The microscopic theory of a mixture of superfluid liquids

Yu. A. Nepomnyashchiĭ

A. M. Gor'kiĭ Permian State University

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We use a diagram method to obtain an exact solution of the problem of the excitation spectrum of a mixture of two Bose-liquids with a condensate in the long-wavelength limit at $T=0$.

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1. INTRODUCTION

The recently significantly increased probability for obtaining experimentally a mixture of superfluid liquids has promoted an appreciably increased theoretical interest in that system.[1]

To begin with, already in 1957 Khalatnikov,[2] applying a hydrodynamic approach, suggested the study of a mixture of two superfluid liquids as a peculiar quantum system. A microscopic approach was proposed in[3] which makes it possible to validate the stability of a state with independently moving condensates of the components of the mixture (the region of admissible velocities was determined in the Bogolyubov approximation); another specific feature of the system was also noted in the same approximation—neither of the two sound branches of its spectrum is described in the general case at $T=0$ by the formula for the classical elastic continuum $u^2=\partial P/\partial\rho$.

Andreev and Bashkin[1] indicated an effect which is not taken into account in Khalatnikov's hydrodynamic theory and which goes beyond the framework of the Bogolyubov approximation: each superfluid motion is accompanied by the transfer of a well-defined fraction of mass of both components of the mixture (in the microscopic theory: the motion of the condensate of one component drags along with it in a well-defined way the particles of both components of the mixture which are outside the condensate). Correspondingly, both velocities of the superfluid motions are introduced in Khalatnikov's set of hydrodynamic equations in the expression for the mass fluxes:

$$\mathbf{j}_i=\sum_k \rho_{ik}\mathbf{v}_k \tag{1}$$

(the system of reference is given by the condition $\mathbf{v}_n=0$). It is important that there is on the phenomenological level no unique definition of the superfluid velocities (in particular, a definition which masks the drag effect, $\mathbf{v}_i=\mathbf{j}_i/\rho$, is permissible). The meaning and necessity of the modification of the expression for $\mathbf{j}_i$ are explained in the microscopic approach: in the framework of a BCS theory it was shown in[1] that the condition for potential flow

$$\operatorname{rot}\mathbf{v}_i=0\ , \tag{2}$$

which is taken into account in the set of hydrodynamic equations is in agreement only with (1).

There is, however, also another relation which is included in the hydrodynamical equations and which needs a microscopic basis:

$$\partial E/\partial\mathbf{v}_i=\mathbf{j}_i. \tag{3}$$

It is impossible to obtain this relation by fixing the definition of the velocities $\mathbf{v}_i$ merely by the requirement (2)—the remaining ambiguity (the possibility of arbitrary linear transformations of the $\mathbf{v}_i$ with constant coefficients) is not permissible in (3). If we take (3) as a phenomenological definition of the $\mathbf{v}_i$, there remains the open question of them satisfying the condition (2).

The fact that (3) is not obvious is underlined by the "drag effect." One might even prove that the phenomenological expression for $\partial E/\partial\mathbf{v}_i$ must be generalized (like $\mathbf{j}_i$):

$$\partial E/\partial\mathbf{v}_i=\sum_k \eta_{ik}\mathbf{j}_k.$$

Indeed, if, for instance, the equations $\mathbf{j}_1=\rho_{11}\mathbf{v}_1$ and $\mathbf{j}_2=\rho_{21}\mathbf{v}_1$, which describe a state with a moving condensate of the first component, corresponded to a drift of the classical particles (with pair interactions) with velocity $\mathbf{v}_1$, instead of (3) the equation

$$\frac{\partial E}{\partial\mathbf{v}}=\frac{\partial E_{\text{kin}}}{\partial\mathbf{v}}=\mathbf{j}_1+\mathbf{j}_2$$

would be satisfied.

Therefore, turning to the microscopic approach in the case of a mixture of superfluid liquids is not only required to give a foundation for the possibility of introducing non-operator densities, fluxes, and so on, which describe long-wavelength excitations (as in the case of a single-component Bose-liquid) but also for the construction itself of a correct set of hydrodynamic equations.

We give in the present paper a microscopic theory of a mixture of two Bose-liquids with a condensate in the long-wavelength limit at $T=0$. We use a Green function, determined from the Dyson equation using a method analogous to the derivation of the Hugenholtz-Pines[4] relations, to find the excitation spectrum. The result is exact, i.e., it is not connected with any assumptions whatever about the relative concentrations in the mixture or the weakness of the interactions. When the relative concentrations are not too different we confirm the result obtained in[3] in the framework of the Bogolyubov approximation: none of the sound velocities is connected in the normal way with the compressibility. Equation (3) is rigorously proven using the definition of the $\mathbf{v}_i$ as the velocities of the condensates. We note the role of the density fluctuations as a source for the drag effect. The result for the spectrum is the same as the one obtained at $T=0$ from the set of hydrodynamic equations used in[1].

