Theory of weak ferromagnetism of a Fermi liquid

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(Submitted December 16, 1975)
Zh. Eksp. Teor. Fiz. 70, 1987–2005 (May 1976)

A microscopic quantum theory of a weakly ferromagnetic Fermi liquid is proposed. The spin-excitation spectrum, consisting of spin fluctuations of the paramagnon type and a transverse spin-wave branch, is determined. It is shown that at temperatures $T \ll T_C^2/\epsilon_F$ (T_C is the Curie temperature, ϵ_F is the Fermi energy) in the absence of a magnetic field ($H=0$) the spin-density dependence $S(T)$ is determined by the spin-wave contribution and is described by the law $S(T)-S(0)\sim T^{3/2}$. In the region $T_C^2/\epsilon_F \ll T < T_C$ the paramagnon contribution is dominant, leading to the formula $S(T)=S(0)[1-(T/T_C)^{4/3}]^{1/2}$. For $T_C \ll T \ll \epsilon_F$ the susceptibility varies as $\chi \sim T^{-4/3}$. The longitudinal susceptibility for $T<T_C$, $H\to 0$ diverges like $H^{-1/2}$. The contribution of the spin excitations to the specific heat for $T \gg T_C^2/\epsilon_F$ is proportional to $T \ln T$. The calculations are performed by the methods of quantum field theory.

PACS numbers: 75.10.Dg, 75.30.Fv

1. INTRODUCTION

In most ferromagnets the magnetic order is the result of the exchange interaction. The Heisenberg model based on this mechanism gives an explanation of many properties of nonconducting magnetic systems, at both low and high temperatures. The interpretation of the properties of metallic ferromagnets follows from the consideration of ferromagnetic Fermi liquids.[1–3]

A special place amongst ferromagnetic metals is occupied by weak band ferromagnets. The most well-known of these are the intermetallic compounds $ZrZn_2$ and Sc_3In. A characteristic feature of such systems is that their average magnetic moment per atom is found to be considerably smaller than its nominal value at all temperatures. The Curie temperature T_C is well small compared with the Fermi energy ε_F: $T_C \ll \varepsilon_F$. The magnetic susceptibility as $T\to 0$ is large compared with the Pauli susceptibility. The increase in magnetic moment that occurs with increasing intensity of the applied magnetic field does not cease, right up to the strongest fields that have been used in the experiments.

The first attempt at a theoretical treatment of weak band ferromagnets was undertaken in the papers[4,5] of Edwards and Wohlfarth, who assumed that the main contribution to the thermodynamics of these systems is given by the thermal one-electron excitations. Despite the agreement of the theory of [4,5] with the experimental data on the magnetization in strong fields, the temperature dependences stemming from the theory turned out to be substantially weaker than those observed experimentally. A result of this deficiency was, in particular, a considerable overestimation, by a factor of almost ε_F/T_C, of the magnetic susceptibility in the paramagnetic region of temperatures.

Murata and Doniach[6] pointed out that the principal temperature dependence of the quantities for a weak ferromagnet arises from the thermal excitation of spin fluctuations. However, they ignored the quantum effects, which are important because of the low-temperature character of the magnetic transition. In a subsequent paper by Murata,[7] the quantum effects were taken into account qualitatively by the introduction of a cutoff in the integration over the wave-vectors of the spin fluctuations. In both papers[6,7] their authors started from a model classical Hamiltonian describing a one-component fluctuating field.

In the present paper a microscopic quantum theory of a weakly ferromagnetic Fermi liquid is proposed. In Sec. 2 the spectrum of the spin excitations is found. It consists of spin fluctuations of the paramagnon type and a transverse spin-wave branch. In Secs. 3 and 4 the thermodynamic properties of the system are considered. For $T \ll T_C^2/\varepsilon_F$, in the absence of an external magnetic field ($H=0$), the temperature variation of the spin density $S(T)$ is determined by the spin-wave contribution and is described by the well-known law

$$S(T)-S(0)\sim T^{3/2}.$$

In the region $T_C^2/\varepsilon_F \ll T < T_C$, $H=0$, the spin density is given by the formula

$$S(T)=S(0)[1-(T/T_C)^{4/3}]^{1/2}.$$

The susceptibility of the paramagnetic phase in weak fields for $T_C \ll T \ll \varepsilon_F$ varies according to the law

$$\chi\sim T^{-4/3}.$$

The longitudinal susceptibility for $T<T_C$ has a divergence $\sim H^{-1/2}$, due to the spin-wave contribution. The contribution of the spin fluctuations to the specific heat C_s of a weakly ferromagnetic Fermi liquid for $T \gg T_C^2/\varepsilon_F$ is proportional to $T\ln T$. In Sec. 5 the limits of applicability of the theory developed are established. In the Appendix we give the derivation of a number of relations for the vertex parts determining the interaction between the spin excitations. In treating the spin density and susceptibility we neglect the contribution of the thermal Fermi excitations, which is essentially small compared with the contribution of the spin excitations. We also disregard the influence of interactions of a magnetic nature that are small compared with the exchange interaction.

It should be noted that dependences for $S(T)$ ($T \gg T_C^2/$

ε_F), χ and C_s analogous to those given above have also been obtained in the paper[7] by Murata. However, the classical model which he used did not give the possibility of establishing the numerical coefficients in χ and C_s or of finding the quantitative relationship between $S(0)$ and the parameters characterizing the system. In this sense, unlike ours, the results of Murata have a qualitative character. Because of the one-component nature of the fluctuations considered in his paper, [7] the effect of transverse spin-waves on the thermodynamic properties of the ferromagnet was also taken into account in[7].

2. SPIN-EXCITATION SPECTRUM

The spectrum of the single-particle excitations of a Fermi liquid is given by the poles of the Green function $G_{\alpha\beta}(p)$ [8] ($p=\{\varepsilon, \mathbf{p}\}$ is the {energy, momentum}: α, β are the spin variables). For $\varepsilon\to 0$, $|\mathbf{p}|\to p_+, p_-$ (p_+ and p_- are the Fermi momenta of electrons with opposite spin projections), the G-function has the form[2]:

$$G_{\alpha\beta}(p)\approx\frac{a_+}{\varepsilon-v_+(|\mathbf{p}|-p_+)+i\delta\operatorname{sign}\varepsilon}\left(\frac{1}{2}\delta_{\alpha\beta}+\mathbf{n}\mathbf{S}_{\alpha\beta}\right)+\frac{a_-}{\varepsilon-v_-(|\mathbf{p}|-p_-)+i\delta\operatorname{sign}\varepsilon}\left(\frac{1}{2}\delta_{\alpha\beta}-\mathbf{n}\mathbf{S}_{\alpha\beta}\right),\quad \delta\to+0. \tag{2.1}$$

Here $\mathbf{S}_{\alpha\beta}$ are the spin-$\frac{1}{2}$ matrices, $\mathbf{n}$ is the unit vector in the direction of the total spin of the system, $\varepsilon_\pm(\mathbf{p})=v_\pm(|\mathbf{p}|-p_\pm)$ are the energies of the quasi-particles, v_+ and v_- are their velocities, and a_+ and a_- are renormalization constants.

The weakness of the ferromagnetism corresponds to the condition

$$\Delta\equiv p_+-p_-\ll p_+,\ p_-.$$

In the calculation of the spin-excitation spectrum the difference in the quasi-particle energies $\varepsilon_+(\mathbf{p})$ and $\varepsilon_-(\mathbf{p})$ is substantial only in the region of momenta $|\mathbf{p}|\sim p_+$, p_-, where $|\varepsilon_+(\mathbf{p})-\varepsilon_-(\mathbf{p}|\propto|\varepsilon_\pm(\mathbf{p})|$. Inasmuch as the velocity difference $(v_+-v_-)/v_\pm\propto\Delta/p_\pm$ leads to relative corrections to the difference $\varepsilon_+(\mathbf{p})-\varepsilon_-(\mathbf{p})$ of the order of $\Delta/p_\pm\ll 1$, we shall put $v_+\approx v_-\approx v$. For an analogous reason, we shall assume that $a_+\approx a_-\approx a$.

By virtue of the fact that the difference in the volumes of the Fermi surfaces of electrons with opposite directions of the spin projection is relatively small, local variations of spin density lead to a redistribution of the Fermi quasi-particles only in the vicinity of the Fermi surfaces, where, because of the small damping, the concept of Fermi quasi-particles remains valid. Therefore, the spin oscillations in a weakly ferromagnetic Fermi liquid can be considered as collective excitations in a system of Fermi quasi-particles. This means that the singularities of the two-particle vertex part Γ that correspond to these excitations[8] are due to the same mechanism as zero-sound in a nonferromagnetic Fermi liquid.[9] Namely, the source of the singularities of the function $\Gamma_{\alpha\beta\gamma\delta}(p,p';k)$ in the momentum transfer $k=\{\omega,\mathbf{k}\}$ ($p+k$, p' are the four-momenta of the quasi-particles before the scattering, and p, $p'+k$ are those after the scattering) is the diagrams containing sections with two G-lines with momenta $p+k$ and p.

