\mathcal{K}^1_{\alpha\beta\gamma\delta}. \tag{24}$$

The first integral corresponds to the Coulomb interaction. Separating the angle variables with the aid of Eq. (9) and summing over the spin variables, we obtain the factors $\delta_{\alpha\beta}$ and $\delta_{\gamma\delta}$. Only the spherical part remains in the expansion of $1/|\mathbf{r}-\mathbf{r}'|$ in terms of spherical harmonics of the angle between $\mathbf{r}$ and $\mathbf{r}'$. We obtain

$$\mathcal{K}^0_{\alpha\beta\gamma\delta}=\delta_{\alpha\beta}\delta_{\gamma\delta}\int\frac{(F^2+G^2)_r(F^2+G^2)_{r'}}{r_>}\,dr\,dr'=\delta_{\alpha\beta}\delta_{\gamma\delta}I_0. \tag{25}$$

where $r_>$ denotes the larger of the quantities r and r'.

The calculation of the second term is somewhat more complicated. The first factor in the numerator of the integrand takes the form

$$ir^2FG\{\Omega_\alpha^*\sigma_i\Omega_\beta'-\Omega_\alpha'^*\sigma_i\Omega_\beta\},$$

where Ω and Ω' denote the spherical spinors corresponding to the first and second components of Ψ^0_α. Having made use of the relationship (10), we obtain

$$\{\ldots\}=-\{\Omega_\alpha^*\sigma_i\sigma\mathbf{n}\Omega_\beta-\Omega_\alpha^*\sigma\mathbf{n}\sigma_i\Omega_\beta\}=-2ie_{ikl}n_k\Omega_\alpha^*\sigma_l\Omega_\beta.$$

The second factor of the numerator is written down in similar fashion. Thus, the numerator contains a factor of the form $n_k n'_r$. Therefore, only the term

$$\frac{r_<}{r_>^2}(\mathbf{nn}')$$

is left in the expansion of $1/|\mathbf{r}-\mathbf{r}'|$ in terms of harmonic functions, where $r_>$ is the larger and $r_<$ is the smaller of the quantities r and r'. Averaging over the angles between $\mathbf{r}$ and $\mathbf{r}'$ gives

$$\overline{\{\}\cdot\{\}}'=-{}^1/_9e_{ikl}e_{ikm}(\sigma_l)_{\alpha\beta}(\sigma_m)_{\gamma\delta}=-{}^8/_9(\sigma)_{\alpha\beta}(\sigma)_{\gamma\delta}.$$

We have utilized the relationships $\overline{n_k n_r}=(1/3)\delta_{kr}$, $e_{ikl}e_{ikm}=2\delta_{lm}$, and also the form of Ω_α:

$$\Omega_{1/2}=\begin{Bmatrix}1\\0\end{Bmatrix},\quad \Omega_{-1/2}=\begin{Bmatrix}0\\1\end{Bmatrix}.$$

Substituting the obtained result into $\mathcal{K}_{\alpha\beta\gamma\delta}$, we find

$$\mathcal{K}^1_{\alpha\beta\gamma\delta}=-\frac{8}{9}\int\frac{(FG)_r(FG)_{r'}r_<}{r_>^2}\,dr\,dr'(\sigma)_{\alpha\beta}(\sigma)_{\gamma\delta}=-{}^8/_9I_1(\sigma)_{\alpha\beta}(\sigma)_{\gamma\delta}. \tag{26}$$

If $\alpha=\beta$, then $\sigma_{\alpha\beta}$ differs from zero only for σ_z and, therefore, $\gamma=\delta$; if $\gamma=\alpha$ then $\sigma_{\alpha\beta}\sigma_{\gamma\delta}=1$; however, if $\gamma\neq\alpha$ then $\sigma_{\alpha\beta}\sigma_{\gamma\delta}=-1$.

The value of $\mathcal{K}^1_{\alpha\beta\gamma\delta}$ is analogously determined even for the case $\alpha\neq\beta$. Thus, all matrix elements are expressed in terms of two integrals containing the radial functions of the K shell for $Z=Z_c$. For $r>R$ these functions are determined in terms of the Macdonald function of imaginary argument[3]:

$$G(r)=a\mathcal{K}_{i\nu}(\sqrt{8\xi r}),\quad F(r)=a\xi^{-1}(rG'-G). \tag{27}$$

Here $\nu=2\sqrt{\xi^2-1}$; the normalization constant a is determined from the condition $\int(G^2+F^2)\,dr=1$; $\mathcal{K}_{i\nu}$ is determined by the integral

$$\mathcal{K}_{i\nu}(x)=\int_0^\infty e^{-x\,\mathrm{ch}\,t}\cos\nu t\,dt.$$

In the region $r<R$ the solution is distorted by the influence of the nucleus. The condition for matching the solution (25) with the solution for $r<R$ gives a trancendental equation for the determination of ξ_c. The result weakly depends on the details of the charge density distribution inside the nucleus. The value $Z_c\approx 170$[2,3] cited above is obtained by exactly such a method. Evaluation of the integrals I_0 and I_1 on a computer gave $I_0=4.52$ and $I_1=0.560$.