2. THE GREEN FUNCTION EXPRESSED IN TERMS OF FORMAL THERMODYNAMIC DERIVATIVES

A mixture of two Bose-liquids with pair interactions is described by the Hamiltonian

$$H=\sum_{i=1}^{2}\left\{\psi_i^+(\mathbf{r})\left(-\frac{\hbar^2\nabla^2}{2m}\right)\psi_i(\mathbf{r})\,d\mathbf{r}+\frac{1}{2}\int\psi_i^+(\mathbf{r})\psi_i^+(\mathbf{r}')\right.$$
$$\left.\times v_i(\mathbf{r}-\mathbf{r}')\psi_i(\mathbf{r}')\psi_i(\mathbf{r})\,d\mathbf{r}\,d\mathbf{r}'\right\}+\quad\psi_1^+(\mathbf{r})\psi_1(\mathbf{r})v(\mathbf{r}-\mathbf{r}')\psi_2^+(\mathbf{r}')\psi_2(\mathbf{r}')\,d\mathbf{r}\,d\mathbf{r}'.$$

Let $T=0$ and let the concentrations n_{0i} of particles in the condensate be non-vanishing. The Green function $\hat{G}$ and the irreducible self-energy part $\hat{\Sigma}$ form a 4×4 matrix corresponding to the four possibilities for incoming and outgoing lines: two directions and two kinds of particle.

From the Dyson matrix equation (the upper indexes characterize the number of the components and the lower ones the directions):

$$G_{mn}^{ik}=G_{0mn}^{ik}+G_{0mr}^{il}\Sigma_{rs}^{ll}G_{sn}^{lk},$$
$$G_{0mn}^{ik}(p)\equiv\begin{pmatrix}\hat{G}_0^{(1)}(p) & 0\\ 0 & \hat{G}_0^{(2)}(p)\end{pmatrix},\quad p\equiv(\varepsilon,\mathbf{p}),$$
$$\hat{G}_0^{(i)}(p)\equiv\begin{pmatrix}(\varepsilon-\varepsilon_{pi}{}^0+\mu_i+i\delta)^{-1} & 0\\ 0 & (-\varepsilon-\varepsilon_{pi}{}^0+\mu_i+i\delta)^{-1}\end{pmatrix},$$
$$i,\ k,\ldots=(1,\ 2),\quad \varepsilon_{pi}{}^0\equiv p^2/2m_i,$$

we find

$$\hat{G}(p)=\begin{pmatrix}\hat{G}_0^{(1)-1}-\hat{\Sigma}^{(1)} & -\hat{\Sigma}^{(12)}\\ -\hat{\Sigma}^{(21)} & \hat{G}_0^{(2)-1}-\hat{\Sigma}^{(2)}\end{pmatrix}^{-1} \tag{4}$$
$$(G^{ii}\equiv G^{(i)},\quad \Sigma^{ii}\equiv\Sigma^{(i)}).$$

We can use the generalized Hugenholtz-Pines and Gavoret-Nozières relations[4] for a mixture (see Appendix, Sec. 1 for a derivation) to determine the asymptotic form of $\hat{G}(p)$ as $p\to0$. These relations establish a connection between the terms of the expansion

$$\hat{\Sigma}(p)=\hat{\Sigma}(0)+\Delta_1\hat{\Sigma}+\Delta_2\hat{\Sigma}+\ldots$$

(the $\Delta_n\hat{\Sigma}$ contain in general non-analyticities and are determined by the conditions $\lim_{p\to0}\Delta_n\hat{\Sigma}/p^{n-1}=0$ and $\lim\Delta_n\hat{\Sigma}/p^n\neq0$) and the derivatives of the Hugenholtz-Pines[4] thermodynamic potential:

$$E'(n_{0i},\mu_i,\mathbf{p}_i)=E(n_i,\mathbf{p}_i)-\mu_1n_1'-\mu_2n_2' \tag{5}$$

(the volume $\Omega=1$; $n_i'=n_i-n_{0i}$ are the concentrations of particles outside the condensates, μ_i the chemical potentials, and $\mathbf{p}_i$ the momenta of the particles in the condensates)$^{1)}$;

$$\Sigma_{11}^{(1)}(0)=\mu_1+n_{01}a_{11},\quad \Sigma_{12}^{(1)}(0)=n_{01}a_{11},\quad \Sigma_{mn}^{12}(0)=[n_{01}n_{02}]^{1/2}a_{13},$$
$$\Delta_1\Sigma_{11}^{(1)}=\varepsilon a_{12},\quad \Delta_2[\Sigma_{11}^{(1)}-\Sigma_{12}^{(1)}]=\frac{1}{2n_{01}}\left(\varepsilon^2a_{22}+p^2\frac{n_1'}{\tilde{m}_1}\right), \tag{6}$$

$$\Delta_1\Sigma_{11}{}^{12}=\frac{\varepsilon}{2}\left[\left(\frac{n_{02}}{n_{01}}\right)^{1/2}a_{23}+\left(\frac{n_{01}}{n_{02}}\right)^{1/2}a_{14}\right],$$
$$\Delta_1\Sigma_{12}{}^{12}=\frac{\varepsilon}{2}\left[\left(\frac{n_{02}}{n_{01}}\right)^{1/2}a_{23}-\left(\frac{n_{01}}{n_{02}}\right)^{1/2}a_{14}\right];$$
$$\Delta_2[\Sigma_{11}{}^{12}-\Sigma_{12}{}^{12}]=\frac{1}{2\sqrt{n_{01}n_{02}}}(\varepsilon^2a_{24}+p^2\alpha); \tag{7}$$

$$a_{ik}\equiv\partial^2E'/\partial x_i\partial x_k,\quad x_{1,3}\equiv n_{0(1,2)},\quad x_{2,4}\equiv\mu_{1,2}. \tag{8}$$

The constants $\tilde{m}_i$ and α can be expressed in terms of the derivatives of the energy of the system with respect to the condensate velocities $\mathbf{v}_i$ (or the momenta $\mathbf{p}_i=m_i\mathbf{v}_i$ of the condensate particles, see Appendix, Sec. 1). The formulae for $\Sigma^{(2)}(0)$, $\Delta_n\Sigma^{(2)}$, $\Sigma^{21}(0)$, and $\Delta_n\Sigma^{21}$ are obtained from (6) and (7) directly through the substitutions $1,2\rightleftarrows3,4$ and $n_{01}\rightleftarrows n_{02}$ in the a_{ik}.