We first consider the singularities of the transverse (in the spin) component $\Gamma_\perp(p,p';k)\equiv\Gamma_{-++-}(p,p';k)$ of the vertex part at zero temperature. In analogy with the case of a nonferromagnetic Fermi liquid[9], with the aid of the equality (2.1) we separate the singular element of the diagrams for $\Gamma_\perp$—the product $G_-(p+k)\times G_+(p)$—into singular and regular terms:

$$G_-(p+k)G_+(p)=2\pi ia^2\delta(\varepsilon)\delta(\varepsilon(\mathbf{p}))\frac{\omega}{\omega-v\Delta-\mathbf{vk}}+\overline{G_-(p+k)G_+(p)}. \tag{2.2}$$

The first term on the right here corresponds to the singular (rapidly varying with ω and $\mathbf{k}$) contribution, arising from the integration over the vicinities of the Fermi surfaces, in the integral of the product of G-functions over $\mathbf{p}$ and ε; $\varepsilon(\mathbf{p})=v(|\mathbf{p}|-p_F)$; $p_F=(p_++p_-)/2$. The regular term $\overline{G_-G_+}$ corresponds to the integration over regions of ε and $\mathbf{p}$ far from the Fermi surfaces. The decomposition (2.2) has been performed in such a way that the quantities G_-G_+ and $\overline{G_-G_+}$ coincide for $\omega=0$.

We denote by $\overline{\Gamma}_\perp(p,p';k)$ the function defined by the set of those diagrams for $\Gamma_\perp$ in which the singular sections $G_-(p+k)G_+(p)$ are replaced by $\overline{G_-(p+k)G_+(p)}$. When (2.2) is taken into account the equation relating $\overline{\Gamma}_\perp$ and $\Gamma_\perp$ has the form

$$\mathcal{T}_\perp(\mathbf{p},\mathbf{p}';k)=\overline{\mathcal{T}}_\perp(\mathbf{p},\mathbf{p}';k)+\nu\int\frac{dO_1}{4\pi}\overline{\mathcal{T}}_\perp(\mathbf{p},\mathbf{p}_1;k)\frac{\omega}{\omega-v\Delta-\mathbf{v}_1\mathbf{k}}\mathcal{T}_\perp(\mathbf{p}_1,\mathbf{p}',k) \tag{2.3}$$

Here we have introduced the notation

$$\mathcal{T}_\perp(\mathbf{p},\mathbf{p}';k)=a^2\Gamma_\perp(p,p';k)|_{\varepsilon=\varepsilon'=0},$$
$$\overline{\mathcal{T}}_\perp(\mathbf{p},\mathbf{p}';k)=a^2\overline{\Gamma}_\perp(p,p';k)|_{\varepsilon=\varepsilon'=0}, \tag{2.4}$$

$\nu=p_F^2/2\pi^2v$ is the density of quasi-particle states at one of the Fermi surfaces (their difference must be neglected), $\mathbf{v}_1=v\mathbf{p}_1/|\mathbf{p}_1|$, and do_1 is the element of solid angle in the direction of the vector $\mathbf{p}_1$.

We now separate the function $\overline{\mathcal{T}}_\perp(\mathbf{p},\mathbf{p}';k)$ into isotropic and anisotropic parts with respect to the variables $\mathbf{p}$ and $\mathbf{p}'$:

$$\overline{\mathcal{T}}_\perp(\mathbf{p},\mathbf{p}';k)=\overline{\mathcal{T}}_\perp^s(\mathbf{p},\mathbf{p}';k)+\overline{\mathcal{T}}_\perp^a(\mathbf{p},\mathbf{p}';k). \tag{2.5}$$

The anisotropic part $\overline{\mathcal{T}}_\perp^a$ remains finite as $\omega,\mathbf{k}\to 0$. In view of the fact that its $(\omega,\mathbf{k})$-dependence arising from the integration over regions far from the Fermi surfaces is much weaker than that in the kernel of Eq. (2.3), in the low-frequency and long-wavelength limit the function $\overline{\mathcal{T}}_\perp^a$ must be assumed to be independent of ω and $\mathbf{k}$:

$$\overline{\mathcal{T}}_\perp^a(\mathbf{p},\mathbf{p}';k)\approx\overline{\mathcal{T}}_\perp^a(\mathbf{p},\mathbf{p}'). \tag{2.6}$$

As regards the isotropic component $\overline{\mathcal{T}}_\perp^s$ of $\overline{\mathcal{T}}_\perp$, it becomes infinite as $k\to 0$. This is connected with the fact that for $\omega=0$ and $\mathbf{k}\to 0$ the isotropic part of the function $\mathcal{T}_\perp$, with which, according to (2.3), $\overline{\mathcal{T}}_\perp$ coincides in this case, behaves like $1/\mathbf{k}^2$. This follows from the fact that the transverse component of the uniform static susceptibility, which is related linearly to $\mathcal{T}_\perp$, is equal to infinity in the absence of magnetic anisotropy. Thus, the quantity $\overline{\mathcal{T}}_\perp^s$ can be written in the form

$$\overline{\mathcal{T}}_\perp^s(p, p'; k) = -\nu^{-1}/b^2\mathbf{k}^2. \tag{2.7}$$

Here b is a constant with the dimensions of length ($\sim p_F^{-1}$). The negative sign of $\overline{\mathcal{T}}_\perp^s$ is dictated by the requirement that the ground state be stable. The order of magnitude of the constant b is due to the fact that $\overline{\mathcal{T}}_\perp$ is determined by the properties of the system far from the Fermi surfaces, and, consequently, depends weakly on their spin splitting. In principle, the quantity $\overline{\mathcal{T}}_\perp^s$ could also contain dependence on the frequency ω, but in solving Eq. (2.3) this must be neglected in comparison with the stronger dependence of the kernel of the equation.

We shall seek the solution of (2.3) in the form

$$\mathcal{T}_\perp(\mathbf{p}, \mathbf{p}'; k) = \mathcal{T}_\perp^s(k) + \mathcal{T}_\perp^a(\mathbf{p}, \mathbf{p}'; k), \tag{2.8}$$

Analysis of Eq. (2.3), with allowance for the equalities (2.5), (2.7), and (2.8) and the fact that we are interested in its solution near the poles of $\mathcal{T}_\perp$ in ω and $\mathbf{k}$, leads to the conclusion that, to within quantities of relative order $b^2\mathbf{k}^2$, it decomposes into two equations:

$$\mathcal{T}_\perp^s(k) = -\frac{\nu^{-1}}{b^2\mathbf{k}^2} - \frac{1}{b^2\mathbf{k}^2}\int\frac{dO_1}{4\pi}\frac{\omega}{\omega - v\Delta - \mathbf{v}_1\mathbf{k}}\mathcal{T}_\perp^s(k), \tag{2.9}$$

$$\mathcal{T}_\perp^a(\mathbf{p}, \mathbf{p}'; k) = \overline{\mathcal{T}}_\perp^a(\mathbf{p}, \mathbf{p}') + \int\frac{dO_1}{4\pi}\overline{\mathcal{T}}_\perp^a(\mathbf{p}, \mathbf{p}_1)\frac{\omega}{\omega - v\Delta - \mathbf{v}_1\mathbf{k}}\mathcal{T}_\perp^a(\mathbf{p}_1, \mathbf{p}'; k). \tag{2.10}$$

The solution of (2.9) has the form

$$\mathcal{T}_\perp^s(k) = \nu^{-1}\left[\frac{\omega}{2v|\mathbf{k}|}\ln\left(\frac{\omega - v(\Delta + |\mathbf{k}|) + i\delta}{\omega - v(\Delta - |\mathbf{k}|) + i\delta}\right) - b^2\mathbf{k}^2\right]^{-1}, \quad \delta \to +0. \tag{2.11}$$

The infinitesimal extra term $i\delta$ under the logarithm ensures, as usual, the correct choice of the imaginary part of the logarithm.

In the region of wave-vectors $|\mathbf{k}| \lesssim \Delta$ the function $\mathcal{T}_\perp^s(k)$ (2.11) has a real pole

$$\omega_s(\mathbf{k}) \approx 2v|\mathbf{k}|b^2\mathbf{k}^2/\ln\left(\frac{\Delta + |\mathbf{k}|}{\Delta - |\mathbf{k}|}\right). \tag{2.12}$$

It corresponds to the usual spin-wave branch, with a quadratic spectrum for $|\mathbf{k}| \ll \Delta$:

$$\omega_s(\mathbf{k}) \approx v\Delta b^2\mathbf{k}^2, \quad |\mathbf{k}| \ll \Delta. \tag{2.13}$$

It follows from the expression (2.11) and from (2.12) that the quantity $\omega_s(\mathbf{k}) \to 0$ as $|\mathbf{k}| \to \Delta$. A more detailed analysis with allowance for the difference in the velocities of quasi-particles with opposite spin directions leads to the conclusion that the quantity $\omega_s(\mathbf{k})$ does not reach zero as $|\mathbf{k}| \to \Delta$ but tends to a finite limit $\omega_s^{\min}$:

$$\omega_s^{\min} \sim \omega_s^{\max}/\ln(p_F/\Delta),$$

where $\omega_s^{\max} \sim v\Delta b^2\Delta^2$ is the maximum value taken by the frequency $\omega_s(\mathbf{k})$ for $|\mathbf{k}| < \Delta$.

