Evolution of a modulated wave pulse in a medium with a saturated non-linearity

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We consider the evolution of a strong modulated wave pulse in media with complex models of the non-linear response which describe the non-uniform dependence of the dispersive properties of the medium on the wave intensity. We find analytically stationary field distribution classes that depend continuously on the non-uniformity parameter which corresponds to the amplification or saturation of the quadratic-approximation effects. We obtain exact analytical solutions of the non-linear geometric optics equations which characterize the localization of the region where the pulse is transformed in a medium with non-linear saturation. We consider in the framework of the same approach the set of phenomena of field self-modulation which are connected with the dissipation of the pulse energy and with the non-linear acceleration of the wave.

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1. INTRODUCTION

The aim of the present paper is an analytical study of the self-action of a strong wave pulse in a medium with a complex form of non-linear response. Typical of such a self-action is the connection of the non-linear and dissipative properties of the medium, which lead to the formation of local distortions of the wave field, with the dispersion that causes spreading of the pulse. The simultaneous action of these factors stimulates a related change in the amplitude and frequency envelopes of the pulse, and such a field-transformation process is sensitive to the actual form of the non-linear response and to the field distribution at the boundary of the medium. Simple models of the non-linearity, taking into account not merely the dependence of the dispersive properties of the medium on the wave intensity, but also the non-uniformity of that dependence, enable us to analyze a number of tendencies of the non-linear field transformation. We can then describe the stationary propagation of modulated waves in a medium with a saturated non-linearity (Sec. 2) and the localization in such a medium of the region where the pulse is transformed (Sec. 3). We consider the effects of the stimulation and quenching of a number of self-modulation phenomena under the action of the dissipation of the field energy (Sec. 4) and of the non-linear acceleration of the wave (Sec. 5).

It is well known that the non-linear evolution of the pulse field E in a dispersive medium is described by the equation[1]

$$2ik\frac{\partial E}{\partial z}+b^2\frac{\partial^2 E}{\partial \xi^2}-\frac{k^2(i\varepsilon_1+\varepsilon_2)}{\varepsilon_0}E=0. \quad (1)$$

Here z is the direction of the wave propagation, $\xi=t-zv_0^{-1}$, v_0 the group velocity; the dielectric permittivity ε of the medium has the form $\varepsilon=\varepsilon_0+i\varepsilon_1+\varepsilon_2$, where ε_0 is the linear value of $\mathrm{Re}\,\varepsilon$; $\varepsilon_1=\mathrm{Im}\,\varepsilon$; the function $\varepsilon_2(W)$ depends on the intensity W; $k=\omega c^{-1}\varepsilon_0^{1/2}$, where c is the velocity of light and ω the frequency of the wave. The parameter b^2 is connected with the dispersion of the medium and with the refractive index $n_0=\varepsilon_0^{1/2}$ through the formula

$$b^2=-\frac{k^2}{\omega n_0}\frac{\partial^2(\omega n_0)}{\partial\omega^2}.$$

The function $\varepsilon_2(W)$ describes the non-linear response of the medium. Introducing the non-linearity parameter β:

$$\beta=\left.\frac{\partial\varepsilon_2}{\partial W}\right|_{W=0}, \quad (2)$$

we can characterize the "quadratic" model by the formula $\varepsilon_2=\beta W$. We shall here consider a "power-law" model

$$\varepsilon_2=\beta W+\delta W^2 \quad (3)$$

and a model of a medium with saturation of the non-linearity[1)]

$$\varepsilon_2=\beta W(1+\delta W)^{-1}. \quad (4)$$

For small values of the parameter δ ($|\delta|\ll 1$) the forms (3) and (4) reduce to the "quadratic" approximation. Depending on the signs of the parameters β and δ Eqs. (3) and (4) can span a wide variety of possibilities for non-linear self-action of the wave field.