In the determination of the matrix elements h_{nm} the majority of the operator quartets vanish in the given facings, and h_{nm} is expressed in terms of a few terms of the form (24). In order to clarify this point, let us consider the matrix element $\langle 0|H'|4\rangle\equiv h_{04}$ between the vacuum state and the state with two pairs:

$$|4\rangle=a_1^+a_2^+b_1^+b_2^+|0\rangle.$$

There are only four terms in H' which give nonvanishing contributions to h_{04}:

$$b_1a_1b_2a_2h_{21}^{12}+b_1a_2b_1a_1h_{22}^{11}+b_2a_1b_1a_2h_{11}^{22}+b_2a_2b_1a_1h_{12}^{21}.$$

Hence we obtain

$$\langle 0|H'|4\rangle=-h_{21}^{12}+h_{22}^{11}+h_{11}^{22}-h_{12}^{21}.$$

Using formulas (24)–(26) we find

$$h_{22}^{11}=h_{11}^{22}={}^1/_2(I_0-{}^8/_9I_1)e^2,\quad h_{21}^{12}=h_{12}^{21}={}^1/_2({}^{16}/_9I_1)e^2$$

and the following answer is obtained for the matrix element:

$$\langle 0|H'|4\rangle=e^2f_{04}=e^2(I_0-{}^8/_3I_1)\approx 3e^2.$$

All remaining matrix elements are calculated in similar fashion. In the case of the state containing a single pair there are several independent state, among which one should choose that one where the total spin of the pair and its projection are equal to zero. As one can easily verify, with the aid of creation operators this state is written in the following form:

$$\Psi_{s^2=0,\ s_z=0}=2^{-1/2}(a_1^+b_2^++a_2^+b_1^+)|0\rangle.$$

The states with spin 1 have the form

$$\Psi_{s^2=1,\ s_z=1,0,-1}=\begin{cases}a_1^+b_1^+|0\rangle\\ 2^{-1/2}(a_1^+b_2^+-a_2^+b_1^+)|0\rangle.\\ a_2^+b_2^+|0\rangle\end{cases}$$

In all other cases there is only one independent state.

Let us present the matrix elements calculated in such a manner. The number f^{ee}, determining the energy of the state with two electrons in the K shell (formula (23)), is given by

$$f^{ee}=I_0-{}^8/_3I_1\approx 3.$$

In the case of charge $-e$ in the K shell, we have

$$f_{11}=-f_{33}=-{}^1/_2(I_0+{}^6/_3I_1)\approx 3, \quad f_{13}=f_{31}=I_1-{}^3/_3I_1\approx 3.$$

The following result is obtained for an uncharged "K shell":

$$f_{00}=-f_{11}=I_0+{}^6/_3I_1\approx 6, \quad f_{02}=f_{20}=f_{21}=f_{12}=0,$$
$$f_{04}=f_{40}=f_{22}=I_0-{}^3/_3I_1\approx 3.$$

Substitution into Eq. (22) gives the following energy values of the corresponding three levels for the channel $Q=0$:

$$\tilde{E}_0=-6.7\ e^2=-6.7\ mc^2/137=-25\ \text{keV},$$
$$\tilde{E}_2=3.0\ e^2=11\ \text{keV}, \quad \tilde{E}_4=6.7\ e^2=25\ \text{keV}.$$

Here the system energy is chosen equal to zero for $Z=Z_c-0$. Upon a transition through Z_c the energy $\tilde{E}_0$ of the state without pairs is decreased by an abrupt change to 25 keV. All three energies do not depend on Z. In the case of the channel $Q=-e$ we have

$$\tilde{E}_1=-1-\beta(\zeta-\zeta_c)-4.25\ e^2, \quad \tilde{E}_3=-1-\beta(\zeta-\zeta_c)+4.25\ e^2.$$

Finally, in the case of a filled K shell (formula (23)) we find

$$\tilde{E}^{ee}=-2-2\beta(\zeta-\zeta_c)+3\ e^2.$$

These results are shown in the figure. As is clear from the figure, for $Z-Z_c<Z_c'-Z_c\approx 0.31$ the lowest state of the vacuum is the state with charge equal to zero; in the interval $Z_c''>Z>Z_c'$ the lowest state of the vacuum has charge $-e$, and finally for $Z>Z_c''$, $Z_c''-Z\approx 0.87$ the ground state corresponds to charge $-2e$. We note that in the channel with charge $Q=-e$ the lower of the two states is stationary over the entire interval from Z_c to Z_c''.

