and Yu. A. Shishkov, Soobshch. OIYaI (JINR Commun.) R8-8563, 1975.
[10]E. Segrè (ed.), Experimental Nuclear Physics, John Wiley, New York, 1959 (Russ. Transl., Vol. 3, IIL, M., 1961, p. 13).
[11]T. A. Carlson, C. W. Nestor, N. Wasserman, and J. D. McDowell, At. Data 2, 63 (1970).
[12]A. Salop, Phys. Rev. A 9, 2496 (1974).
[13]Y. Hahn and K. N. Watson, Phys. Rev. A 7, 491 (1973).
[14]W. Lotz, Z. Phys. 216, 241 (1968).

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On Coulomb polarization of vacuum

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The charge density induced by a Coulomb field is represented as the absorptive part of a certain operator, which can be reduced in turn to a finite-rotation operator of the Coulomb dynamic group $O(2,1)$. In this way the total induced charge can be calculated and the cause of suppression of the contribution from higher partial waves can be ascertained.

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INTRODUCTION

The polarization of a vacuum of charged particles by an external electromagnetic field was first considered by Dirac, Heisenberg, Serber, and Uehling in the weak-field approximation.[1] They obtained an expression for the induced charge density

$$\rho=-\alpha\Box\rho_0/15\pi m^2 \tag{1}$$

in terms of the charge density ρ_0 that produces the external field.

Subsequently Weisskof and Schwinger[2] presented a general expression suitable for fields of arbitrary strength, but this expression turned out to be too complicated and yielded a result in explicit form only in some particular cases. In addition, the important case of a Coulomb field could not be handled by this method.

Wichmann and Kroll[3] have therefore returned to the direct calculation methods and, by using very complicated computations, obtained corrections to the Uehling formula. These calculations were recently radically improved by Brown *et al.*[4]

The question of the calculation of the Coulomb polarization of vacuum has been under lively discussion in recent times, since this effect turned out to be particularly noticeable in heavy μ-mesic atoms.[5] However, even after the publication of[4], the theory of Coulomb polarization remains rather cumbersome. The reason, in our opinion, is the neglect of the symmetry of the Coulomb field.

We describe below a calculation method that takes into account this important property explicitly.

GENERAL RELATIONS

We first transform the well known formula[2] for the induced current

$$j_\mu=\mathrm{tr}\,\gamma_\mu G, \tag{2}$$

which contains for an electron in the external field, a Green's function satisfying the equation

$$[m-\gamma(p-eA)]G=1. \tag{3}$$

In operator form ($\Pi=p-eA$) we have

$$j_\mu=\mathrm{tr}\,\gamma_\mu(m-\hat\Pi)^{-1}=\mathrm{tr}\,\gamma_\mu(m+\hat\Pi)(m^2-\hat\Pi^2)^{-1}=\mathrm{tr}\,\gamma_\mu\hat\Pi(m^2-\hat\Pi^2)^{-1} \tag{4}$$

(the term $\sim m$ has dropped out, since it contains an odd number of Dirac matrices). Transferring $\hat\Pi$ to the right, we obtain the expression

$$j_\mu=\mathrm{tr}\,\gamma_\mu(m^2-\hat\Pi^2)^{-1}\hat\Pi,$$

which when summed with (4) yields

$$j_\mu=\mathrm{tr}\,\Pi_\mu(m^2-\hat\Pi^2)^{-1}=\Pi_\mu\int_0^\infty ds\,\mathrm{tr}\exp\{-s(m^2-\hat\Pi^2]\}. \tag{5}$$

The charge density induced by the static field is equal to

$$\rho_E=(E-eA_0)\int_0^\infty ds\,\mathrm{tr}\exp\{-s[m^2+\mathbf{p}^2-(E-eA_0)^2-e\sigma_{\mu\nu}F_{\mu\nu}/2]\} \tag{6}$$

or

$$\rho_E=\frac{1}{2}\frac{d}{dE}\int_0^\infty\frac{ds}{s}\,\mathrm{tr}\exp\{-s[m^2-\hat\Pi^2]\}. \tag{6a}$$