It is convenient for an analysis of Eq. (1) to write the complex field amplitude in the form $E=Ae^{is}$ and, neglecting the dispersion of the non-linearity, to change from (1) to a set of two equations[4]:

$$\frac{\partial v}{\partial\zeta}+v\frac{\partial v}{\partial q}-\beta^{-1}\frac{\partial\varepsilon_2}{\partial W}\frac{\partial W}{\partial q}-\frac{1}{2}\gamma\frac{\partial}{\partial q}\left[\frac{1}{\sqrt{W}}\frac{\partial^2\sqrt{W}}{\partial q^2}\right]=0, \quad (5)$$

$$\frac{\partial W}{\partial\zeta}+\frac{\partial(Wv)}{\partial q}=-\varkappa W,\quad \varkappa=\frac{k\varepsilon_1}{2b^3c^2\varepsilon_0\sqrt{|\beta|}}. \quad (6)$$

ζ and q in Eqs. (5) and (6) are independent variables which are connected with the variables z and ξ through the relations $q=(bc)^2\xi$, $\zeta=b^3c^2z|\beta|^{1/2}$, where the parameter β is defined by (2), while the parameter γ is $\gamma=(bc)^4b^2k^{-2}|\beta|^{-1}$. The dimensionless functions W and

v describe the distribution of the intensity I and the modulation v in the pulse:

$$W = II_0^{-1}, \quad v = \frac{b}{k\sqrt{|\beta|}} \frac{\partial s}{\partial \xi} \operatorname{sign} b^2. \tag{7}$$

Here I_0 is the maximum value of the intensity of the pulse at the boundary of the non-linear medium ($\zeta = 0$).

We use the set (5), (6) to consider some stationary and non-stationary regimes of the propagation of a wave in a non-linear medium.

2. STATIONARY MODULATED WAVES

We shall construct in this section stationary solutions of the self-action Eqs. (5), (6) in a conservative medium ($\varkappa = 0$), the non-linear properties of which are described by model (3). In this model the non-linear response of the medium with increasing intensity changes non-uniformly ($\partial\varepsilon_2/\partial W \neq \text{const}$) which leads to an appreciable widening of the classes of stationary waves.

In the stationary case the set of Eqs. (5), (6) has the first integrals

$$\frac{v^2}{2} - \frac{\varepsilon_2}{\beta} - \frac{1}{2}\gamma \frac{\partial^2 \sqrt{W}}{\partial q^2} = -C_1, \quad vW = C_0.$$

Assuming that the maximum value of the function W of (7) in the maximum of the distribution equals unity we can write the stationary profile in implicit form:

$$\frac{t}{T_0} = \frac{1}{2} \int\limits_{\sqrt{W_{min}}}^{\sqrt{W}} dW \{W(1-W)[\alpha W^2 + (\alpha \pm 1) W + B_\mp^\pm]^{-1/2}\}. \tag{8}$$

The + and − signs correspond here respectively to the cases $b^2\varepsilon_2 > 0$ and $b^2\varepsilon_2 < 0$; in those cases the values of the parameters α, B_-^+ have the form

$$B_-^+ = \alpha - 2C_1 + {}^1\!/_2 C_0^2 \pm 1, \quad \alpha = {}^2\!/_3 \delta |\beta^{-1}|. \tag{9}$$

The characteristic time T_0 is equal to

$$T_0 = b(bc)^2/k\sqrt{|\beta|}.$$

Depending on the relation (9) between the parameters of the medium and of the pulse the solution of (8) can describe stationary regimes in the form of solitary waves and oscillating field distributions. We consider examples of such regimes under the conditions $b^2\varepsilon_2 > 0$ and $b^2\varepsilon_2 < 0$. In the case $b^2\varepsilon_2 > 0$ we are interested in the situation which is connected with the saturation of the non-linearity in the field of a strong pulse in the model (4); this corresponds to different signs of β and δ. In that case we have under the conditions

$$(1+\alpha)^2 = 4\alpha B^+, \quad -1 < \alpha < {}^1\!/_3$$

the possibility of a one-parameter family of stationary field distributions in the form of a solitary wave:

$$W = \mathrm{ch}^2\left(\frac{q}{T_0} M\right)\left[1 - 2\alpha(1+\alpha)^{-1} \mathrm{sh}^2\left(\frac{q}{T_0} M\right)\right]^{-1}, \quad M = [-1 - 2\alpha(1+\alpha)^{-1}]^{1/2}.$$