We can construct the quantity E' as follows. If the condensate and the system of particles outside the condensate did not exchange particles, their concentrations should be considered to be independent variables so that

$$dE(n_{0i},n_i')=\sum_i(\mu_{0i}dn_{0i}+\mu_i'dn_i'),$$
$$\left(\frac{\partial E}{\partial n_{0i}}\right)_{n_i'}=\mu_{0i},\quad \left(\frac{\partial E}{\partial n_i'}\right)_{n_{0i}}=\mu_i';$$

in another thermodynamic representation

$$E'(n_{0i},\mu_i')=E-\sum_i\mu_i'n_i',\quad dE'=\sum_i(\mu_{0i}dn_{0i}-n_i'd\mu_i'),$$
$$\left(\frac{\partial E'}{\partial n_{0i}}\right)_{\mu_i'}=\mu_{0i},\quad \left(\frac{\partial E'}{\partial \mu_i'}\right)_{n_{0i}}=-n_i'. \tag{9}$$

We can retain the function $E'(n_{0i},\mu_i')$ as a special thermodynamic characteristic of the system and when exchange of particles is present between the condensate and the system outside the condensate we need only introduce the additional requirement

$$\left(\frac{\partial E}{\partial n_{0i}}\right)_{n_i}=0,\quad\text{i.e.,}\quad \left(\frac{\partial E}{\partial n_{0i}}\right)_{n_i'}+\left(\frac{\partial E}{\partial n_i'}\right)_{n_{0i}}\frac{dn_i'}{dn_{0i}}=\mu_{0i}-\mu_i'=0. \tag{10}$$

By virtue of (10) we can not distinguish μ_{0i} and μ_i', and we denote them by μ_i. We construct a simple diagram technique for

$$\hat{H}'=\hat{H}-\sum_i\mu_i\hat{n}_i'$$

($\hat{H}'|\Phi\rangle=E'|\Phi\rangle$) (formally, as for the Hamiltonian of the system of particles outside the condensate; the n_{0i} play the role of parameters); the quantity E' corresponds to the sum of the vacuum diagrams.[4]

Equations (6) to (8) determine the asymptotic form of the matrices occurring in (4):

$$\hat{G}_0^{(1)-1}-\hat{\Sigma}^{(1)}=(-1)\begin{pmatrix}n_{01}a_{11}-(1-a_{12})\varepsilon+\Delta_2\Sigma_{11}^{(1)}+\varepsilon_{p1}{}^0; & n_{01}a_{11}+\Delta_2\Sigma_{12}^{(1)}\\ n_{01}a_{11}+\Delta_2\Sigma_{12}^{(1}; & n_{01}a_{11}+(1-a_{12})\varepsilon+\Delta_2\Sigma_{11}\ +\varepsilon_{p1}{}^0\end{pmatrix}$$
$$\Sigma_{ik}{}^{12}=(n_{01}n_{02})^{1/2}a_{13}+(-1)^k\frac{\varepsilon}{2}\left[\left(\frac{n_{02}}{n_{01}}\right)^{1/2}a_{23}+(-1)^i\left(\frac{n_{0i}}{n_{02}}\right)^{1/2}a_{14}\right]+\Delta_2\Sigma_{ik}{}^{12}, \tag{11}$$

while

$$\Delta_2\Sigma_{11}{}^{12}=\Delta_2\Sigma_{22}{}^{12}, \qquad \Delta_2\Sigma_{12}{}^{12}=\Delta_2\Sigma_{21}{}^{12};$$

for $\hat{G}_0^{(2)-1}-\hat{\Sigma}^{(2)}$ and $\hat{\Sigma}^{21}$ the relations differ only in the replacements $1,2\rightleftarrows 3,4$ and $n_{01}\rightleftarrows n_{02}$ in the a_{ik}.

Substituting (11) in (4) and after that evaluating the inverse 4×4 matrix up to terms of lowest order in p and ε we get

$$G_{mn}{}^{ik}(p)=\frac{b_{ik}c_{mn}}{A\varepsilon^4-B\varepsilon^2p^2+Cp^4}, \tag{12}$$

$$b_{ik}=\begin{pmatrix} n_{01}[M_1\varepsilon^2-L_1p^2], & \sqrt{n_{01}n_{02}}[N\varepsilon^2+\alpha Rp^2] \\ \sqrt{n_{01}n_{02}}[N\varepsilon^2+\alpha Rp^2]. & n_{02}[M_2\varepsilon^2-L_2p^2] \end{pmatrix},$$

$$c_{mn}=\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad L_1=\frac{n_2}{m_2^*}R, \quad L_2=\frac{n_1}{m_1^*}R,$$

$$R=a_{11}a_{33}-a_{13}^2, \qquad M_2=M_1(a_{ik(1,2\rightleftarrows 3,4)}),$$

$$B=\frac{n_1}{m_1^*}M_1+\frac{n_2}{m_2^*}M_2+2\alpha N, \quad C=\left(\frac{n_1n_2}{m_1^*m_2^*}-\alpha\right)R, \quad \frac{n_i}{m_i^*}\equiv\frac{n_{0i}}{m_i}+\frac{n_i'}{\tilde{m}_i}.$$