For $|\mathbf{k}| > \Delta$ the pole of $\mathcal{T}_\perp^s(k)$ becomes complex. In the region $|\mathbf{k}| \gg \Delta$ it is found to be pure-imaginary. In this case $\mathcal{T}_\perp^s(k)$ has the form

$$\mathcal{T}_\perp^s(k) = -\frac{\nu^{-1}}{b^2\mathbf{k}^2 - i\pi\omega/2v|\mathbf{k}|}, \quad |\mathbf{k}| \gg \Delta. \tag{2.14}$$

This expression describes the paramagnon branch of spin fluctuations, the "dispersion law" of which is

$$\omega_p^\perp = 2\pi^{-1}vb^2|\mathbf{k}|^3, \quad |\mathbf{k}| \gg \Delta. \tag{2.15}$$

The solution of Eq. (2.10) can be sought by expanding the quantities appearing in it in spherical harmonics. Here it must be remembered that, in accordance with the definition of the function $\overline{\mathcal{T}}_\perp^a$, the isotropic harmonic ($l = 0$) is absent in its expansion. The frequencies of the spin oscillations determined by the poles of the solution of (2.10) remain finite as $\mathbf{k} \to 0$ and are equal to $v\Delta$ in order of magnitude. Those oscillations for which the corresponding frequencies $\omega_l(0) < v\Delta$ have a spectrum that falls off with increasing $|\mathbf{k}|$. For $|\mathbf{k}| \to \Delta$

$$\omega_l(k) \to \omega_l^{\min} \sim v\Delta/\ln(p_F/\Delta).$$

For $|\mathbf{k}| > \Delta$ the poles of the function $\overline{\mathcal{T}}_\perp^a$ that correspond to these branches are complex with real and imaginary parts that are equal in order of magnitude. The important point is that both parts are much larger than the paramagnon frequencies (2.15). The anisotropic spin-wave branches, for which $\omega_l(0) > v\Delta$, have an increasing spectrum. At $|\mathbf{k}| \gg \Delta$ they emerge into the linear dispersion law characteristic of spin-waves of the zero-sound type. [9]

The treatment of the longitudinal spin component of the vertex part,

$$\Gamma_\parallel = \tfrac{1}{2}(\Gamma_{++++} + \Gamma_{----} - \Gamma_{+-+-} - \Gamma_{-+-+}),$$

is analogous to that of the transverse component. The singular sections of the graphs for $\Gamma_\parallel$, unlike those for the transverse component, consist of G-functions with the same direction of the spin projections. Therefore, in going over from $\Gamma_\perp$ to $\Gamma_\parallel$ in formula (2.2) and, correspondingly, in the equation of the type (2.3), we must put $\Delta = 0$. If, as we did for $\overline{\mathcal{T}}_\perp$, we separate the function $\overline{\mathcal{T}}_\parallel$ analogous to it into isotropic and anisotropic parts ($\overline{\mathcal{T}}_\parallel^s$, $\overline{\mathcal{T}}_\parallel^a$), then, in view of the regular character of $\overline{\mathcal{T}}_\parallel^a$, its dependence on the argument k can be omitted and, to within quantities $\sim(\Delta/p_F)^2$, we can put it equal to $\overline{\mathcal{T}}_\perp^a$. The symmetric part $\overline{\mathcal{T}}_\parallel^s$ of the function $\overline{\mathcal{T}}_\parallel$, being related to the longitudinal susceptibility, is anomalously large in a weak ferromagnet. The relative displacement Δ of the Fermi surfaces plays the role of the inverse correlation length in the case of longitudinal spin fluctuations. Therefore, for $|\mathbf{k}| \gg \Delta$, the functions $\overline{\mathcal{T}}_\perp$ and $\overline{\mathcal{T}}_\parallel$, being determined by regions of integration far from the Fermi surfaces, should coincide, and, consequently, with allowance for the equality (2.7) the quantity $\overline{\mathcal{T}}_\parallel^s$ can be written in the form

$$\bar{\mathcal{T}}_{\parallel}{}^{s} = -\frac{\nu^{-1}}{\alpha + b^2\mathbf{k}^2}, \quad \alpha \sim \left(\frac{\Delta}{p_F}\right)^2 \ll 1. \tag{2.16}$$

After this, the function $\mathcal{T}_{\parallel}(\mathbf{p}, \mathbf{p}'; k)$, defined analogously to $\mathcal{T}_{\perp}(\mathbf{p}, \mathbf{p}'; k)$ (2.4), can be decomposed into a sum of isotropic $\mathcal{T}_{\parallel}^{s}$ and anisotropic $\mathcal{T}_{\parallel}^{a}$ components, which, in analogy with the case of $\mathcal{T}_{\perp}^{s}$ and $\mathcal{T}_{\perp}^{a}$, satisfy two independent equations of the type (2.9), (2.10), to within quantities of relative order $\sim(\alpha + b^2\mathbf{k}^2)$. The solution for the symmetric part $\mathcal{T}_{\parallel}^{s}$ has the form

$$\mathcal{T}_{\parallel}{}^{s} = -\frac{\nu^{-1}}{\alpha + b^2\mathbf{k}^2 - i\pi\omega/2v|\mathbf{k}|}. \tag{2.17}$$

It determines the longitudinal paramagnon spin-fluctuation branch with the "dispersion law"

$$\omega_p{}^{\parallel} = 2\pi^{-1}v|\mathbf{k}|(\alpha + b^2\mathbf{k}^2). \tag{2.18}$$

For $|\mathbf{k}| \gg \Delta$ the frequencies of the longitudinal (2.18) and transverse (2.15) branches of the fluctuations coincide.

The poles of the anisotropic part $\mathcal{T}_{\parallel}^{a}$ of $\mathcal{T}_{\parallel}$ divide into two groups. One of these determines the spectrum of longitudinal spin-waves of the zero-sound type, with frequencies $\sim v|\mathbf{k}|$. The second corresponds to complex poles, the real and imaginary parts of which substantially exceed the frequencies of the paramagnon branch. For this reason, as in the case of the transverse modes, the anisotropic spin fluctuations at temperatures $T \ll \varepsilon_F$ give a negligibly small contribution to the thermodynamics of a weak ferromagnet.

In the following we shall need expressions for the Green functions constructed from the spin-density operators $\hat{S}_i(x)$:

$$D_{ij}(x-x') = -i\langle (T((\hat{S}_i(x) - S_i)(\hat{S}_j(x') - S_j)))\rangle.$$

Here $\hat{S}_i(x) = \hat{\psi}_\alpha^+(x) S^i_{\alpha\beta} \hat{\psi}_\beta(x)$; $\hat{\psi}_\alpha^+$ and $\hat{\psi}_\beta$ are electron creation and annihilation operators; $x = \{\mathbf{r}, t\}$ is the space-time coordinate; $\langle \ldots \rangle$ is the symbol for averaging over the ground state; T is the time-ordering operator; $S_i = \langle \hat{S}_i(x)\rangle$ is the average value of the spin-moment density of the system. The retarded Green function D^R_{ij} corresponding to D_{ij} determines the linear response of the spin density to an external magnetic field.

According to the general rules of the diagram technique of [8], the Fourier transform $D_{ij}(\omega, \mathbf{k})$ of the function $D_{ij}(x - x')$ is expressed by the equality

$$D_{ij}(\omega,\mathbf{k}) = -i\int \frac{d^4p}{(2\pi)^4}\Big\{S_{\alpha\beta}{}^{i} G_{\beta\gamma}(p+k) G_{\delta\alpha}(p) \times \Big[S_{\gamma\delta}{}^{j} - i\int \frac{d^4p'}{(2\pi)^4}\Gamma_{\gamma\varkappa\delta\sigma}(p,p',k) G_{\sigma\sigma'}(p'+k) S^{j}_{\sigma'\varkappa'} G_{\varkappa'\varkappa}(p')\Big]\Big\}.$$

Dividing the integration over the momenta into regions lying in the vicinity of and far from the Fermi surface and taking into account the equalities (2.2) and (2.4), Eq. (2.3) and the equality

$$D_{ij}(\omega, 0) = e_{ijl}\frac{S_l}{\omega},$$

which follows from the conservation of the total spin, we obtain

$$D_{xx}(\omega,\mathbf{k}) = D_{yy}(\omega,\mathbf{k}) = \frac{\nu^2}{4}[\mathcal{T}_{\perp}{}^{s}(\omega,\mathbf{k}) + \Gamma_{\perp}{}^{s}(-\omega,\mathbf{k})],$$
$$D_{xy}(\omega,\mathbf{k}) = -D_{yx}(\omega,\mathbf{k}) = -i\frac{\nu^2}{4}[\mathcal{T}_{\perp}{}^{s}(\omega,\mathbf{k}) - \mathcal{T}_{\perp}{}^{s}(-\omega,\mathbf{k})],$$
$$D_{zz}(\omega,\mathbf{k}) = \frac{\nu^2}{2}\mathcal{T}_{\parallel}{}^{s}(\omega,\mathbf{k}); \quad D_{iz} = D_{zi} = 0, \quad i \neq z. \tag{2.19}$$