Here and henceforth the factor M takes into account the difference between the non-linearity model (3) and the "quadratic" model. It is important that the effect of the saturation of the non-linearity leads, in contrast to the "quadratic" non-linearity model, to a finite value of the intensity, even in the limit as $q \to \pm\infty$:

$$\lim W|_{q\to\pm\infty} = W_0 = -{}^1\!/_2(1+\alpha^{-1}),$$

where $0 \leqslant W \leqslant 1$ (Fig. 1). We note the possibility, characteristic for model (3), of a "transitional" regime ($B^+ = 0$, $\alpha = -0.5$) which is connected with a monotonic distribution (see Fig. 1):

$$W = \left[1 + \left(\frac{1-W_0}{W_0}\right) \times \exp\left(-\frac{q}{T_0} M\right)\right]^{-1}, \quad M = 2^{1/2}.$$

On the other hand, in the parameter range $B^+ = 0$, $\alpha > -0.5$ a stationary intensity distribution is possible which tends to zero at the edges of the pulse:

$$W = \left[\mathrm{ch}^2\left(\frac{q}{T_0} M\right) + \alpha(1+\alpha)^{-1} \mathrm{sh}^2\left(\frac{q}{T_0} M\right)\right]^{-1}, \quad M = (1+\alpha)^{1/2}. \tag{10}$$

The solution (10) can exist in a medium without saturation ($\alpha > 0$).

Equation (8) describes for suitable choice of the parameters also a number of oscillating regimes. For instance, when $\alpha > 0$, $(1+\alpha)^2(4\alpha)^{-1} > B^+ > 0$ a persistent amplitude-modulated profile is connected with the Jacobi function:

$$W = \mathrm{cn}^2\left(\frac{q}{T_0} M\right), \quad M = \left\{B^+ + \frac{1+\alpha}{2} + \frac{1}{2}((1+\alpha)^2 - 4\alpha B^+)^{1/2}\right\}^{1/2}.$$

The case $b^2\varepsilon_2 < 0$ can be analyzed similarly to the case $b^2\varepsilon_2 > 0$; without writing down the cumbersome general formulae we note that in the case $B_- = 0$, $\alpha < \frac{1}{3}$ a "soliton" field distribution is possible:

$$W = \mathrm{th}^2\left(\frac{q}{T_0} M\right)\left[1 - \alpha(1-2\alpha)^{-1} \mathrm{ch}^{-2}\left(\frac{q}{T_0} M\right)\right]^{-1}, \quad M = (1-3\alpha)^{1/2}. \tag{11}$$

This distribution is characterized by a minimum in the center and an asymptotic tending to a finite value at infinity ($\lim W|_{q\to\pm\infty} = 1$).

The above-mentioned solutions describe both an amplification and a depression of the effects of the quadratic approximation where stationary waves of envelopes are also possible; the distributions (10) and (11) here generalize the results of the "quadratic" model, which corresponds to the particular case $\delta = 0$ and leads to the well known solutions W_1[5] and W_2[6]:

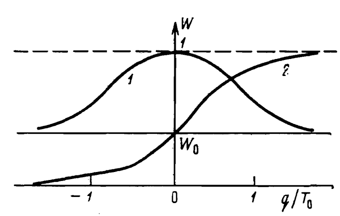

FIG. 1. Stationary profiles of solitary waves in a medium with saturation of the non-linearity: 1—solitary wave on a finite background, 2—"transitional" regime.

$W_1 = \mathrm{ch}^{-2}(q/T_0), \quad W_2 = \mathrm{th}^2(q/T_0).$

However, the non-uniform dependence of the non-linear response of the medium on the wave intensity which is taken into account in model (3) by the quantity δ manifests itself in the existence of whole families of stationary field distributions which depend continuously on the parameter which characterizes the field modulation and the non-uniformity of the response $\varepsilon_2(W)$.

3. LARGE-SCALE PICTURE OF THE EVOLUTION OF A WAVE IN A MEDIUM WITH SATURATION OF THE NON-LINEARITY

We shall consider in this section the non-stationary case of the large-scale non-linear evolution of a wave pulse in a medium with saturation. The pace of the evolution is determined by the competition between the processes of the non-linear self-action and the dispersive spreading out. It is sufficient for evolution of strong fields that the non-linear distortions of the field structure under the action of the inhomogeneity of the amplitude-phase distribution accumulates faster than the dispersive spreading out of the initial field distribution occurs. Moreover, in a number of problems the size of the region occupied by the non-linear medium is limited so that the amplitude and phase distribution in the field at the entrance to the medium plays an important role in the self-action process for the field in the whole of this region. It makes sense to consider for the evolution of a pulse under such conditions simplified problems which are connected with an application of geometric optics. For a geometric optics description of the evolution of pulses with a sufficiently long duration t the dispersive effects are weak so long as $t^2k^2|b^{-2}||\varepsilon_2| \gg 1$. In that case, neglecting the last term in (5) we get the eikonal equation from non-linear geometric optics in the form

$$\frac{\partial v}{\partial \zeta_1} + v\frac{\partial v}{\partial q_1} - |\beta^{-1}|\frac{\partial \varepsilon_2}{\partial W}\frac{\partial W}{\partial q_1} = 0. \tag{12}$$