$$M_1=-a_{44}(a_{11}a_{33}-a_{13}^2)+(1-a_{34})^2a_{11}+2(1-a_{34})a_{14}a_{13}+a_{14}^2a_{33},$$

$$N=a_{24}(a_{11}a_{33}-a_{13}^2)+(1-a_{12})a_{33}a_{14}+(1-a_{34})a_{11}a_{23}+(1-a_{12})(1-a_{34})a_{13}+a_{14}a_{23}a_{13},$$

$$A=(a_{11}a_{33}-a_{13}^2)(a_{22}a_{44}-a_{24}^2)+[(1-a_{12})(1-a_{34})-a_{14}a_{23}]^2-2(1-a_{12})(a_{33}a_{14}a_{24}+a_{13}a_{44}a_{23})-2(1-a_{34})(a_{11}a_{23}a_{24}+a_{13}a_{22}a_{14})-2(1-a_{12})(1-a_{34})a_{13}a_{24}-(1-a_{12})^2a_{33}a_{44}-(1-a_{34})^2a_{11}a_{22}-a_{11}a_{44}a_{23}^2-a_{33}a_{22}a_{14}^2-2a_{13}a_{23}a_{14}a_{24}.$$

We draw attention to the multiplicative character of the way the G_{mn}^{ik} depend on the indexes. It is important that in $\hat{G}$ only such combinations of the $\Delta_n\Sigma$ entered as are contained in Eqs. (6) to (8), i. e., they have an analytical form and can be expressed in terms of a_{ik}, m_i^*, and α.

3. TRANSITION TO PHYSICAL DERIVATIVES

We show now that the combinations of formal derivatives a_{ik} in $\hat{G}$ in (12) have a simple physical meaning. To do this we express the physical derivatives $\partial\mu_i/\partial n_k$ ($\mu_i=\partial n_i$) in terms of the formal a_{ik}. It follows from (8) to (10) that

$$d\mu_1=a_{11}dn_{01}+a_{12}d\mu_1+a_{13}dn_{02}+a_{14}d\mu_2,$$
$$d\mu_2=a_{31}dn_{01}+a_{32}d\mu_1+a_{33}dn_{02}+a_{34}d\mu_2,$$
$$dn_1=(1-a_{21})dn_{01}-a_{22}d\mu_1-a_{23}dn_{02}-a_{24}d\mu_2,$$
$$dn_2=-a_{41}dn_{01}-a_{42}d\mu_1+(1-a_{43})dn_{02}-a_{44}d\mu_2,$$

i. e.,

$$\hat{A}\,d\hat{\mu}=\hat{B}\,d\hat{n}_0, \qquad \hat{A}^T d\hat{n}_0=d\hat{n}+\hat{C}\,d\hat{\mu}, \tag{13}$$

$$\hat{A}=\begin{pmatrix} 1-a_{12} & -a_{14} \\ -a_{23} & 1-a_{34} \end{pmatrix}, \quad \hat{B}=\begin{pmatrix} a_{11} & a_{13} \\ a_{13} & a_{33} \end{pmatrix},$$

$$\hat{C}=\begin{pmatrix} a_{22} & a_{24} \\ a_{24} & a_{44} \end{pmatrix}, \quad d\hat{\mu}=\begin{pmatrix} d\mu_1 \\ d\mu_2 \end{pmatrix}, \quad d\hat{n}_0=\begin{pmatrix} dn_{01} \\ dn_{02} \end{pmatrix}.$$

Eliminating $d\hat{n}_0$ from (13) we find

$$d\hat{n}=\hat{D}\,d\hat{\mu}, \quad \hat{D}=-\hat{C}+\hat{A}^T\hat{B}^{-1}\hat{A}.$$

The evaluation of $\hat{D}$ leads to the following equations (see (12), (13)):

$$\begin{gathered} D_{11}=M_2/\det\hat{B}, \quad D_{22}=M_1/\det\hat{B}, \\ D_{12}=D_{21}=-N/\det\hat{B}, \quad \det\hat{D}=A/\det\hat{B}. \end{gathered} \tag{14}$$

Substituting (14) into (12) and using the obvious relations

$$\frac{\partial\mu_1}{\partial n_1}=\frac{D_{22}}{\det\hat{D}} \quad \frac{\partial\mu_2}{\partial n_2}=\frac{D_{11}}{\det\hat{D}} \quad \frac{\partial\mu_1}{\partial n_2}=\frac{\partial\mu_2}{\partial n_1}=-\frac{D_{12}}{\det\hat{D}} \tag{15}$$

enables us to eliminate the quantities a_{ik} from $\hat{G}$ in (12):

$$\hat{G}=\frac{1}{Z}\hat{P}\hat{Q},$$

$$P=\begin{pmatrix} n_{01}\left[\frac{\partial\mu_1}{\partial n_1}\varepsilon^2-\frac{n_2}{m_2^*}dp^2\right], & \sqrt{n_{01}n_{02}}\left[\frac{\partial\mu_1}{\partial n_2}\varepsilon^2+\alpha\,dp^2\right] \\ \sqrt{n_{01}n_{02}}\left[\frac{\partial\mu_1}{\partial n_2}\varepsilon^2+\alpha\,dp^2\right], & n_{02}\left[\frac{\partial\mu_2}{\partial n_2}\varepsilon^2-\frac{n_1}{m_1^*}dp^2\right] \end{pmatrix},$$

$$Q=\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad Z=\varepsilon^4-\varepsilon^2p^2\left(\frac{n_1}{m_1^*}\frac{\partial\mu_1}{\partial n_1}+\frac{n_2}{m_2^*}\frac{\partial\mu_2}{\partial n_2}-2\alpha\frac{\partial\mu_1}{\partial n_2}\right)$$
$$+p^4\left(\frac{n_1n_2}{m_1^*m_2^*}-\alpha\right)d \quad \left(d\equiv\det\frac{\partial\mu_i}{\partial n_k}\right).$$

Apart from the usual thermodynamic derivatives $\partial^2E/\partial n_i\partial n_k$ there occur in the expression for the long-wavelength spectrum derivatives of the energy (see (A2)) with respect to the velocities of the superfluid motion (which can be expressed in terms of a single "drag parameter" α—see below). The velocities of the superfluid motions play the role of peculiar additional thermodynamic variables, the occurrence of which is connected with the stability of the relative superfluid motion.