(The z-axis is chosen in the direction of the resultant spin of the ferromagnet.) Taking into account the equalities (2.11)–(2.16) and going over from the function D_{ij} to the retarded Green function D^R_{ij}, defined in the usual way, [8] from (2.19) we obtain

$$D_{xx}{}^{R}(\omega,\mathbf{k}) = D_{yy}{}^{R}(\omega,\mathbf{k}) = -S\frac{\omega_s(\mathbf{k})}{(\omega+i\delta)^2 - \omega_s{}^2(\mathbf{k})}, \quad |\mathbf{k}| \ll \Delta; \tag{2.20}$$

$$D_{xx}{}^{R}(\omega,\mathbf{k}) = D_{yy}{}^{R}(\omega,\mathbf{k}) = -\frac{\nu}{2}\frac{1}{b^2\mathbf{k}^2 - i\pi\omega/2v|\mathbf{k}|}, \quad |\mathbf{k}| \gg \Delta; \tag{2.21}$$

$$D_{zz}{}^{R}(\omega,\mathbf{k}) = -\frac{\nu}{2}\frac{1}{\alpha + b^2\mathbf{k}^2 - i\pi\omega/2v|\mathbf{k}|}. \tag{2.22}$$

In deriving formulas (2.19)–(2.22) we have taken into account that the absolute magnitude $S \equiv |\mathbf{S}|$ of the average spin-density vector is determined by the semi-difference of the volumes of the Fermi surfaces[10] and is equal to

$$S = \frac{1}{2}\int_{p_- < |\mathbf{p}| < p_+} \frac{d\mathbf{p}}{(2\pi)^3} \approx \frac{\nu v \Delta}{2}. \tag{2.23}$$

Up to now it has been assumed that an external magnetic field does not act on the system. The switching-on of a static uniform magnetic field of intensity $\mathbf{H}$ leads primarily to dependence on $H = |\mathbf{H}|$ of the relative displacement Δ of the Fermi surfaces that appears in the above formulas. In addition, the magnetic field leads to renormalization of the quantities $\bar{\mathcal{T}}_{\perp}$ and $\bar{\mathcal{T}}_{\parallel}$. Inasmuch as the longitudinal magnetic susceptibility $\sim \Delta^{-2}$, besides the appearance of a dependence of the constant α on H in (2.16) and (2.22) the possibility of the appearance of a term $\sim H/\Delta$ in the denominator of $\bar{\mathcal{T}}_{\perp}^{s}$ (2.7) is not ruled out. In fact, this happens, and follows from the formula (A.9) obtained in the Appendix:

$$\lim_{k\to 0} D_{xx}{}^{R}(0,\mathbf{k}) = \lim_{k\to 0} D_{yy}{}^{R}(0,\mathbf{k}) = -\frac{S}{2\mu_0 H},$$

where μ_0 is the Bohr magneton. Hence, taking into account the first formula (2.19), the equalities $D^R_{ij}(0,\mathbf{k}) = D_{ij}(0,\mathbf{k})$ and $\mathcal{T}_{\perp}^{s}(0,\mathbf{k}) = \bar{\mathcal{T}}_{\perp}^{s}(0,\mathbf{k})$, and formula (2.7), we obtain the expression for the function $\bar{\mathcal{T}}_{\perp}^{s}$ in the presence of a magnetic field:

$$\bar{\mathcal{T}}_{\perp}{}^{s} = -\frac{\nu^{-1}}{b^2\mathbf{k}^2 + 2\mu_0 H/v\Delta}.$$

Correspondingly, formula (2.11) for the function $\mathcal{T}_{\perp}^{s}$ acquires a term $-2\mu_0 H/v\Delta$ in the denominator of the right-hand side when the field is switched on, and the formula for the spin-wave spectrum for $|\mathbf{k}| \ll \Delta$ (2.13) takes the form

$$\omega_s(k) \approx 2\mu_0 H + v\Delta b^2\mathbf{k}^2, \quad |\mathbf{k}| \ll \Delta. \tag{2.24}$$

The formulas (2.20)–(2.22) for $D^R_{ij}(\omega,\mathbf{k})$ preserve their form when we go over to the case $H \neq 0$. Here it must be recalled that the spin-wave frequency appearing in (2.20) is determined by the equality (2.24), and the quantities Δ and α are functions of the field strength H.

Formally, the results obtained are not changed when we go over to nonzero temperatures, under the condition $T \ll \varepsilon_F$. Because of the small size of the contribution of the thermal Fermi excitations the modification of the formulas (2.20)–(2.22) for $D^R_{ij}(\omega, \mathbf{k})$ for $T \neq 0$, $\omega \sim T$ consists only in taking into account the temperature dependence of the quantities Δ and α, and this will be considered in the next section.

In the paramagnetic temperature region ($T > T_C$) for $H = 0$ the function D^R_{ij} becomes isotropic: $D^R_{ij} = D^R \delta_{ij}$. The quantity D^R coincides in form with the function D^R_{zz} (2.22) in the ferromagnetic temperature region. Thus, the low-frequency spin fluctuations at $T > T_C$ reduce to three degenerate paramagnon branches.

3. SUSCEPTIBILITY AND SPIN DENSITY FOR $T \neq 0$

The longitudinal differential susceptibility to a static uniform magnetic field is determined by the equality

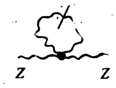
$$\chi = -(2\mu_0)^2 \lim D_{zz}''(0, \mathbf{k}; T) = -(2\mu_0)^2 \mathscr{D}_{zz}(0, 0; T). \tag{3.1}$$

Here $\mathscr{D}_{ij}(i\omega_n, \mathbf{k}; T)$ is the spin Green function, corresponding to D_{ij}, in the Matsubara representation[8]; $\omega_n = 2\pi n T$. To find the temperature dependence of the function $\mathscr{D}_{ij}$ we shall use the method described in the book[8] for calculating the temperature corrections. Since the principal temperature dependence arises on account of the thermal spin excitations, the main contribution to $\mathscr{D}_{zz}(0, 0; T) - \mathscr{D}_{zz}(0, 0, 0)$ will be given by those diagrams which contain internal lines $\mathscr{D}_{ij}(i\omega_n, \mathbf{k}; T)$ with frequencies $\omega \sim T$. Instead of summing over the frequencies of these lines ($T\Sigma$) we must apply the operation

$$T \sum_{\omega}^{\omega} - \int \frac{d\omega}{2\pi}$$

Integration is performed over all the remaining frequencies.

We shall start the calculation from the paramagnetic temperature region $T > T_C$, for $H = 0$. We introduce the quantity $\overline{\mathscr{D}}$, which is obtained from $\mathscr{D}_{zz}(0, 0; T)$ by eliminating the temperature-dependent contribution of the thermal spin fluctuations. If we neglect the contribution of the thermal Fermi excitations the quantity $\overline{\mathscr{D}}$ is a temperature-independent constant. After singling out one internal $\mathscr{D}$-line with frequencies $\omega \sim T$ from the graphs for $\mathscr{D}_{zz}(0, 0; T)$, we obtain the following diagrammatic expression for $\overline{\mathscr{D}}^{-1} - \mathscr{D}^{-1}_{zz}(0, 0; T)$:

$\overline{\mathscr{D}}^{-1} - \mathscr{D}^{-1}_{zz}(0, 0; T) =$ 

$T > T_c, \quad H = 0.$ (3.2)

The wavy line corresponds to the function $\mathscr{D}_{lm}$. The slash indicates that the operation

$$T \sum_{\omega} - \int \frac{d\omega}{2\pi}.$$

should be applied to this function. The blob on the graph denotes the four-point vertex $\tilde{\gamma}_{ijlm}$ of the interaction of the spin fluctuations. At temperatures T much lower than the temperature T_0 defined by the maximum spin-fluctuation frequency, the dependence of the quantity $\tilde{\gamma}_{ijlm}$ on the frequencies, wave-vectors and temperature must be neglected. With neglect of relativistic magnetic interactions, the tensor structure of the quantity $\tilde{\gamma}_{ijlm}$ has the form

$$\tilde{\gamma}_{ijlm} = {}^1/_3 \gamma (\delta_{ij}\delta_{lm} + \delta_{il}\delta_{jm} + \delta_{im}\delta_{jl}). \tag{3.3}$$

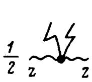
When we take this equality into account the analytic expression of the equality (3.2) takes the form

$$\overline{\mathscr{D}}^{-1} - \mathscr{D}^{-1}_{zz}(0, 0; T) = \gamma\beta(T), \quad T > T_c, \quad H = 0; \tag{3.4}$$

$$\beta(T) = -\frac{1}{3}\left(T \sum_{\omega} - \int \frac{d\omega}{2\pi}\right) \int \frac{d\mathbf{k}}{(2\pi)^3} (\mathscr{D}_{ll}(i\omega, \mathbf{k}; T) + 2\mathscr{D}_{zz}(i\omega, \mathbf{k}; T)). \tag{3.5}$$

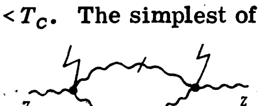
Except in the region $T \to T_C$ of critical spin fluctuations (the corresponding conditions will be found below), diagrams for $\overline{\mathscr{D}}^{-1} - \mathscr{D}^{-1}_{zz}(0, 0; T)$ with more than one internal $\mathscr{D}$-line with $\omega \sim T$ have a small statistical weight for $T_C > T > T_0$ and can be omitted. We note that the quantity $\overline{\mathscr{D}}$ does not coincide with $\mathscr{D}_{zz}(0, 0; T)$ for $T = 0$, $H = 0$. The reason for this is the appearance of the spontaneous spin moment as we move into the region $T < T_C$.