We have introduced here the dimensionless variables $\zeta_1 = \zeta/T$, $q_1 = q/T$; in (6) we have then $\varkappa_1 = \varkappa T$, where T is the characteristic length of the pulse. Equations (12) and (6) form a set describing the simultaneous change in the intensity and the spectrum of the pulse.

It is important that the non-linear geometric-optics set of equations (6) and (12) admits of a wide class of exact analytical solutions. For the construction of such solutions it is convenient to transform this set to a linear one, by regarding W and v as independent variables and $\zeta_1(W, v)$ and $q_1(W, v)$ as functions. The set of two first-order differential equations obtained this way can be reduced to one second-order differential equation which expresses the quantities ζ_1 and q_1 in terms of some function $\psi(W, v)$ through relations such as the Legendre transformation. This can be accomplished in two different ways by choosing the function ψ to satisfy exactly either an eikonal equation, or the transfer equation in (W, v) space. Of course, the eigenfunctions ψ are in each case different. Depending on the actual form of the non-linear response of the medium and the initial shapes of the amplitude and frequency envelopes one or the other method is the convenient one to use. In particular, in the "quadratic" model of the non-linearity exact analytical solutions of the equations for the self-action of the pulse[7] are known which are obtained using a function ψ which satisfies the eikonal equation exactly. Similar transformations are also used to describe the one-dimensional motion of a compressible gas.[8] In contrast to this, in our problem we use another transformation, which satisfies exactly the transfer equation in (W, v) space. The functions $\zeta_1(W, v)$ and $q_1(W, v)$ are then introduced by the formulae (for the sake of simplicity we have dropped here and henceforth the indices of the dimensionless quantities ζ_1 and q_1)

$$\zeta = \frac{\partial \psi}{\partial v}, \quad q = -\psi - W\frac{\partial \psi}{\partial W} + v\frac{\partial \psi}{\partial v}. \tag{13}$$

The equation for the function ψ which follows from the eikonal equation then has the form ($W = p^2$, $v = 2u$)

$$\frac{\partial^2 \psi}{\partial p^2} + \frac{3}{p}\frac{\partial \psi}{\partial p} + |\beta^{-1}|\frac{\partial \varepsilon_2}{\partial W}\frac{\partial^2 \psi}{\partial u^2} = 0. \tag{14}$$

The boundary conditions for this equation are connected with the structure of the field of the pulse at the boundary of the non-linear medium ($\zeta = 0$):

$$\left.\frac{\partial \psi}{\partial u}\right|_{u=0} = 0, \quad q(p)|_{u=0} = -\psi - \frac{1}{2}p\left.\frac{\partial \psi}{\partial p}\right|_{u=0}. \tag{15}$$

In the approximation considered the function ψ satisfying Eq. (14) and the condition (15) contains in it all the information about the non-linear evolution of the pulse inside the medium.

We consider now the model (4) of a medium with saturation of the non-linearity, using (14) and (15). After separating the variables $\psi(p, u) = f(p)F(u)$ and introducing a new variable $p = (1 - y)^{1/2}(\delta y)^{-1/2}$ we get a set of equations for the functions $f(p)$ and $F(u)$:

$$\partial^2 F/\partial u^2 - k^2 F = 0,$$

$$y(1-y)\frac{\partial^2 f}{\partial y^2} - 2y\frac{\partial f}{\partial y} + \frac{k^2}{4\delta}f = 0.$$

The function f is determined by the hypergeometric equation, which indicates the possibility of constructing a number of analytical solutions of Eq. (14). In particular, a simple solution, corresponding to the value $k^2 = 8\delta$, has the form

$$\psi = -(1+\delta p^2)^{-1}[A\,\mathrm{ch}(2u\sqrt{2\delta}) + B\,\mathrm{sh}(2u\sqrt{2\delta})]. \tag{16}$$