This means that the total charge density is equal to

$$\rho=\int_{-i\infty}^{+i\infty}\rho_E\frac{dE}{2\pi i}=(2\pi i)^{-1}[W(E)-W(-E)]_{E\to i\infty}, \tag{7}$$

so that the problem reduces to the calculation of

$$W(E)=\frac{1}{2}\int_0^\infty \frac{ds}{s}\,\mathrm{tr}\exp\{s[(E-eA_0)^2+e\sigma F/2-m^2-\mathbf{p}^2]\}. \quad (8)$$

COULOMB SYMMETRY

In a Coulomb field $eA_0=-Z\alpha/r$, $e\sigma F/2=-\gamma_0\boldsymbol{\gamma}Z\alpha\mathbf{r}/r^3$. the argument of the exponential in (8) is

$$(E^2-m^2)-p_r^2+\frac{2Z\alpha E}{r}-\frac{\mathbf{L}^2-(Z\alpha)^2+Z\alpha\gamma_0\boldsymbol{\gamma}\mathbf{n}}{r^2}. \quad (9)$$

The operator

$$N=\mathbf{L}^2-(Z\alpha)^2+Z\alpha\gamma_0\boldsymbol{\gamma}\mathbf{n} \quad (10)$$

commutes with the angular momentum $\mathbf{J}^2$, with its projection J_z, and with the parity P. Its eigenvalues are doubly degenerate and are equal to $\gamma(\gamma\pm1)$, where[6]

$$\gamma=[(j+{}^1/_2)^2-(Z\alpha)^2]^{1/2}, \quad (11)$$

and the ± sign is the same as in the relation $P=\pm(-1)^{j+1/2}$.

Since N is a unit matrix in formula (8), the determination of the trace reduces to summation over the two parity signs and to multiplication by two. Thus,

$$W_{jMP}(E)=\int_0^\infty s^{-1}ds\exp\left\{s\left[(E^2-m^2)-p_r^2+\frac{2Z\alpha E}{r}-\frac{\gamma(\gamma\pm1)}{r^2}\right]\right\}. \quad (12)$$

Replacing the integration variable s by sr, we reduce the quantity in the square brackets to the form

$$2Z\alpha E+r(E^2-m^2)-rp_r^2-\gamma(\gamma\pm1)r^{-1}\equiv2Z\alpha E-\beta K_0-\lambda K_1, \quad (13)$$

where

$$K_0=[rp_r^2+\gamma(\gamma\pm1)r^{-1}+rk^2](2k)^{-1},$$
$$K_1=[rp_r^2+\gamma(\gamma\pm1)r^{-1}-rk^2](2k)^{-1}, \quad (14)$$
$$\beta=k+(m^2-E^2)/2k,\quad \lambda=k-(m^2-E^2)/2k. \quad (15)$$

The operators K_0, K_1, and

$$K_2=rp_r-i \quad (16)$$

form an $O(2,1)$ algebra (see[7]):

$$[K_0,K_1]=iK_2,\quad [K_1K_2]=-iK_0,\quad [K_2K_0]=iK_1, \quad (17)$$

with a Casimir operator

$$C^{(2)}=K_0^2-K_1^2-K_2^2=\gamma(\gamma\pm1). \quad (18)$$

It follows from (17) that the operator $K_+=K_1+iK_2$ increases the eigenvalue of the operator K_0 by unity, so that the spectrum of K_0 takes a ladder form:

$$(K_0)'=\varkappa+n,\quad n=0,1,2,\ldots. \quad (19)$$

The maximum eigenvalue $\varkappa$ is determined from (18):

$$\varkappa(\varkappa-1)=\gamma(\gamma\pm1),$$
$$\varkappa=\begin{cases}\gamma & P=-(-1)^{j+1/2}\\ \gamma+1 & P=(-1)^{j+1/2}\end{cases}. \quad (20)$$