Going over to the ferromagnetic temperature region $T < T_C$, on an equal footing with the temperature corrections arising on account of the spin excitations we shall also take into account the temperature dependence due to the spontaneous spin moment $S(T) \equiv \langle \hat{S}_z(x) \rangle$. Here it is convenient to regard the quantity $S(T)$ as the condensate of the field $\hat{S}_z(x)$. A small value of $S(T)$ allows us to confine ourselves to taking account of graphs with the smallest number of condensate lines. The simplest graph of this type for $\overline{\mathscr{D}}^{-1} - \mathscr{D}^{-1}_{zz}(0, 0; T)$ has the form

$\frac{1}{2}$  $= \frac{1}{2}\gamma S^2$ (3.6)

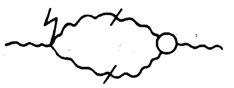
The jagged lines correspond to the condensate. The four-point vertex here denotes the vertex part that is irreducible with respect to internal cuts through one $\mathscr{D}$-line. To within higher orders in the temperature and powers of the spin density, it coincides with the quantity $\tilde{\gamma}_{zzzz}$ appearing in the equality (3.2).

Of the diagrams that are more complicated than (3.2) and (3.6), those containing sections with two $\mathscr{D}$-lines in the horizontal direction merit special treatment for $T < T_C$. The simplest of them has the form

 (3.7)

Owing to the pole character of the functions $\mathscr{D}_{xx}$ and

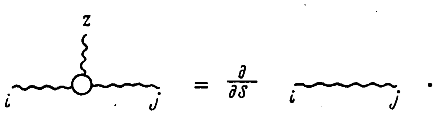
$\mathscr{D}_{yy}$ for $\omega = 0$, $\mathbf{k} \to 0$ (cf. (2.20), (2.24)), this graph diverges like $H^{-1/2}$ in the limit $H \to 0$. This means that, besides (3.7), it is formally necessary to sum all diagrams with an arbitrary number of sections with two $\mathscr{D}$-lines, as a result of which (3.7) is replaced by the diagram



(3.8)

in which the circle denotes the effective three-point vertex that takes into account the temperature corrections arising from the arbitrary number of intermediate singular sections.

In order to simplify the analytic expression for the graph (3.8), we shall make use of the identity (in which the vertical $\mathscr{D}$-line corresponds to $\omega = 0$, $\mathbf{k} \to 0$)



(3.9)

This follows from the fact that its left-hand side can be obtained from the function $\mathscr{D}_{ij}$ by singling out one condensate line from the graphs for it and then replacing this line by an insertion for the interaction with the field $\hat{S}_z$ for $\omega = 0$, $\mathbf{k} \to 0$. Comparing (3.8), (3.9), (3.2) and (3.4), we arrive at the conclusion that the analytic expression for the diagram (3.8) is equal to

$$\gamma S \frac{\partial}{\partial S} \beta(T). \qquad (3.10)$$

Collecting now (3.4), (3.6) and (3.10), we obtain the final expression for the quantity $\bar{\mathscr{D}}^{-1} - \mathscr{D}_{zz}^{-1}(0, 0; T)$ for $T < T_C$:

$$\bar{\mathscr{D}}^{-1} - \mathscr{D}_{zz}^{-1}(0,0;T) = \gamma \frac{\partial}{\partial S}(S\beta(T)) + \frac{1}{2}\gamma S^2. \qquad (3.11)$$

We can convince ourselves that, at temperatures not too close to T_C, the diagrams not taken into account by this expression have a small statistical weight; we have therefore discarded them.

For $H = 0$, $T > T_C$, the spin density vanishes and the expression (3.11) goes over into (3.4). Inasmuch as the interaction of the magnetic field with the system occurs directly through the field $\hat{S}_z$, the equality (3.11) is also valid for the paramagnetic temperature region in the presence of a magnetic field, when $S \neq 0$.

It should be noted that those divergences in the diagrams which are due to transverse spin-waves do not have the fundamental character inherent in the divergences induced by the critical fluctuations as $T \to T_C$. The point is that the exact vertices for the interaction of transverse spin-waves with each other, which take into account also the exchange of virtual fluctuations of the longitudinal spin-component, vanish in the long-wavelength and low-frequency limit for $H = 0$ (see formula (A.17)). As a result, the divergences associated with the transverse spin-waves cancel.

Further transformation of the expression (3.11) is necessary to determine the equilibrium spin density $S = S(T, H)$. With this purpose we shall consider the function $\tilde{F}(T, S)$, which is obtained from $F(T, H)$, the free energy per unit volume of the system, defined in the usual way, by changing to the variables T, S:

$$\tilde{F} = F + 2\mu_0 SH.$$

The function $\tilde{F}(T, S)$ satisfies the condition

$$\partial\tilde{F}(T, S)/\partial S = 2\mu_0 H. \qquad (3.12)$$

The quantity $\tilde{F}(T, S)$ is determined by the set of closed connected diagrams containing, in particular, the interaction with the condensate field S. The functions $\tilde{F}(T, S)$ and $\mathscr{D}_{zz}(0, 0; T)$ are connected by the obvious relation

$$\frac{\partial^2 \tilde{F}(T,S)}{\partial S^2} = (2\mu_0)^2 \chi^{-1} = -\mathscr{D}_{zz}^{-1}(0,0;T). \qquad (3.13)$$

Substituting the equality (3.11) into this, and then comparing (3.13) with (3.12), we obtain

$$-\bar{\mathscr{D}}^{-1} S + {}^1/_6 \gamma S^3 + \gamma S \beta(T) = 2\mu_0 H. \qquad (3.14)$$

Taking into account that $\beta \to 0$ as $T \to 0$, from (3.14) we obtain

$$\bar{\mathscr{D}}^{-1} = {}^1/_6 \gamma S_0^2, \quad S_0 \equiv S(T=0,\ H=0). \qquad (3.15)$$

When this equality is taken into account, the relation (3.14) takes the form

$$ {}^1/_6 \gamma S^3 - {}^1/_6 \gamma S_0^2 S + \gamma S \beta(T) = 2\mu_0 H. \qquad (3.16)$$

Using the condition $S \to 0$ for $H = 0$, $T \to T_C - 0$, from this we find

$$S_0^2 = 6\beta_C, \quad \beta_C \equiv \beta|_{T=T_C,\ H=0}. \qquad (3.17)$$

At $T = 0$ the quantity β and its derivative with respect to S vanish. Therefore, it follows from (3.16) that

$$\chi_0 \equiv \chi(T=0, \quad H=0) = 3(2\mu_0)^2/\gamma S_0^2. \qquad (3.18)$$

Using the equalities (3.17) and (3.18), we can rewrite Eq. (3.16), relating the quantities S, H and T, in the form

$$S^2 = S_0^2 \left[1 - \frac{\beta(T)}{\beta_C} + \frac{\chi_0 H}{\mu_0 S}\right]. \qquad (3.19)$$

We next study the calculation of the function $\beta(T)$ appearing in the last equality. Changing in formula (3.5) from the summation to an integration

$$T \sum_{\omega} \to \int \frac{d\omega}{4\pi i} \operatorname{cth} \frac{\omega}{2T},$$

over a contour enclosing the poles of the function $\coth(\omega/2T)$, and then, as usual,[8] deforming the integration contour on to the real axis of the variable ω, we obtain

$$\beta(T) = -\int_0^\infty \frac{d\omega}{2\pi}\frac{1}{e^{\omega/T}-1}\int\frac{d\mathbf{k}}{(2\pi)^3}\operatorname{Im}[D_{zz}^R(\omega,\mathbf{k})+{}^2/{}_3D_{xx}^R(\omega,\mathbf{k})]. \quad (3.20)$$

The main contribution to the integral here arises from frequencies $\omega \sim T$. At such frequencies and at temperatures not too close to T_C, diagrams of the type (3.7) and (3.8) give a negligibly small contribution. Therefore, in place of the function D_{ij}^R in (3.20) we must substitute the expressions (2.20)–(2.22) in which, according to (3.11) and (3.15), the quantity α is equal to

$$\alpha = {}^1/{}_2\nu({}^1/{}_2\gamma S^2 - {}^1/{}_6\gamma S_0^2 + \gamma\beta(T)). \quad (3.21)$$

Using the relation (3.16), we can show that in the limiting cases $2\mu_0 H \ll \gamma S^3(T,0)/6$ and $2\mu_0 H \gg S^3(T,0)/6$ the quantity α (3.21) is equal to

$$\begin{aligned}&\alpha = {}^1/{}_6\nu\gamma S^2, \quad 2\mu_0 H \ll {}^1/{}_6\gamma S^3(T,0);\\ &\alpha = {}^1/{}_4\nu\gamma S^2, \quad 2\mu_0 H \gg {}^1/{}_6\gamma S^3(T,0).\end{aligned} \quad (3.22)$$

In accordance with the existence of the two types of spin excitations, we separate the quantity $\beta(T)$ (3.20) into a spin-wave contribution β_{sw} and a paramagnon contribution β_p:

$$\beta = \beta_{sw} + \beta_p,$$

In the calculation of β_{sw} the term D_{zz}^R in (3.20) must be omitted, and in place of D_{xx}^R we must substitute the expression (2.20) and limit the integration over $\mathbf{k}$ by the condition $|\mathbf{k}| < \Delta$:

$$\beta_{sw}(T) = \frac{1}{3}S\int\frac{d\mathbf{k}}{(2\pi)^3}\frac{1}{\exp(\omega_s(\mathbf{k})/T)-1}. \quad (3.23)$$