The possibility to connect the long-wavelength spectrum with the thermodynamic potential $E(n_i, \mathbf{v}_i)$ (although through the enlarged inclusion of the variables $\mathbf{v}_i$) is not self-evident for a quantum system; for instance, for a Fermi-system (zero sound) or for a quantum (coherent) crystal[6] one needs a more detailed characteristic of the system than that given by $E(n)$.

When there is no interaction between the components of the mixture Eq. (15) would for each of the liquids lead to the well-known expression[4]

$$\hat{G}^{(i)}=n_{0i}\frac{\partial\mu_i}{\partial n_i}\left(\varepsilon^2-\frac{n}{m}\frac{\partial\mu_i}{\partial n_i}p^2\right)^{-1}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

4. CONNECTION BETWEEN THE DERIVATIVES OF THE ENERGY WITH RESPECT TO THE CONDENSATE VELOCITIES

Using the diagram technique for a system of two Bose-liquids with moving condensates[3] we can easily evaluate the derivatives $\partial E(\mathbf{v}_1, \mathbf{v}_2)/\partial\mathbf{v}_i$. Differentiation of the contribution E' of any of the vacuum diagrams with respect to $\mathbf{v}_i$ leads to a set of diagrams which differ from the original one by the replacement of one of the zero-order Green functions of the i-th component $G_0^{(i)}(p) = (\varepsilon-\mathbf{p}\cdot\mathbf{v}_i-\varepsilon_p^0+\mu_i)^{-1}$ by the expression

$$\partial G_0^{(i)}(p)/\partial\mathbf{v}_i=\mathbf{p}G_0^{(i)}(p)^2.$$

Hence it follows that

$$\frac{\partial E'}{\partial\mathbf{v}_i}=i\int\mathbf{p}[G_{11}^{(i)}(p)-G_0^{(i)}(p)]\frac{d^4p}{(2\pi)^4}$$
$$=i\int\mathbf{p}G_{11}^{(i)}(p)\frac{d^4p}{(2\pi)^4}=\int\mathbf{p}n_i(p)\frac{d^3p}{(2\pi)^3}=\mathbf{j}_i'$$

($\mathbf{j}_i'$ is the mass flux of the i-th component particles outside the condensate). Using (A1) and also the equations $\mathbf{p}_i=m_i\mathbf{v}_i$ we have

$$\mathbf{j}_1'=\frac{m_1^2n_1'}{\tilde{m}_1}\mathbf{v}_1+\alpha m_1m_2\mathbf{v}_2, \quad \mathbf{j}_2'=\frac{m_2^2n_2'}{\tilde{m}_2}\mathbf{v}_2+\alpha m_1m_2\mathbf{v}_1.$$

For the total mass fluxes we get (see (A2))

$$\mathbf{j}_1=n_{01}m_1\mathbf{v}_1+\mathbf{j}_1'=\frac{m_1^2n_1}{m_1^*}\mathbf{v}_1+\alpha m_1m_2\mathbf{v}_2=\frac{\partial E}{\partial \mathbf{v}_1},$$
$$\mathbf{j}_2=n_{02}m_2\mathbf{v}_2+\mathbf{j}_2'=\frac{m_2^2n_2}{m_2^*}\mathbf{v}_2+\alpha m_1m_2\mathbf{v}_1=\frac{\partial E}{\partial \mathbf{v}_2}. \qquad (16)$$

Using the law for the transformation of the $\mathbf{j}_i$ when we change to another inertial frame of reference, we find

$$\frac{m_{1,2}^2n_{1,2}}{m_{1,2}^*}+\alpha m_{1,2}m_{2,1}=m_{1,2}n_{1,2}, \qquad (17)$$

whence

$$m_{1,2}^*=\frac{m_{1,2}}{1-\alpha m_{2,1}/n_{1,2}}. \qquad (18)$$

Substitution of (18) into the equation for the spectrum (see (15)) gives

$$\varepsilon^4-[\rho_1b_{11}+\rho_2b_{22}-\alpha m_1m_2(b_{11}+b_{22}-2b_{12})]\varepsilon^2p^2$$
$$+(\rho_1\rho_2-\alpha m_1m_2\rho)(b_{11}b_{22}-b_{12}^2)p^4=0. \qquad (19)$$

$$\rho_i\equiv n_im_i,\qquad b_{ik}\equiv\frac{\partial\mu_i'}{\partial\rho_k},\qquad \mu_i'\equiv\frac{\partial E}{\partial\rho_i}=\frac{\mu_i}{m}.\qquad \rho\equiv\rho_1+\rho_2$$