For each value of Z, the distance between the various curves give the energies of the positrons which are emitted from the corresponding quasistationary states. As is clear from the figure, in the case $Q=0$ there are six possible values for the energy of an emitted positron, starting with $Z=Z_c'''$, $Z_c'''-Z_c\approx 1.33$; these energies correspond to transitions from the three levels of the channel $Q=0$ to the two levels of the channel $Q=-e$. In the case of the channel $Q=-e$, starting with Z_c'' there are two positron lines corresponding to transitions from the two states of the channel $Q=-e$ to the single state of the channel $Q=-2e$. From the figure one can easily determine the number of possible positron lines for all values of $Z-Z_c$.

POSITRONS PRODUCTION WHEN NUCLEI APPROACH EACH OTHER

As was shown by Gershteĭn and Popov,[11] a state corresponding to the free K shell of the combined nucleus appears with high probability during the adiabatic approach of heavy nuclei, when the incident nucleus has no electrons in its K shell. At distances $R\ll 1$ the system can be regarded as a nucleus with charge Z_1+Z_2 and having an effective radius $\sim R$. In this connection the system may turn out to be in a subcritical state, notwithstanding the fact that $Z_1+Z_2>Z_c$, because the distance R is much larger than the radius of the nucleus. The distance R plays the role of a critical parameter. The curves shown in the figure are given as functions of $Z-Z_c$, but can easily be obtained as functions of R_c-R. At small distances $(R_c-R)/R_c\ll 1$ the conversion will be linear. The barrier penetration depends on R_c-R exponentially, and therefore positrons will be emitted for values of R close to R_{min}, and such positrons will have rather monochromatic energies which can be determined from the figure for the value of Z corresponding to R_{min}.

From what has been said above, it follows that three states appear in the initial stage of the approach, when the effective Z is close to Z_c; the lowest of these states is stationary and the upper two states, which correspond to one pair and to two pairs, are quasistationary. As the two nuclei come closer together, all three levels become quasistationary. The separation between the levels is of the order of $\Delta E\sim 10$ to 20 keV. However, if the collision of the nuclei takes place sufficiently slowly, so that the condition for adiabaticity is satisfied, that is, $v/R_c<\Delta E$, where v is the velocity of the nucleus, then only the lowest state of the channel $Q=0$ would be realized during the approach. The rate of approach is estimated in[11] for the case of two uranium nuclei having energies sufficient for an approach to the critical distance R_c:

$$v_0={}^1/_{70}\sqrt{\zeta/R_c}.$$

Here $\zeta=Z/137=\frac{2}{3}$. As is clear from the cited expression, adiabaticity is apparently not satisfied and in the collision process an intermixing of all three states takes place. In the case of strong nonadiabaticity, the weight of each state is equal to $\frac{1}{3}$.

Thus, the probability of positron production depends on the extent of the adiabaticity of the approach. For sufficient initial energy of the nuclei, six positron lines should be emitted which differ in energy and in intensity. If there is a single electron in the K shell in the initial state, two positron lines should be observed. The splitting of the lines is of the order of 10 to 30 keV.

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[1]I. Pomeranchuk and Ya. Smorodinsky, J. Phys. USSR 9, 97 (1945).
[2]W. Pieper and W. Greiner, Z. Phys. 218, 327 (1969).
[3]V. S. Popov, Yad. Fiz. 12, 429 (1970) [Sov. J. Nucl. Phys. 12, 235 (1971)].
[4]A. B. Migdal, Zh. Eksp. Teor. Fiz. 61, 2209 (1971) [Sov. Phys.-JETP 34, 1184 (1972)].
[5]Ya. B. Zel'dovich and V. S. Popov, Usp. Fiz. Nauk 105, 403 (1971) [Sov. Phys.-Uspekhi 14, 673 (1972)].
[6]A. I. Akhiezer and V. B. Berestetskiĭ, Kvantovaya élektrodinamika (Quantum Electrodynamics), Nauka, 1969 [Interscience].
[7]A. B. Migdal, A. M. Perelomov, and V. S. Popov, Yad. Fiz. 14, 874 (1971) [Sov. J. Nucl. Phys. 14, 488 (1972)].