At temperatures $T \ll \alpha^{3/2}vb^{-1}$ the paramagnon contribution $\beta_p(T)$ is proportional to T^2 and thereby turns out to be indistinguishable from the T^2 contribution of the Fermi excitations that we have neglected. Therefore,

$$\beta(T) \approx \beta_{sw}(T), \quad T \ll T^*, \quad T^* \equiv \alpha^{3/2}vb^{-1}. \quad (3.24)$$

In the temperature region $T \gg T^*$, in (3.20) we can, with a sufficient degree of accuracy, assume that $D_{xx}^R = D_{zz}^R$ and extend the integration over $\mathbf{k}$ to the whole space:

$$\beta_p(T) = -\frac{5}{3\pi}\int_0^\infty d\omega\frac{1}{e^{\omega/T}-1}\int\frac{d\mathbf{k}}{(2\pi)^3}\operatorname{Im}D_{zz}^R(\omega,\mathbf{k}),$$
$$T \gg T^*.$$

Substitution of formula (2.22) into this and calculations give

$$\beta_p(T) \approx \frac{5\Gamma({}^4/{}_3)\zeta({}^4/{}_3)}{36\pi^2\sqrt{3}}\frac{\nu T_0}{b^3}\left(\frac{T}{T_0}\right)^{4/3}, \quad (3.25)$$
$$T_0 = 2\pi^{-1}vb^{-1}, \quad T \gg T^*,$$

where $\Gamma(x)$ is the gamma function and $\zeta(x)$ is the Riemann zeta-function. The terms discarded in (3.25) have relative order $(T^*/T)^{1/3}$. The contribution of the spin waves for $T \gg T^*$ corresponds to the same order. Therefore,

$$\beta(T) \approx \beta_p(T), \quad T \gg T^*. \quad (3.26)$$

We now calculate the spin density in different limiting cases. For $H = 0$, from (3.19) we find

$$S(T,0) = S_0(1-\beta(T)/\beta_C)^{1/2}. \quad (3.27)$$

Hence, using formulas (3.15), (3.17), (3.23) and (3.24)–(3.26), we shall have

$$S(T,0) = S_0 - \int_{|\mathbf{k}|<\Delta}\frac{d\mathbf{k}}{(2\pi)^3}\frac{1}{\exp(\omega_s(\mathbf{k})/T)-1}, \quad T \ll T^*; \quad (3.28)$$

$$S(T,0) = S_0[1-(T/T_C)^{4/3}]^{1/2}, \quad T^* \ll T < T_C. \quad (3.29)$$

When the expression (2.13) for the spin-wave spectrum at $H = 0$ is taken into account, the formula (3.28) leads to the well-known $T^{3/2}$ law for the temperature dependence of the spin density for $T \ll T^*$:

$$S(T,0) = S_0 - \frac{\zeta({}^3/{}_2)}{8\pi^{3/2}}\left(\frac{T}{v\Delta b^2}\right)^{3/2}.$$

From formulas (3.17), (3.25) and (3.26) we obtain an expression for the spin density at T, $H = 0$:

$$S_0^2 = \frac{5\Gamma({}^4/{}_3)\zeta({}^4/{}_3)}{6\pi^2\sqrt{3}}\frac{\nu T_0}{b^3}\left(\frac{T_c}{T_0}\right)^{4/3}. \quad (3.30)$$

With the assumption that the energy corresponding to the vertex γ and also $T_0 = 2\pi^{-1}vb^{-1}$ are of the order of ε_F, we find from (3.22) and (3.30) the estimates:

$$T^* = \alpha^{3/2}vb^{-1} \sim T_C^2/\varepsilon_F, \quad H = 0;$$
$$\frac{\Delta(T=0, H=0)}{p_F} \approx \frac{2\nu^{-1}S_0}{\varepsilon_F} \sim \left(\frac{T_C}{\varepsilon_F}\right)^{2/3}. \quad (3.31)$$

The second of these estimates leads to the conclusion that $\omega_s^{max} \sim v\Delta b^2\Delta^2$, the maximum spin-wave frequency at $T = 0$, $H = 0$, has the same order of magnitude as T^*.

We shall consider now the case $H \neq 0$. In weak magnetic fields, satisfying the condition

$$2\mu_0 H \ll {}^1/{}_6\gamma S^3(T,0),$$

taking into account the expression (3.23) and the weak dependence of the quantity $\beta_p(T)$ on H, we obtain

$$S(T,H) = S(T,0) + \frac{1}{2\mu_0}\tilde{\chi}H - \delta S_{sw}. \quad (3.32)$$

Here we have introduced the notation:

$$\tilde{\chi} = \frac{3(2\mu_0)^2}{\gamma S^2(T,0)} = \frac{(2\mu_0)^2}{2\gamma(\beta_C-\beta(T))}, \quad (3.33)$$

$$\delta S_{sw} = \int_{|\mathbf{k}|<\Delta}\frac{d\mathbf{k}}{(2\pi)^3}\left[\frac{1}{\exp(\omega_s(\mathbf{k},H)/T)-1} - \frac{1}{\exp(\omega_s(\mathbf{k},0)/T)-1}\right]. \quad (3.34)$$

In the limiting cases $2\mu_0 H \ll T$ and $2\mu_0 H \gg T$, we obtain for the quantity δS_{sw} the expressions

$$\delta S_{sw} = -\frac{1}{4\pi b^3}\frac{T(2\mu_0 H)^{1/2}}{(v\Delta)^{1/2}}, \quad T \gg 2\mu_0 H;$$

$$\delta S_{sw}=-\frac{1}{8\pi^{3/2}b^3}\left(\frac{T}{v\Delta}\right)^{3/2}\left(\exp\left(-\frac{2\mu_0 H}{T}\right)-\zeta\left(\frac{3}{2}\right)\right),\quad T\ll 2\mu_0 H. \qquad (3.35)$$

In the limit of strong magnetic fields, satisfying the inequality $2\mu_0 H\gg\gamma S^3(T,0)/6$, according to (3.16) the spin density is described by the expression

$$S(T,H)\approx(12\mu_0 H/\gamma)^{1/3},\quad 2\mu_0 H\gg{}^1/_6\gamma S^3(T,0). \qquad (3.36)$$

We return now to the question of the magnetic susceptibility. According to (3.1), (3.11) and (3.15),

$$\chi=(2\mu_0)^2\left[\frac{1}{2}\gamma S^2-\frac{1}{6}\gamma S_0^2+\gamma\frac{\partial}{\partial S}(S\beta(T))\right]^{-1}. \qquad (3.37)$$

In the limiting case of weak fields,

$$2\mu_0 H\ll{}^1/_6\gamma S^3(T,0),\quad T<T_c,$$

using the relation (3.16) and the obvious identity

$$\frac{\partial\beta}{\partial S}=2\mu_0\chi^{-1}\frac{d\beta}{dH},$$

we rewrite (3.37) in the form

$$\chi=\tilde\chi-\frac{6\mu_0}{S}\frac{d\beta}{dH}. \qquad (3.38)$$

The quantity $\tilde\chi$ is defined by formula (3.33). In the differentiation of β with respect to H we must take into account only the contribution of the spin-waves to which the singularity of the diagram (3.8) corresponds. With the aid of formula (3.23), with the supplementary condition $2\mu_0 H\ll T$, from (3.38) we find

$$\chi=\tilde\chi+\frac{1}{8\pi b^3}\frac{T}{(v\Delta)^{3/2}(2\mu_0 H)^{1/2}},\quad 2\mu_0 H\ll{}^1/_6\gamma S^3. \qquad (3.39)$$

We recall that the quantity $v\Delta$ is related to S by the equality $S=\nu v\Delta/2$. It follows from (3.39) that the longitudinal susceptibility becomes infinite as $H\to 0$. The second term in (3.39) becomes the main contribution for

$$2\mu_0 H\ll\frac{T_c^{2/3}T^2}{\varepsilon_F^{4/3}}\frac{S}{S_0}.$$

For $T\ll 2\mu_0 H\ll\gamma S^3/6$ the second term in (3.38) can be neglected and then

$$\chi=\tilde\chi=\frac{3(2\mu_0)^2}{\gamma S^2(T,0)},\quad T\ll 2\mu_0 H\ll{}^1/_6\gamma S^3(T,0). \qquad (3.40)$$

In the case of strong magnetic fields ($2\mu_0 H\gg\gamma S^3(T,0)/6$), according to (3.37) and (3.16) the susceptibility acquires the form

$$\chi=\frac{2(2\mu_0)^2}{\gamma S^2(T,H)},\quad 2\mu_0 H\gg{}^1/_6\gamma S^3(T,0). \qquad (3.41)$$

We shall now consider the paramagnetic temperature region $T>T_C$. It follows from Eq. (3.16) and the equality (3.17) that, in weak magnetic fields $2\mu_0 H\ll\gamma(\beta-\beta_C)^{3/2}$, the spin density is linear in the field: $S=(2\mu_0)^{-1}\chi H$. In this case the susceptibility is equal to

$$\chi=\frac{(2\mu_0)^2}{\gamma(\beta(T)-\beta_C)},\quad 2\mu_0 H\ll\gamma(\beta-\beta_C)^{3/2}. \qquad (3.42)$$

For $T>T_C$ the condition $T\gg T^*\equiv\alpha^{3/2}vb^{-1}$ is fulfilled and, therefore, the function $\beta(T)$ is determined by the paramagnon contribution. Using formulas (3.25) and (3.42) we obtain

$$\chi=\frac{(2\mu_0)^2}{\gamma\beta_C[(T/T_C)^{4/3}-1]}$$
$$\beta_C=\frac{5\Gamma(^4/_3)\zeta(^4/_3)}{36\pi^2\sqrt{3}}\frac{\nu T_0}{b^3}\left(\frac{T_C}{T_0}\right)^{4/3}. \qquad (3.43)$$

In the temperature region $T_C\ll T\ll\varepsilon_F$ the formula (3.43) takes the form

$$\chi\approx\frac{(2\mu_0)^2}{\gamma\beta_C}\left(\frac{T_C}{T}\right)^{4/3}.$$

In the case of strong magnetic fields $2\mu_0 H\gg\gamma(\beta-\beta_C)^{3/2}$, the formulas for the spin density and susceptibility for $T>T_C$ reduce to (3.36) and (3.41).