or, after some transformations

$$\varepsilon^4-\left[\left(\frac{\partial P}{\partial\rho}\right)_c+\left\{c(1-c)-\frac{\alpha m_1m_2}{\rho}\right\}\left(\frac{\partial\zeta}{\partial c}\right)_\rho\right]\varepsilon^2p^2$$
$$+\left\{c(1-c)-\frac{\alpha m_1m_2}{\rho}\right\}\left(\frac{\partial\zeta}{\partial c}\right)_P\left(\frac{\partial P}{\partial\rho}\right)_c=0. \qquad (20)$$
$$\zeta\equiv\mu_1'-\mu_2'\qquad c\equiv\rho_1/\rho,$$

where (see (A2), (17))

$$c(1-c)-\frac{\alpha m_1m_2}{\rho}=\frac{1}{\rho^2}\left[\frac{\partial^2E}{\partial v_1^2}\frac{\partial^2E}{\partial v_2^2}-\left(\frac{\partial^2E}{\partial v_{1x}\partial v_{2x}}\right)^2\right]>0$$

(stability condition). It is clear that as

$$\left(\frac{\partial\zeta}{\partial c}\right)_\rho-\left(\frac{\partial\zeta}{\partial\rho}\right)_P=\rho^2\left(\frac{\partial\zeta}{\partial\rho}\right)_c^2\Big/\left(\frac{\partial P}{\partial\rho}\right)_c\neq0,$$

none of the sound velocities is in the general case described by the formula for a classical elastic continuum $u^2=(\partial P/\partial\rho)_c$. For weak mixtures ($c\ll1$) we find

$$u_1\approx\left(\frac{\partial P}{\partial\rho}\right)_c+(\rho_1-\alpha m_1m_2)\rho\left(\frac{\partial\zeta}{\partial\rho}\right)_c^2\Big/\left(\frac{\partial P}{\partial\rho}\right)_c,$$
$$u_2\approx(\rho_1-\alpha m_1m_2)\left(\frac{\partial\zeta}{\partial c}\right)_P\Big/\rho.$$

One verifies easily that the same result for the spectrum follows from a set of hydrodynamic equations[1] at $T=0$:

$$\dot\rho_i+\operatorname{div}\mathbf{j}_i=0,\qquad \dot{\mathbf{v}}_i+\nabla\mu_i'=0, \qquad (21)$$
$$\mathbf{j}_i=\frac{\partial E}{\partial\mathbf{v}_i},\quad \mu_i'=\frac{\partial E}{\partial\rho_i},\quad \Delta E=\frac{1}{2}(\rho_{11}v_1^2+2\rho_{12}\mathbf{v}_1\mathbf{v}_2+\rho_{22}v_2^2).$$

Indeed, from (21) follows the dispersion equation for $u=\omega/k$:

$$u^4-(\rho_{11}b_{11}+\rho_{22}b_{22}+2\rho_{12}b_{12})u^2+(\rho_{11}\rho_{22}-\rho_{12}^2)(b_{11}b_{22}-b_{12}^2)=0,$$

which is equivalent to (19) (here $\rho_{11}=n_1m_1^2/m_1^*$, $\rho_{22}=n_2m_2^2/m_2^*$, $\rho_{12}=\alpha m_1m_2$, see (16), (17)).

5. DRAG EFFECT

The effect of dragging by the superfluid motion of a well-defined fraction of the mass of both components of the mixture from the point of view of a microscopic theory means the following: if the condensate of one of the components is "put into a state of motion," the particles of both components outside the condensates obtain a non-vanishing average velocity. In the diagram language the necessity of this effect is clear from the fact that when a moving condensate is present the diagram for the Green function $\hat G^{(i)}(p)$, and hence also the distribution functions $n_i(\mathbf{p})$ of both components become anisotropic.[3]

In a microscopic approach the drag effect (a quantitative expression for which is given by Eq. (16)) emerges as the consequence of the appearance of a "cross" term in the energy of the state with a moving condensate (A2), i.e., of the non-additivity of the contributions to the energy of the superfluid motions. This is a quantum effect: although the interaction Hamiltonian can be expressed only in terms of the particle number density operators while the average densities are not changed in a uniform superfluid motion, the average value of the interaction energy depends on $|\mathbf{v}_1-\mathbf{v}_2|$—an important contribution to it comes from the density fluctuations which depend on the excitation spectrum and hence also on the relative velocity of the condensates. It is noteworthy that the constant α in the "cross" term serves as an exhaustive characteristic of all manifestations of the drag effect—in the expression (16) for the mass fluxes $\mathbf{j}_i$, in Eq. (20) for the spectrum, in the difference (18) between the "effective" masses m_i^* and the "bare" ones m_i (the constants m_i^* play the role of their own kind of "effective" masses of particles when there is superfluid motion if we use Eq. (A2) for the energy, but not if we use Eq. (16) for the mass fluxes).

It is interesting to note that in the limit of a strong rarefraction of one of the components of the mixture (let it be number 1) the constant m_1^* acquires the role of an effective quasi-particle mass in the usual sense. To wit, using the generalization of equations of the Hugenholtz-Pines and Gavoret-Nozières type we can show (see Appendix, Sec. 2) that as $n_{01}\to0$ one of the branches of the spectrum of the mixture tends to the behavior

$$\varepsilon_p^{(1)}=\frac{p^2}{2m_1^*}+O(\sqrt{n_{01}})p.$$

Neglecting the drag effect ($\alpha=0$) leads to agreement of the spectrum with the results of the theory in[2] and to the absence of a renormalization of the particle mass of the rarefied impurity $m_1^*=m_1$.

We express our deep gratitude to A. F. Andreev for discussing this paper and for important hints.