[8]V. S. Popov, Zh. Eksp. Teor. Fiz. 59, 965 (1970) [Sov. Phys.-JETP 32, 526 (1971)].
[9]L. D. Landau and E. Ml Lifshitz, Kvantovaya mekhanika (Quantum Mechanics), Nauka, 1974 [Pergamon].
[10]G. W. Erickson, Phys. Rev. Lett. 27, 780 (1971); A. M. Desiderio and W. R. Johnson, Phys. Rev. A3, 1267 (1971).
[11]S. S. Gershtein and V. S. Popov, Nuovo Cimento Lett. 6, 593 (1973); V. S. Popov, Zh. Eksp. Teor. Fiz. 65, 35 (1973) [Sov. Phys.-JETP 38, 18 (1974)]; V. S. Popov, Yad. Fiz. 19, 155 (1974) [Sov. J. Nucl. Phys. 19, 81 (1974)].

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Polarization of secondary protons in the $\gamma+p\to\pi^0+p$ reaction at $E_\gamma=536$–640 MeV

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Measurements are reported of the polarization of protons from the $\gamma+p\to\pi^0+p$ reaction at photon energies of 540, 560, 585, 610, and 640 MeV at pion angle of emission of 90° in the center of mass system. The angular dependence of the proton polarization has been investigated for this reaction at a photon energy 600 MeV. The data obtained are compared with the results of the phenomenological analysis reported by Metcalf and Walker (Preprint CALT-68-425, 1974).

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Studies of the photoproduction of pions on nucleons play an important role in the systematics of resonances. The energy behavior of resonance multipoles, their relative contribution to observed quantities, and the parametrization of small phonon amplitudes combine into one of the important problems in the theoretical analysis of the photoproduction of pions on nucleons.

Several theoretical papers[1–5] have been published in recent years, where different methods (resonance model, dispersion relations, energy-independent multipole analyses, and so on) were used to calculate the partial amplitudes and their contributions to the different observed quantities. Comparison of experimental results with theoretical predictions may provide information on the degree of validity of any particular method. However, the amount of information available on polarization, which is one of the main characteristics involved in the study of the photoproduction of pions on nucleons, is currently quite inadequate. Despite the fact that the polarization of protons, especially in region II of πN resonances in the $\gamma+p\to\pi^0+p$ reaction was measured quite a long time ago, these measurements are neither systematic enough nor accurate enough to enable us to decide in favor of any particular theoretical approach.

In this paper we report the results of measurements of the polarization of protons emitted in the $\gamma+p\to\pi^0+p$ reaction at 90° in the center of mass system in the region of the π resonances, and the angular dependence of the polarization at $E_\gamma=600$ MeV.

1. ENERGY DEPENDENCE OF POLARIZATION AT 90° CM AT PHOTON ENERGIES BETWEEN 536 AND 640 MeV

The experiment was performed with the photon beam of the Khar'kov 2-GeV linear accelerator. The proton polarization was measured by a telescope consisting of optical spark chambers mounted at the exit from a magnetic spectrometer.[10] The telescope contained three four-gap spark chambers with aluminum electrodes, 0.15 mm thick (IK-4), and one 42-gap spark chamber with graphite electrodes (IK-42) of $350\times350\times7$ mm. The set of graphite electrodes of the IK-42 chamber was used as the polarization analyzer. The spark chambers IK-4, which were located in front of the IK-42, were used to determine the proton track direction prior to scattering in the graphite electrodes.

Since the focal line of the magnetic spectrometer passed through the first gap of the IK-42, this enabled us to determine the momentum of protons recorded on stereophotographs to an accuracy of better than 0.5%, and to assign it to the corresponding kinematic interval of the reaction under investigation. The total momentum range in the experiment was 7%.

The proton polarization was determined from the scattering asymmetry for a carbon target, using the maximum likelihood method. The analyzing power was taken from the paper of Peterson.[11] The following results were obtained for the polarization of protons from the $\gamma+p\to\pi^0+p$ reaction at 90° in the center of mass system:

$E_\gamma\pm\Delta E_\gamma$, MeV:	536 ± 10	560 ± 11	585 ± 12	610 ± 13	640 ± 28
$P\pm\Delta P$:	-0.48 ± 0.09	-0.54 ± 0.10	-0.59 ± 0.09	-0.72 ± 0.08	-0.74 ± 0.07

The uncertainties are statistical.

Figure 1 shows the experimental polarizations and the calculations of Metcalf and Walker[2] based on the resonance model. Satisfactory agreement is observed between the experimental energy dependence of polarization at 90° and the dependence predicted theoretically[2] in region II of the πN resonances.