4. SPECIFIC HEAT

The contribution of the spin excitations to the temperature-dependent part of the free energy F per unit volume for $T\ll T_0$ is given by the expression

$$\delta F_s=\frac{1}{2}\left(T\sum_\omega-\int\frac{d\omega}{2\pi}\right)\int\frac{d\mathbf{k}}{(2\pi)^3}\operatorname{Sp}\ln[-\hat{\mathscr{D}}^{-1}(i\omega,\mathbf{k};T)]. \qquad (4.1)$$

The symbol Sp ($\equiv$ Tr) refers here to the indices labeling the components of the function $\mathscr{D}_{ij}$. After the change from the summation over discrete frequencies to integration the equality (4.1) acquires the form

$$\delta F_s=\int_0^\infty\frac{d\omega}{\pi}\frac{1}{e^{\omega/T}-1}\int\frac{d\mathbf{k}}{(2\pi)^3}\operatorname{Im}\operatorname{Sp}\ln[-\hat{D}_R^{-1}(\omega,\mathbf{k})]. \qquad (4.2)$$

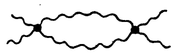
Substituting the expressions (2.20)–(2.22) for the function D^R_{ij} into this and performing the integration over ω and $\mathbf{k}$, we obtain the following expression for the specific heat due to the spin excitations:

$$C_s=-T\frac{\partial^2}{\partial T^2}\delta F_s\approx\begin{cases}\dfrac{1}{6\pi b^3}\dfrac{T}{T_0}\ln\dfrac{T_0}{T^*}, & T\ll T^*\\[2ex] \dfrac{1}{6\pi b^3}\dfrac{T}{T_0}\ln\dfrac{T_0}{T}, & T\gg T^*.\end{cases} \qquad (4.3)$$

In calculating C_s we have neglected the spin-wave contribution, which is small compared with (4.3) ($\sim T^{3/2}$ for $T\ll T^*$), and the nonlogarithmic terms $\sim b^{-3}T/T_0$.

5. CONCLUSION

In the preceding sections we have developed a theory of a weakly ferromagnetic Fermi liquid under the assumption that the interaction between the spin fluctuations weakly renormalizes the initial quantities. This assumption is certainly violated in the critical region $T\to T_C$. In order to find the criterion for applicability of the theory we shall consider the renormalization of the four-point vertex of the interaction of the spin fluctuations, defined by the constant γ. The simplest graph describing the temperature correction to γ is the following:



(5.1)

Inasmuch as the phase-space volume corresponding to the spin-waves tends to zero as $T \to T_C$, the intermediate $\mathscr{D}$-lines in the graph essentially correspond to paramagnons. With the aid of the expressions (2.21) and (2.22), we find that the relative correction to γ arising from (5.1) has the order $\nu^2 \gamma b^{-3} T_C \alpha^{-1/2}$. In order that it be considerably smaller than unity it is necessary that $\alpha \gg (T_C/\varepsilon_F)^2$. In the limit $T \to T_C$, according to the equalities (3.22) and (3.29) the quantity α is equal to $(T_C/T_0)^{4/3} |\tau|$ in order of magnitude ($\tau = (T - T_C)/T_C$). Therefore, the condition for the renormalization of the quantity γ to be small takes the form

$$|\tau| \gg (T_C/\varepsilon_F)^{2/3}. \qquad (5.2)$$

(As earlier, we assume that the energies corresponding to γ, T_0 and ε_F are of the same order.) The condition (5.2) corresponds, in essence, to the Ginzburg criterion[11] as applied to a weakly ferromagnetic Fermi liquid.

APPENDIX

We shall study the proof of a number of relations between the vertex parts determining the mutual scattering of spin excitations. With this purpose we shall consider the responses, of different orders, of the spin density of the ferromagnet to an additional external static uniform magnetic field of intensity $\delta\mathbf{H}$. We shall assume that for $\delta\mathbf{H} = 0$ the system is in a constant uniform magnetic field $\mathbf{H}$.

To within quantities $|\delta\mathbf{H}|^4$ the spin-density change δS_i induced by the field $\delta\mathbf{H}$ can be written in the form

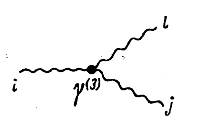
$$\delta S_i = -\mathscr{D}_{ij}\delta\mathscr{H}_j + \frac{1}{2}\mathscr{D}_{ijl}\delta\mathscr{H}_j\delta\mathscr{H}_l - \frac{1}{3!}\mathscr{D}_{ijlm}\delta\mathscr{H}_j\delta\mathscr{H}_l\delta\mathscr{H}_m, \quad \delta\mathscr{H}_i = 2\mu_0\delta H_i. \qquad (A.1)$$

where μ_0 is the Bohr magneton. The quantity $\mathscr{D}_{ij}$ determines the linear response and obviously coincides with the retarded Green function $D^R_{ij}(\omega, \mathbf{k})$ in the limit $\omega = 0$, $\mathbf{k} \to 0$. If we take into account that the Hamiltonian of the interaction of the system with the magnetic field has the form

$$\hat{H}_{int} = \int dr \hat{S}_i(x) \mathscr{H}_i$$

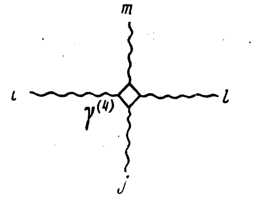
($\hat{S}_i$ is the Heisenberg operator of the spin density), the quantities $\mathscr{D}_{ijl}$, $\mathscr{D}_{ijlm}$ expressing the second- and third-order responses have the following graphical representation:

D_{ijl} =



(A.2)

D_{ijlm} =



(A.3)

The wavy lines on the graphs depict the quantity $\mathscr{D}_{ij}$. The blob in (A.2) and the square in (A.3) respectively denote the three-point $\gamma^{(3)}_{ijl}$ and four-point $\gamma^{(4)}_{ijlm}$ vertices for the interaction of the spin fluctuations. The frequencies and wave-vectors, on which these quantities depend in the general case, are equal to zero.

To determine $\gamma^{(3)}$ and $\gamma^{(4)}$ we note that the quantities $\mathscr{D}_{ij}$, $\mathscr{D}_{ijl}$ and $\mathscr{D}_{ijlm}$ appearing in the equality (A.1) can be expressed in an obvious way in terms of derivatives of the spin density S_i with respect to the quantity $\mathscr{H}_i = 2\mu_0 H_i$:

$$\mathscr{D}_{ij} = -\frac{\partial S_i}{\partial \mathscr{H}_j}, \quad \mathscr{D}_{ijl} = \frac{\partial^2 S_i}{\partial \mathscr{H}_j \partial \mathscr{H}_l}, \quad \mathscr{D}_{ijlm} = -\frac{\partial^3 S_i}{\partial \mathscr{H}_j \partial \mathscr{H}_l \partial \mathscr{H}_m}. \qquad (A.4)$$

By virtue of the exchange character of the interaction, the total spin of the ferromagnet is oriented along the external magnetic field:

$$S_i = S n_i, \quad n_i = \mathscr{H}_i / |\mathscr{H}|.$$

Differentiating S_i with respect to $\mathscr{H}_j$ and then substituting into the equality (A.4), we obtain

$$\mathscr{D}_{ij} = -\frac{S}{\mathscr{H}}\delta_{ij}^{\perp} - S' n_i n_j, \qquad (A.5)$$

$$\mathscr{D}_{ijl} = \left(\frac{S}{\mathscr{H}}\right)' (\delta_{ij}^{\perp} n_l + \delta_{il}^{\perp} n_j + \delta_{jl}^{\perp} n_i) + S'' n_i n_j n_l, \qquad (A.6)$$

$$\mathscr{D}_{ijlm} = -\frac{1}{\mathscr{H}}\left(\frac{S}{\mathscr{H}}\right)' (\delta_{ij}^{\perp}\delta_{lm}^{\perp} + \delta_{il}^{\perp}\delta_{jm}^{\perp} + \delta_{im}^{\perp}\delta_{jl}^{\perp}) - \left(\frac{S}{\mathscr{H}}\right)'' (\delta_{ij}^{\perp} n_l n_m + \delta_{il}^{\perp} n_j n_m + \delta_{jl}^{\perp} n_i n_m + \delta_{im}^{\perp} n_j n_l + \delta_{jm} n_i n_l + \delta_{lm}^{\perp} n_i n_j) - S''' n_i n_j n_l n_m. \qquad (A.7)$$

Here the prime denotes differentiation with respect to the quantity $\mathscr{H}$; $\delta^{\perp}_{ij}$ is the two-dimensional unit tensor in the plane perpendicular to the vector $\mathscr{H}$:

$$\delta^{\perp}_{ij} = \delta_{ij} - n_i n_j. \qquad (A.8)$$

From the equality (A.5) it follows, in particular, that

$$\lim_{\omega=0,\, k\to 0} D_{xx}^R(\omega, \mathbf{k}) = \lim_{\omega=0,\, k\to 0} D_{yy}^R(\omega, \mathbf{k}) = -\frac{S}{2\mu_0 H} \qquad (A.9)$$

(the z-axis is assumed to be directed along $\mathbf{H}$).