One sees easily that the function ψ of (16) describes a pulse with an envelope which at the boundary of the non-linear medium ($z = 0$) when we change the variable q from $q = A$ to $q = 0$ has the form

$$W|_{z=0} = \delta^{-1}[(q_0/A)^{-1/2} - 1]. \tag{17}$$

The distribution (17) characterizes a one-parameter family of pulses (A is the parameter); while at the boundary of the medium there is no modulation ($u|_{z=0} = 0$). Using Eq. (13) we can construct the expressions $\zeta(W, u)$

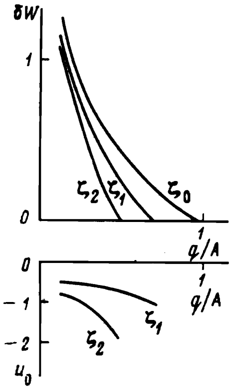

FIG. 2. Localization of the self-modulation region inside a medium with saturation of the non-linearity ($\zeta_2 = 2\zeta_1$, $u_0 = 4A\delta\zeta_1^{-1}u$).

and $q(W, u)$, which implicitly describe the evolution of the original field distribution inside the medium:

$$\frac{\zeta}{A} = -\frac{\sqrt{2\delta}\,\mathrm{sh}\,(2u\sqrt{2\delta})}{1+\delta p^2} \qquad \frac{q}{A} = -\frac{2u\zeta}{A} + \frac{\mathrm{ch}\,(2u\sqrt{2\delta})}{(1+\delta p^2)^2}. \tag{18}$$

Equations (18) characterize the simultaneous development of the processes of the self-compression of the pulse (17), the formation of its front, and the development of the modulation when the pulse progates in a medium with the non-linear saturation (4). We show in Fig. 2 an example of such a restructuring of a pulse. We see here the effect of the localization of the region of the non-linear restructuring of the field near the leading edge of the pulse. Indeed, we can write the solution (18) on the periphery of the pulse in the form

$$\zeta = -4\delta u, \quad q = q_0 + \zeta u,$$

where the distribution of q_0 is defined by (17) on the surface $\zeta = 0$. Such an asymptotic behavior corresponds to the well known expression for the "linear" profile[7] in the "quadratic" model of the non-linear response. In the region mentioned ($\delta W \ll 1$) the slope of the front of the pulse increases, and the modulation grows proportional to the path traversed. However, in contrast to the "quadratic" model the solution (18) describes the gradual weakening of the above-mentioned non-linear effects when the pulse passes into the saturation region for large values of the intensity.

In the particular case when the saturation is small so that in model (4) $|\delta W| \ll 1$ the transformation (13) leads to the five-dimensional axisymmetric Laplace equation

$$\frac{\partial^2\psi}{\partial p^2} + \frac{3}{p}\frac{\partial\psi}{\partial p} + \frac{\partial^2\psi}{\partial u^2} = 0. \tag{19}$$

Introducing an ellipsoidal set of coordinates (θ, η) in (p, u) space, by using the formulae[4]

$$W = (1+\theta^2)(1-\eta^2), \quad u = \theta\eta,$$

where $\theta \geq 0$, $-1 \leq \eta \leq 1$, we can write Eq. (19) in a symmetrical form:

$$(1+\theta^2)\frac{\partial^2\psi}{\partial\theta^2} + 4\theta\frac{\partial\psi}{\partial\theta} + (1-\eta^2)\frac{\partial^2\psi}{\partial\eta^2} - 4\eta\frac{\partial\psi}{\partial\eta} = 0. \tag{20}$$

The solution of Eq. (20) has the form

$$\psi = \sum_n \{T_n(\eta)[A_nT_n(i\theta) + B_nR_n(i\theta)] + R_n(\eta)[C_nT_n(i\theta) + D_nR_n(i\theta)]\}; \tag{21}$$

here R_n and T_n are Jacobi polynomials and functions, and A_n, B_n, C_n, and D_n are coefficients determined from the initial conditions (15),