APPENDIX

1. The Hugenholtz-Pines kind of formulae (6) for the mixture are derived by complete analogy with the one Bose-liquid case.[4] The generalization (7) of the Gavoret-Nozières relations needs some additional remarks which we give below.

If we change in the vacuum diagram E' the "energy factor" of one pair of entry-exit condensate lines of any of the mixture components ($1 \to e^{\mp i\varepsilon t}$) the changes affect the internal Green functions of both components. However, when summing the contributions from the set of diagrams $\Sigma_{mn}^{(i)}(0, \varepsilon)$ constructed on the base of some vacuum diagram E' we get by means of a discussion similar to the one given in[4] a result which is larger by a well-defined factor than the change in the contribution of the diagram E' when simultaneously we make the change $1 \to e^{\mp i\varepsilon t}$ in all condensate lines of the given component of the mixture, which is equivalent to changing the chemical potential $\mu_i \to \mu_i + \varepsilon$ in the internal Green functions of only that component. This leads to equations for $\Sigma_{mn}^{(i)}(0, \varepsilon)$ which are analogous to the ones obtained in[4].

When deriving the relations for $\Delta_1\Sigma_{11}^{12}(0, \varepsilon)$ we assume that the change $1 \to e^{\mp i\varepsilon t}$ simultaneously affects the condensate lines of both components; after that we subtract from the total sum

$$\tfrac{1}{2}\{[n_{01}\Sigma_{11}^{(1)}(0,\varepsilon)+n_{02}\Sigma_{11}^{(2)}(0,\varepsilon)+\sqrt{n_{01}n_{02}}(\Sigma_{11}{}^{12}(0,\varepsilon)+\Sigma_{11}{}^{21}(0,\varepsilon))]$$
$$-[\varepsilon\to-\varepsilon]\}=\varepsilon\left(n_{01}\frac{\partial}{\partial n_{01}}+n_{02}\frac{\partial}{\partial n_{02}}\right)\left(\frac{\partial E'}{\partial\mu_1}+\frac{\partial E'}{\partial\mu_2}\right)$$

the change in $\Sigma_{11}^{(1)}$ and $\Sigma_{11}^{(2)}$.

The derivation of the equations for $\Delta_1\Sigma_{12}^{12}(0, \varepsilon)$ requires a change in the energy factors of the condensate lines which is equivalent to the transformations $\mu_1 \to \mu_1 + \varepsilon$, $\mu_2 \to \mu_2 - \varepsilon$. From the total sum of contributions of entry-exit pairs for the condensate lines of one kind and the entry-entry, exit-exit pairs for different kinds,

$$\tfrac{1}{2}\{[n_{01}\Sigma_{11}^{(1)}(0,\varepsilon)+n_{02}\Sigma_{11}^{(2)}(0,-\varepsilon)+\sqrt{n_{01}n_{02}}(\Sigma_{12}{}^{12}(0,\varepsilon)+\Sigma_{12}{}^{21}(0,-\varepsilon))]$$
$$-[\varepsilon\to-\varepsilon]\}=\varepsilon\left(n_{01}\frac{\partial}{\partial n_{01}}+n_{02}\frac{\partial}{\partial n_{02}}\right)\left(\frac{\partial E'}{\partial\mu_1}-\frac{\partial E'}{\partial\mu_2}\right)$$

we subtract the change in $\Sigma_{11}^{(1)}$ and $\Sigma_{11}^{(2)}$.

We easily obtain the equation for $\Delta_2[\Sigma_{11}^{12}(0, \varepsilon) - \Sigma_{12}^{12}(0, \varepsilon)]$ through the substitution $1 \to e^{\mp i\varepsilon t}$ in all condensate lines; we must from the total sum

$$\tfrac{1}{2}\{[n_{01}(\Sigma_{11}^{(1)}(0,\varepsilon)-\Sigma_{12}^{(1)}(0,\varepsilon)-\mu_1)+n_{02}(\Sigma_{11}^{(2)}(0,\varepsilon)-\Sigma_{12}^{(2)}(0,\varepsilon)-\mu_2)$$
$$+2\sqrt{n_{01}n_{02}}(\Sigma_{11}{}^{12}(0,\varepsilon)-\Sigma_{12}{}^{12}(0,\varepsilon))]+[\varepsilon\to-\varepsilon]\}$$
$$=\frac{\varepsilon^2}{2}\left(\frac{\partial^2E'}{\partial\mu_1^2}+\frac{\partial^2E'}{\partial\mu_2^2}+2\frac{\partial^2E'}{\partial\mu_1\,\partial\mu_2}\right)$$

subtract the terms connected with $\Sigma_{ik}^{(1)}$, $\Sigma_{ik}^{(2)}$.

When deriving the equations for $\Sigma_{mn}^{ik}(p, 0)$ we establish, similar to the preceding, a connection between these quantities and the change in E'—in this case when each of the condensates of the mixture start separately to move (with an infinitesimal velocity):

$$n_{01}[\Sigma_{11}^{(1)}(p,0)-\Sigma_{12}^{(1)}(p,0)-\mu_1]=\Delta E'(p,0),$$
$$n_{02}[\Sigma_{11}^{(2)}(p,0)-\Sigma_{12}^{(2)}(p,0)-\mu_2]=\Delta E'(0,p),$$
$$\sqrt{n_{01}n_{02}}[\Sigma_{11}^{12}(p,0)-\Sigma_{12}^{12}(p,0)]=\tfrac{1}{2}[\Delta E'(\mathbf{p},\mathbf{p})-\Delta E'(p,0)-\Delta E'(0,p)].$$