In view of the fact that there is one singled-out direction in the problem, the quantities $\gamma^{(3)}$ and $\gamma^{(4)}$ appearing in (A.2) and (A.3) have the following tensor structure:

$$\gamma^{(3)}_{ijl} = {}^1/_3 \gamma^{(3)}_{\parallel\perp} (\delta_{ij}^{\perp} n_l + \delta_{il}^{\perp} n_j + \delta_{jl}^{\perp} n_i) + \gamma^{(3)}_{\parallel\parallel} n_i n_j n_l, \qquad (A.10)$$

$$\gamma^{(4)}_{ijlm} = {}^1/_3 \gamma^{(4)}_{\perp\perp} (\delta_{ij}^{\perp}\delta_{lm}^{\perp} + \delta_{il}^{\perp}\delta_{jm}^{\perp} + \delta_{im}^{\perp}\delta_{jl}^{\perp}) + {}^1/_6 \gamma^{(4)}_{\perp\parallel} (\delta_{ij}^{\perp} n_l n_m + \delta_{il}^{\perp} n_j n_m + \delta_{jl}^{\perp} n_i n_m + \delta_{im}^{\perp} n_j n_l + \delta_{jm}^{\perp} n_i n_l + \delta_{lm}^{\perp} n_i n_j) + \gamma^{(4)}_{\parallel\parallel} n_i n_j n_l n_m. \qquad (A.11)$$

Using the formulas (A.5)–(A.7), (A.10) and (A.11), from the graphical equalities (A.2) and (A.3) we find

$$\gamma^{(3)}_{\parallel\perp} = -3\frac{\mathscr{H}^2}{S^2 S'}\left(\frac{S}{\mathscr{H}}\right)', \qquad (A.12)$$

$$\gamma^{(3)}_{\parallel\parallel} = -\frac{1}{S'^3} S'', \qquad (A.13)$$

$$\gamma^{(4)}_{\perp\perp} = -3\frac{\mathscr{H}^3}{S^4}\left(\frac{S}{\mathscr{H}}\right)', \qquad (A.14)$$

$$\gamma^{(4)}_{\perp\parallel} = -6\frac{\mathscr{H}^2}{S^2 S'^2}\left(\frac{S}{\mathscr{H}}\right)'', \qquad (A.15)$$

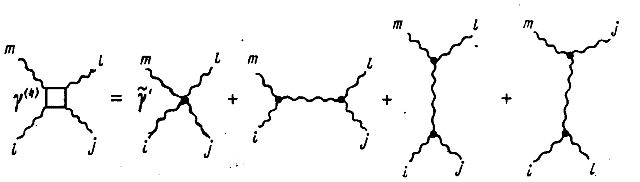
$$\gamma^{(4)}_{\parallel\parallel} = -\frac{1}{S'^4} S'''. \qquad (A.16)$$

With neglect of the magnetic interactions, these relations are completely general, being independent of the temperature and of the assumption that the ferromagnetism is weak. It follows from the equality (A. 14), in particular, that the exact four-point vertex of the interaction of transverse spin excitations vanishes as $H\to 0$:

$$\lim_{H\to 0}\gamma^{(4)}_{\perp\perp} = 0. \qquad (A.17)$$

This conclusion is natural if we take into account that the energy of a ferromagnet does not depend on the orientation of its resultant spin for $H=0$. For nonzero wave-vectors $\{\mathbf{k}_i\}$ of the spin-waves being scattered, the vertex corresponding to them should be proportional to the product of the $\mathbf{k}_i$. This statement can be generalized without difficulty to the case of mutual scattering of an arbitrary number of transverse spin-waves.

We note that, whereas the three-point vertex $\gamma^{(3)}$ has no sections with one $\mathscr{D}$-line, by definition, graphs for $\gamma^{(4)}$ do contain such sections, generally speaking. They can be separated by means of the obvious equality:



(A. 18)

The blob with four entry points for $\mathscr{D}$-lines denotes here the four-point vertex $\tilde{\gamma}^{(4)}_{ijlm}$ that is irreducible with respect to $\mathscr{D}$-lines in any direction. In order to establish the relationship between the vertices $\tilde{\gamma}^{(4)}_{ijlm}$ and $\tilde{\gamma}^{(3)}_{ijl}$ in the case of a weak ferromagnet, we shall use an expression for the free energy $\tilde{F}$, introduced in Sec. 3, at $T=0$; this can be written in the form

$$\tilde{F}(S) = \tilde{F}(0) - \frac{1}{2}lS^2 + \frac{1}{4!}\gamma S^4, \qquad (A.19)$$

where l and γ are constants. The thermodynamic-equilibrium condition $\tilde{F}'(S) = \mathscr{H}$ leads to the equation

$$-lS + \tfrac{1}{6}\gamma S^3 - \mathscr{H} = 0, \qquad (A.20)$$

Hence, for the derivatives of S with respect to $\mathscr{H}$ we obtain the following relations:

$$1/S' = -l + \tfrac{1}{2}\gamma S^2, \qquad (A.21)$$

$$S'' = -\gamma S S'^3, \qquad (A.22)$$

$$S''' = -\gamma S'^4 + 3\gamma^2 S^2 S'^5. \qquad (A.23)$$

Substitution of (A. 20) and (A. 21) into the equalities (A. 12) and (A. 13) gives

$$\gamma^{(3)}_{\parallel\perp} = \gamma^{(3)}_{\parallel\parallel} = \gamma S,$$

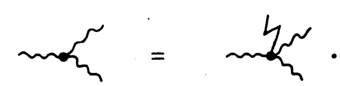
and, consequently, when formula (A. 10) is taken into account, the three-point vertex has the form

$$\gamma_{ijl} = \tfrac{1}{3}\gamma S(\delta_{ij}n_l + \delta_{il}n_j + \delta_{jl}n_i). \qquad (A.24)$$

As a result of substituting the equalities (A. 14)–(A. 16) into (A. 11) and then (A. 11) and (A. 24) into the diagrammatic relation (A. 18), after simple calculations with allowance for (A. 21)–(A. 23) we arrive at an expression for the four-point vertex $\tilde{\gamma}^{(4)}$ irreducible with respect to $\mathscr{D}$-lines:

$$\tilde{\gamma}^{(4)}_{ijlm} = \tfrac{1}{3}\gamma(\delta_{ij}\delta_{lm} + \delta_{il}\delta_{jm} + \delta_{im}\delta_{jl}). \qquad (A.25)$$

We note that formula (A. 24) can be obtained from (A. 25) by means of the diagrammatic relation



(A. 26)

The jagged line corresponds to the condensate S.

[1]A. A. Abrikosov and I. E. Dzyaloshinskiĭ, Zh. Eksp. Teor. Fiz. 35, 771 (1958) [Sov. Phys. JETP 8, 535 (1959)].
[2]P. S. Kondratenko, Zh. Eksp. Teor. Fiz. 46, 1438 (1964) [Sov. Phys. JETP 19, 972 (1964)].
[3]P. S. Kondratenko, Zh. Eksp. Teor. Fiz. 47, 1536 (1964) [Sov. Phys. JETP 20, 1032 (1965)].
[4]D. M. Edwards and E. P. Wohlfarth, Proc. Roy. Soc. London 303, 127 (1968).
[5]E. P. Wohlfarth, J. Appl. Phys. 39, 1061 (1968).
[6]K. K. Murata and S. Doniach, Phys. Rev. Lett. 29, 285 (1972).
[7]K. K. Murata, Phys. Rev. B12, 282 (1975).
[8]A. A. Abrikosov, L. P. Gor'kov and I. E. Dzyaloshinskiĭ, Metody kvantovoĭ teorii polya v statisticheskoi fizike (Quantum Field Theoretical Methods in Statistical Physics), Fizmatgiz, M., 1962 (English translation published by Pergamon Press, Oxford, 1965).
[9]L. D. Landau, Zh. Eksp. Teor. Fiz. 35, 97 (1958) [Sov. Phys. JETP 8, 70 (1959)].
[10]I. E. Dzyaloshinskiĭ, Zh. Eksp. Teor. Fiz. 46, 1722 (1964) [Sov. Phys. JETP 19, 1163 (1964)].
[11]V. L. Ginzburg, Fiz. Tverd. Tela 2, 2031 (1960) [Sov. Phys. Solid State 2, 1824 (1961)].

Translated by P. J. Shepherd