$$\begin{aligned} &R_0(\eta) = 1, \quad T_0(\eta) = \frac{1}{2}\left[\frac{\eta}{1-\eta^2} + \frac{1}{2}\ln\frac{1+\eta}{1-\eta}\right]; \\ &R_1(\eta) = \eta, \quad T_1(\eta) = \frac{1}{2}\left[2 - \frac{\eta^2}{1-\eta^2} - \frac{3}{2}\eta\ln\frac{1+\eta}{1-\eta}\right]; \\ &R_2(\eta) = 1-5\eta^2, \quad T_2(\eta) = \frac{1}{2}\left[15\eta - \frac{2\eta}{1-\eta^2} + \frac{3}{2}R_2(\eta)\ln\frac{1+\eta}{1-\eta}\right]; \\ &R_3(\eta) = 3\eta-7\eta^3, \quad T_3(\eta) = \frac{1}{16}\left[16 - 99\eta^2 + \frac{6\eta^4}{1-\eta^2} - \frac{15}{2}R_3(\eta)\ln\frac{1+\eta}{1-\eta}\right]\ldots \end{aligned} \tag{22}$$

Substituting expression (21) into Eq. (12) we can construct a wide variety of exact analytical solutions of the equations from non-linear geometric optics.

As an example we consider pulses with envelopes which at the boundary of the medium ($\theta = 0$) are described by the complex function (Fig. 3):

$$q|_{\theta=0} = \frac{2\sqrt{1-W_0}}{W_0} - \frac{3}{2}g\sqrt{1-W_0} - \frac{1}{4}(30W_0 - 10 - g)\ln\frac{1+\sqrt{1-W_0}}{1-\sqrt{1-W_0}}. \tag{23}$$

This formula characterizes a one-parameter family (g is the parameter) of intensity distributions $W_0 = 1 - \eta^2$, which decrease monotonically from the center ($g = 0$, $W_0 = 1$) to the periphery ($\lim W_0|_{g\to\pm\infty} = 0$). One sees easily that the relation (23) is single-valued if the values of the parameter g are limited by the condition $-14 < g < -10$. Assuming in what follows that this condition is satisfied we can study the development of the whole family of profiles (23) in a medium with a quadratic non-linearity. Substituting (23) into the boundary condition (15) we find the values of the coefficients in the expression (21): $B_0 = 2 - g$, $C_1 = g - 10$, $B_2 = -1$; the other coefficients zero. The function ψ characterizing the evolution of the profile (23) has the form

$$\psi = (2-g)T_0(\eta)R_0(i\theta) + (g-10)R_1(\eta)T_1(i\theta) - T_2(\eta)R_2(i\theta). \tag{24}$$

One obtains the expressions for the functions $\zeta(\theta, \eta)$ and $q(\theta, \eta)$ which implicitly describe the field structure inside the medium at once by substituting the function ψ of (24) into the formulae (13).

We must note in the case of the "quadratic" non-linearity also another class of exact analytical solutions of the equations from non-linear geometric optics

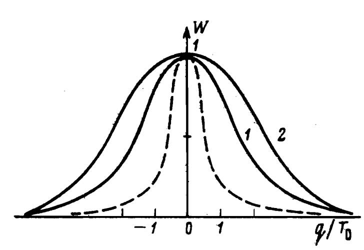

FIG. 3. Family of pulses corresponding to different values of the parameter g (the dashed curve shows the pulse $W = \cosh^{-2}q$); curve 1: $g = -11$, 2: $g = -13$.

which is obtained by using a Legendre type transformation which turns the eikonal equation into an identity. We can in that way describe the non-linear evolution of pulses with an envelope which at the boundary of the medium is a linear combination of Legendre functions. In contrast to this, the transformations (13) indicated here and the solutions (22) enable us to widen the set of boundary conditions which allow an exact analytical solution in the framework of the geometric-optics approach.

It is important that the transformation (13) which is connected with the exact satisfying of the continuity equation in W, v-space does not impose any restrictions on the structure of the non-linear response of the medium $\varepsilon_2(W)$. For different forms of the function $\varepsilon_2(W)$ the application of the transformation (13) to the eikonal equation (12) leads to a linear differential equation in which the function $\varepsilon_2(W)$ is a coefficient. Such a transformation is therefore convenient for an analysis of non-linear wave motions in media characterized by a complex form of the response $\varepsilon_2(W)$.