The linear combination of the non-commuting operators (13) is diagonalized

$$\beta K_0+\lambda K_1=\sqrt{\beta^2-\lambda^2}TK_0T^{-1}=2\sqrt{m^2-E^2}TK_0T^{-1} \quad (21)$$

with the aid of the "tilt" operator[7]

$$T=\exp\left(\frac{i}{2}K_2\ln\frac{m^2-E^2}{k^2}\right). \quad (21a)$$

Consequently we obtain

$$W_{jMP}(E)=\int_0^\infty s^{-1}ds\,e^{\nu s}Te^{-sK_0}T^{-1}. \quad (22)$$

We have introduced here the dimensionless Coulomb parameter

$$\nu=Z\alpha E/\sqrt{m^2-E^2}. \quad (23)$$

VALUE OF THE INDUCED CHARGE

We use the formula (22) obtained above to calculate the induced charge

$$Q=\int\rho\,d\mathbf{r}=\mathrm{Sp}\,\rho=\frac{1}{\pi}\mathrm{Im}\,\mathrm{Sp}\,W(E)|_{E\to i\infty}. \quad (24)$$

The trace is taken over all the variables, i.e., over the spectrum of j, M, P, K_0:

$$Q=\pi^{-1}\sum_j(2j+1)\sum_P\sum_{n=0}^\infty\mathrm{Im}\int_0^\infty s^{-1}ds\,e^{\nu s-s(\varkappa+n)}|_{E\to i\infty}$$
$$=\pi^{-1}\sum_j(2j+1)\mathrm{Im}\int_0^\infty s^{-1}ds\,e^{\nu s-\gamma s}(1+e^{-s})(1-e^{-s})^{-1}|_{E\to i\infty}. \quad (25)$$

The factor $(1+e^{-s})$ is the result of summation over the parity, while the fraction $(1-e^{-s})^{-1}$ is the result of summation over n.

It is now convenient to take the limit as $E\to+i\infty$, since only ν depends on E, with $\nu\to+iZ\alpha$. Putting

$$\eta=\gamma-iZ\alpha=\sqrt{(j+{}^1/_2)^2-(Z\alpha)^2}-iZ\alpha, \quad (26)$$

we obtain

$$Q=\frac{1}{\pi}\sum_j(2j+1)\mathrm{Im}\int_0^\infty\frac{e^{-\eta s}(1+e^{-s})ds}{s(1-e^{-s})}. \quad (27)$$

This expression must be renormalized; this is done, as is known, by subtracting the first terms of the Taylor series, i.e., by replacing $f(x)$ by the residual term of the series

$$f(x)\to\frac{1}{N!}\int_{x_0}^x y^N f^{(N+1)}(y)\,dy. \quad (28)$$

The choice of the number of subtractions N and of the normalization point x_0 is dictated by the actual conditions. In our case it is necessary to subtract a linear polynomial, so that $N=1$ and

$$e^x\to\int_{x_0}^x y(e^y)''dy=e^x(x-1)-e^{x_0}(x_0-1). \quad (29)$$

Thus,

$$Q=\frac{1}{\pi}\sum_j (2j+1)\,\mathrm{Im}\int_0^\infty ds\frac{1+e^{-s}}{s(1-e^{-s})}[e^{-\eta s}(-\eta s-1)+e^{-\eta_0 s}(\eta_0 s+1)]. \tag{30}$$

This integral, being written in the form

$$I=\int_0^\infty ds\left\{\frac{2}{s(1-e^{-s})}[e^{-\eta s}(-\eta s-1)+e^{-\eta_0 s}(\eta_0 s+1)] + \frac{e^{-\eta s}-e^{-\eta_0 s}}{s}+(\eta e^{-\eta s}-\eta_0 e^{-\eta_0 s})\right\} \tag{31}$$

is evaluated by using the formulas (8.341), (8.361), and (3.434) of the handbook[8]:

$$I=2[\eta\psi(\eta)-\ln\Gamma(\eta)]-\ln\eta-(\eta\to\eta_0). \tag{32}$$

The constant normalization η_0 should be chosen such that the renormalized charge coincides with the experimentally observed one. To this end it is necessary that expression (30) not contain the term $\sim Z\alpha$, which is already included in the charge. This requirement is easily satisfied:

$$Q=\frac{2}{\pi}\sum_j (2j+1)\,\mathrm{Im}\left[\eta\psi(\eta)-\ln\Gamma(\eta)-\frac{1}{2}\ln\eta + iZ\alpha\left(j+\frac{1}{2}\right)\psi'\left(j+\frac{1}{2}\right)-\frac{iZ\alpha}{2j+1}\right]. \tag{33}$$

This elegant formula was obtained in[4], and its nonvanishing term

$$Q\approx\frac{2(Z\alpha)^3}{3\pi}\left[\zeta(3)+\frac{7}{6}-\frac{\pi^2}{4}\right]=-0.021(Z\alpha)^3 \tag{34}$$

was obtained even earlier by Wichmann and Kroll.[3] We note that the coefficients of the expansion in $Z\alpha$ decrease slowly; the next term is equal to $-0.007\ (Z\alpha)^5$.