It is clear from symmetry (isotropy of the initial state when the condensates are not moving) that the expansion of $\Delta E'(\mathbf{p}_1, \mathbf{p}_2)$ ($\mathbf{p}_i$ are the particle momenta in the condensate) must start with second-order terms:

$$\Delta E'(\mathbf{p}_1,\mathbf{p}_2)=\frac{n_1'}{2\tilde{m}_1}p_1^2+\frac{n_2'}{2\tilde{m}_2}p_2^2+\alpha\mathbf{p}_1\mathbf{p}_2 \tag{A1}$$

(see Sec. 5 for the necessity of the inequality $\tilde{m}_i \neq m_i$). When calculating $\hat{G}$ in terms $n_i' p^2/2\tilde{m}_i$ occur only in combination with $n_{0i}p^2/2m_i$:

$$p^2\left(\frac{1}{2m_i}+\frac{n_i'}{2\tilde{m}_in_{0i}}\right)\equiv\frac{n_ip^2}{2m_i^*n_{0i}}$$

(see (12)). The physical meaning of m_i^* follows from the expression for the total energy of a system with moving condensates:

$$\Delta E(\mathbf{p}_1,\mathbf{p}_2)=\left(\frac{n_{01}}{2m_1}+\frac{n_1'}{2\tilde{m}_1}\right)p_1^2+\left(\frac{n_{02}}{2m_2}+\frac{n_2'}{2\tilde{m}_2}\right)p_2^2+\alpha\mathbf{p}_1\mathbf{p}_2$$
$$=n_1\frac{p_1^2}{2m_1^*}+n_2\frac{p_2^2}{2m_2^*}+\alpha\mathbf{p}_1\mathbf{p}_2. \tag{A2}$$

2. Let the concentration of one of the components of the mixture tend to zero: $n_{01} \to 0$. Using the equations (see (7), (8))

$$\Sigma_{11}^{(1)}(0)-\Sigma_{12}^{(1)}(0)=\frac{\partial E'}{\partial n_{01}}=\mu_1,\quad \Delta_1\Sigma_{11}^{(1)}=\varepsilon\frac{\partial^2E'}{\partial n_{01}\,\partial\mu_1},$$
$$\Delta_2(\Sigma_{11}^{(1)}-\Sigma_{12}^{(1)})=\frac{\varepsilon^2}{2n_{01}}\frac{\partial^2E'}{\partial\mu_1^2}-\frac{p^2}{2n_{01}\tilde{m}_1}\frac{\partial E'}{\partial\mu_1}$$

and using the fact that $\Sigma_{12}^{(1)} = O(n_{01})$, $\Sigma_{mn}^{12} = O(\sqrt{n_{01}})$, we find

$$\varepsilon\approx\varepsilon_p^0-\mu_1+\Sigma_{11}^{(1)}(\varepsilon,p)\approx\varepsilon_p^0-\mu_1+\Sigma_{11}^{(1)}(\varepsilon,p)-\Sigma_{12}^{(1)}(\varepsilon,p)$$
$$=\varepsilon_p^0+\varepsilon\frac{\partial^2E'}{\partial n_{01}\,\partial\mu_1}-\frac{p^2}{2n_{01}\tilde{m}_1}\frac{\partial E'}{\partial\mu_1}.$$

As $n_{01} \to 0$ the contribution to terms with derivatives of E' come only from diagrams E' with a minimum two non-vanishing number of condensate lines of the 1-st components (diagrams E' without these lines either do not contain either outside-condensate lines of the 1-st component $G_0^{(1)} = (\varepsilon - \varepsilon_p^0 + \mu_1 + i\delta)^{-1}$, or they contain loops of $G_0^{(1)}$ which give zero when integrated over the frequency). As each condensate line introduces a factor $\sqrt{n_{01}}$ we find

$$\frac{\partial^2E'}{\partial n_{01}\,\partial\mu_1}=\frac{1}{n_{01}}\frac{\partial E'}{\partial\mu_1}$$

so that

$$\varepsilon_p^{(1)}=\frac{p^2}{2m_1^*}+O(\sqrt{n_{01}})p.$$

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[1] An analysis of the divergences, similar to the analysis in[5], for a mixture of Bose-liquids will be given separately.

[1] A. F. Andreev and B. P. Bashkin, Zh. Eksp. Teor. Fiz. **69**, 319 (1975) [Sov. Phys. JETP **42**, 164 (1976)].
[2] I. M. Khalatnikov, Zh. Eksp. Teor. Fiz. **32**, 653 (1957); Pis'ma Zh. Eksp. Teor. Fiz. **17**, 534 (1973) [Sov. Phys. JETP **5**, 542 (1957); JETP Lett. **17**, 386 (1973)].
[3] Yu. A. Nepomnyashchiĭ, Teor. Mat. Fiz. **20**, 408 (1974).
[4] N. M. Hugenholtz and D. Pines, Phys. Rev. **116**, 489 (1959), J. Gavoret and P. Nozières, Ann. Phys. (N.Y.) **28**, 349 (1964).
[5] A. A. Nepomnyashchiĭ and Yu. A. Nepomnyashchiĭ, Pis'ma Zh. Eksp. Teor. Fiz. **21**, 3 (1975) [JETP Lett. **21**, 1 (1975)].
[6] D. A. Kirzhnits and Yu. A. Nepomnyashchiĭ, Zh. Eksp. Teor. Fiz. **59**, 2203 (1970) [Sov. Phys. JETP **32**, 1191 (1971)].

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