4. NON-LINEAR EVOLUTION OF A PULSE IN A DISSIPATIVE MEDIUM

The absorption of a wave in a medium which smoothes out the initially inhomogeneous field distribution can lead to an important change in the processes of the restructuring of the amplitude-phase profiles of the pulse. A self-consistent description of such processes which take place in a medium under the influence of non-linearity, dispersion, and absorption is in the geometric optics approximation given by the set of Eqs. (6) and (12). Using the "quadratic" model of the non-linearity we get, by analogy with (14), the equation which characterizes the deformation of the pulse:

$$\frac{\partial^2\psi}{\partial p_1^2}+\frac{3}{p_1}\frac{\partial\psi}{\partial p_1}-\frac{\partial^2\psi}{\partial u^2}\exp\left(-\varkappa_1\frac{\partial\psi}{\partial u}\right)=0. \tag{25}$$

Here $p_1^2=W_1=W\exp(-\varkappa_1\zeta)$ while the function ψ must satisfy the boundary condition (15) with the substitution $p\to p_1$.

In contrast to (14), the Eq. (25) for the function ψ in a dissipative medium is non-linear. However, one of the exact solutions of this equation can be written in the form

$$\psi(p_1,u)=-\frac{Bp_1^2}{2}-\frac{1}{4B\varkappa_1^2}[(1+4B\varkappa_1u)\ln(1+4B\varkappa_1u)-4B\varkappa_1u]. \tag{26}$$

The function ψ in (26) describes the evolution of the perphery of a one-parameter family of pulses with envelopes which at entering the medium change according to the relation

$$W|_{z=0}=B^{-1}q,\quad u|_{z=0}=0,$$

where B is a parameter. Using Eqs. (13) we can find the distribution of the field intensity and modulation inside the medium:

$$\begin{gathered}q=BW\exp(-\varkappa_1\zeta)+(4B\varkappa_1^2)^{-1}[\ln(1+4B\varkappa_1u)-4B\varkappa_1u],\\ u=(4B\varkappa_1)^{-1}[\exp(-2\varkappa_1\zeta)-1].\end{gathered} \tag{27}$$

This distribution characterizes the change in the field gradient at the periphery of the pulse; the wave modulation which occurs tends inside the medium to a constant value which depends on the rate of the energy dissipation $\lim u|_{\zeta\to\infty}=-(4B\varkappa_1)^{-1}$. When the dissipation diminishes ($\varkappa_1\to 0$) Eqs. (27) change to the well known result on the evolution of a "linear" profile.[7]

It is important that after changing the sign of the parameter $\varkappa_1$ Eqs. (27) are valid for an active medium ($\varkappa_1>0$). In that case the slope of the pulse front and its modulation will increase inside the medium.

5. EVOLUTION OF THE SELF-MODULATION WHEN A WAVE IS ACCELERATED NON-LINEARLY

The deformation of a pulse which in a non-linear conservative medium is described by Eqs. (6), (12) with $\varkappa=0$ depends on the form of the envelopes at the boundary of the medium. However, in such a model we neglect the dependence of the group velocity of the wave on its intensity. A more realistic model which takes this dependence into account describes the additional deformation of the pulse which is connected with the displacement of the region of larger intensities relative to the region of smaller intensity. To describe the simultaneous development of both deformations of the pulse we consider a uniform isotropic medium with a non-linear response which is "quadratic" in the field and limited in such a way that $1\gg|\beta|^{1/2}|bv_0-n_0(bc)^{-1}|$. Taking the non-linear wave acceleration into account then leads to a generalization of the self-action Eqs. (6) and (12):

$$\frac{\partial v}{\partial\zeta}+v\frac{\partial v}{\partial q}-\frac{\partial W}{\partial q}+N\frac{\partial(Wv)}{\partial q}=0, \tag{28}$$

$$\frac{\partial W}{\partial\zeta}+\frac{\partial(Wv)}{\partial q}+NW\frac{\partial W}{\partial q}=0, \tag{29}$$

$$N=|\beta|^{1/2}(bv_0)^{-1}(1+nv_0c^{-1})\ \mathrm{sign}\,\varepsilon_2. \tag{30}$$

We can, as in Secs. 3 and 4, construct an exact analytical solution of the set of non-linear Eqs. (28), (29) by taking the functions W and v as new independent variables and the quantities ζ and q as the required functions. However, it is convenient to introduce the function ψ, in contrast to (13), through different formulae:

$$\zeta=-\frac{\partial\psi}{\partial W},\quad q=-v\frac{\partial\psi}{\partial W}-\frac{\partial\psi}{\partial v}+N\left(\psi-W\frac{\partial\psi}{\partial W}+v\frac{\partial\psi}{\partial v}\right). \tag{31}$$