Conversely, the terms of the series (33) in j are greatly different: the term with $j=\frac{1}{2}$ exceeds the sum of all the remaining ones by thirteen times. The contribution of the $P_{1/2}$ wave is $0.72(Z\alpha)^3/3\pi$, and the contribution of the $S_{1/2}$ wave is $0.92(Z\alpha)^3/3\pi$, i.e., terms with opposite parity cancel each other noticeably. Formula (27) offers a good explanation of the suppression of the contribution of the higher partial waves—it is due to the exponential $e^{-\gamma s}$. Consequently, the reason lies in the increase of the centrifugal barrier, which lowers the probability of pair production near the polarizing center.

We note that at $Z=82$ (lead) it is possible, with 5% accuracy, to replace γ by $j+\frac{1}{2}$, starting with $j=\frac{3}{2}$, after which the contribution of all the waves with $j\geq\frac{3}{2}$ can be summed in explicit form

$$Q_{j\geq 3/2}=\frac{2}{\pi}\int_0^\infty ds\frac{\sin(Z\alpha s)-Z\alpha s}{s(1-e^{-s})^4}e^{-2s}(2-e^{-s})(2se^{-s}+e^{-2s}-1). \tag{35}$$

The suppression of the higher partial waves of the virtual particles was noted earlier also in the problem of the Lamb shift.[9]

CONCLUSION

The described method of calculating the polarization of vacuum is based essentially on the representation of the current in the form (5)—it is this which makes it possible to change over to a squared Dirac equation and to express the charge density in terms of the asymptotic form of the operator $W(E)$.

Another important item is the direct utilization of the $O(2,1)$ symmetry of the radial equation of the Coulomb problem, which leads to formula (22). It follows from this formula that $W(E)$ is the Laplace transform of the finite-transformation operator in the $O(2,1)$ group, since $Te^{-sK_0}T^{-1}$ is in fact such an operator. For the calculation of the induced current it is necessary to investigate the asymptotic behavior of the corresponding Wigner functions.

It seems to us that the determination of all these relations will make it possible once more to simplify the theory and to gain a deeper insight in the polarization of vacuum.

[1] P. Dirac, Proc. Camb. Philos. Soc. 30, 150 (1934); W. Heisenberg, Z. Phys. 90, 209 (1934); R. Serber, Phys. Rev. 48, 49 (1935); E. A. Uehling, Phys. Rev. 48, 55 (1935).

[2] V. Weisskopf, K. Dan. Vidensk. 14, 1 (1936); J. Schwinger, Phys. Rev. 82, 664 (1951).

[3] E. Wichmann and N. Kroll, Phys. Rev. 101, 843 (1956).

[4] L. S. Brown, R. N. Cahn, and L. D. McLerran, Phys. Rev. D12, 581, 596, 609 (1975).

[5] S. L. Adler, Phys. Rev. D10, 3714 (1974); L. Tauscher, G. Backenstoss, K. Fransson, H. Koch, A. Hilsson, and J. De Raedt, Phys. Rev. Lett. 35, 410 (1975).

[6] Ya. I. Granovskiĭ and V. I. Nechet, Teor. Met. Fiz. 18, 262 (1974).

[7] V. F. Dmitriev and Yu. B. Rumer, Teor. Mat. Fiz. 5, 276 (1970).

[8] I. S. Gradshteĭn and I. M. Ryzhik, Tablitsy integralov, summ, ryadov i proizvedeniĭ (Tables of Integrals, Sums, Series, and Products), Fizmatgiz, 1962. [Academic 1966].

[9] Ya. I. Granovskiĭ, Yad. Fiz. 15, 340 (1972) [Sov. J. Nucl. Phys. 15, 192 (1972)]; P. Mohr, Ann. Phys. (N.Y.) 88, 26, 52 (1974).

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