The function ψ is then described by the equation ($W=p^2$, $v=2u$)

$$\frac{\partial^2\psi}{\partial p^2}+\frac{1}{p}\frac{\partial\psi}{\partial p}+(1-2Nu)\frac{\partial^2\psi}{\partial u^2}-4N\frac{\partial\psi}{\partial u}=0. \tag{32}$$

The boundary conditions for (32) follows from (31):

$$\left.\frac{\partial\psi}{\partial p}\right|_{u=0}=0,\quad \left.-\frac{1}{2}\frac{\partial\psi}{\partial u}+N\psi\right|_{u=0}=q(p)|_{u=0}.$$

We can write the general solution of Eq. (32) in the form

$$\begin{gathered}\psi=y^{-1}\{I_0(kp)[A_kJ_1(y)+B_kN_1(y)]+K_0(kp)[C_kJ_1(y)+D_kN_1(y)]\},\\ y=kN^{-1}(1-2Nu)^{1/2}.\end{gathered} \tag{33}$$

J_1 and N_1 are here Bessel and Neumann functions of the first order, I_0 and K_0 cylindrical functions of an imaginary argument; A_k, B_k, C_k, D_k, and k constants ($k>0$). In particular, the function ψ from (33) describes for $C_k=D_k=0$, $A_k=RN_1(k/N)$, $R=\text{const}$, $B_k=-RJ_1(k/N)$ the evolution of a family of pulses with the boundary condition

$$q|_{u=0}=RI_0(kp). \quad (34)$$

The field distribution inside the medium which satisfies (34) is given by the formulae

$$\zeta=-\frac{kR}{2p}I_1(kp)\varphi(u),\quad \varphi(u)=y^{-1}[A_kJ_1(y)+B_kN_1(y)]; \quad (35)$$

$$q=R\left\{-\frac{uk}{p}I_1(kp)\varphi(u)-\frac{1}{2}I_0(kp)\frac{\partial\varphi}{\partial u}+N\left[I_0(kp)+\frac{kp}{2}I_1(kp)\right]\varphi(u)+NuI_0(kp)\frac{\partial\varphi}{\partial u}\right\}. \quad (36)$$

These formulae describe the simultaneous development of the self-compression and non-linear acceleration processes of the wave. The tendency to self-compression is thus amplified in a focusing medium ($\varepsilon_2>0$, $N>0$). The increase in the slope of the pulse front is then accelerated leading to the appearance of a region with an appreciable field gradient at the leading edge of the pulse; the values of the variable $u=u_0$ corresponding to this region is determined by substituting the solution (35), (36) into the condition $\partial q/\partial W|_\zeta=0$, $W=0$.

The effect of the non-linear acceleration of the wave on the collection of self-modulation phenomena in a medium with dispersion is here characterized by the parameter N of (30). If the dispersion of the medium or the wave speed are appreciable so that $|N|\ll 1$, the additional shift of the region with an appreciable intensity develops more slowly than the self-compression connected with the non-linear broadening of the pulse spectrum. The right-hand side of Eq. (32) which takes the acceleration of the pulse maximum into account can then be considered to be a perturbation to the background of the "fast" self-compression process. When an initially symmetric pulse, described by functions such as (23), evolves, this perturbation leads, while the maximum increases, to an additional displacement of it and a "slow" loss of the pulse symmetry.[9] In the opposite case, when the dispersion is negligible the above-mentioned shift of the maximum determines the formation of pulse fronts.[10]

The exact analytical solutions of the equations of the self-action of the field which we have considered here indicate the tendency for a controlled restructuring of the field depending on the original form of the pulse and on the non-linear, dispersive, and dissipative properties of the medium. These solutions describe the large-scale evolution of the field which precedes the formation of regions with a large field gradient. After the formation of such regions, when wave effects are important, the asymptotic solution of the problem of the self-modulation of a one-dimensional field distribution was found for the case of a conservative medium with a quadratic non-linearity in[11].

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[1)]The model (4) of the non-linear response of a medium was used in Refs. 2, 3 for a numerical study of the stationary distributions of waves with cylindrical and spherical symmetry.

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