(the suppression of the pionic channels requires that $\bar{n} \gg 1$, or, on account of what was said, $M \gg m_0$).

In conclusion we note that one of the simplest types of coherent states was considered here. It is essential that the modern approach to coherent states (cf. [12]) considerably widens the spectrum of theoretically conceivable models of this type.

The author is indebted to V. A. Karmanov for a valuable remark regarding the calculation of the probability of electromagnetic leptonic decays, as well as to A. E. Kudryavtsev and A. M. Perelomov for useful discussions.

[1)]Equation (1) differs from the corresponding expressions in[3,4]. It corresponds to the operator used in[4] in the special case of real functions $f(x)$.

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Decay of bounded laser beams in nonlinear media

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The stability of propagation of an intense laser beam in a medium with quadratic or cubic optical nonlinearities is investigated. The existence of a discrete spectrum and of a set of natural perturbation functions that correspond to the modes of the "waveguide" produced by the laser beam in the nonlinear medium is found. "Branch points" at which, in contrast to the usual reversal point, no reflection of electromagnetic waves take place, are found to play an important role in the formation of the "waveguide." Dispersion equations for the perturbation growth rates are derived for axisymmetric laser beams of arbitrary and smooth intensity profile. Some simple geometric rules for determining the maximal growth rates are formulated and their dependence on the azimuthal number characterizing the perturbation is found. A limiting transition to the case of an unbounded laser beam considered by Bespalov and Talanov {Pis'ma Zh. Eksp. Teor. Fiz. 3, 471 (1966) [JETP Lett. 3, 307 (1966)]} is analyzed and compared with the results by others.

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Intense laser beams propagating in nonlinear media can be unstable to various types of perturbations. Growth of the perturbations leads to decay of the initial beam. For media with cubic nonlinearity, this effect has been most thoroughly investigated as applied to the self-focusing phenomenon. As shown by Bespalov and Talanov, [1] an intense plane wave is unstable to definite perturbations of its profile, and this causes the plane wave to break up into individual filaments.

Although the theory of Bespalov and Talanov explains the main features of the phenomenon and yields for the self-focusing length an estimate that agrees with experiment, it does not take into account so important a factor as the limited dimensions of real laser beams. Attempts to analyze the stability of a bounded laser beam in a cubic medium were undertaken in several places, the best results being obtained by Lyakhov. [2] This procedure, however, is not sufficiently well founded, and some of his results contradict numerical experiments [3] as well as the ideas that have developed by now.

The fundamental difference between the bounded laser beam and an infinite plane wave is the following: In the case of an intense bounded beam, it becomes possible for perturbations that are bounded in the transverse direction to propagate. Such perturbations correspond to the discrete modes of the "waveguide" produced in the nonlinear medium by the laser beam. The waveguide can be regarded as homogeneous, since the distances of interest to us, which are of the order of the characteristic perturbation growth length, are usually much smaller in real laser beams than the self-focusing length of the beam as a whole. For bounded beams it is precisely the discrete spectrum which corresponds to increasing perturbations. An analysis of the stability of intense and broad laser beams without allowance for the waveguide modes (see, e.g., [4]) is therefore only of limited use.

We report here the results of an investigation of the stability of propagation of an axially-symmetrical beam with an arbitrary smooth intensity profile. In Secs. 1 and 2 we consider a medium with cubic optic nonlinearity. An essentially similar treatment is presented in Sec. 3 for a quadratic medium, where the decay of the initial "pump wave" corresponds, in the given-field approximation, to parametric amplification of weak waves.[5] We use in the analysis essentially a procedure based on a semiclassical (WKB) method,[6,7] the use of which is justified by the slowness of the transverse change in the intensity of the main beam. When determining the indicated discrete spectrum, besides the turning points that are characteristic of second-order differential equations, an essential role is played by the "branch points" first introduced by Rukhadze, Savadchenko, and Triger[8] in investigations of the spectrum of small oscillations of an inhomogeneous plasma in an external magnetic field. Some of the results obtained on the basis of this procedure for a cubic medium are given in [9].

1. RADIAL EQUATIONS AND THEIR WKB SOLUTION

We start with a parabolic equation for the amplitude of the field E:

$$2ik\partial E/\partial z=\Delta_{\perp}E+k^2(\varepsilon_2/\varepsilon_0)E|E|^2, \tag{1}$$

where $k=\omega\varepsilon_0^{1/2}/c$, ω is the optical frequency, and the nonlinear refractive index of the cubic medium is written in the form $\varepsilon=\varepsilon_0+\varepsilon_2|E|^2$, $\varepsilon_2>0$.

We represent the field E in a cylindrical coordinate frame in the form of a sum of a smooth axially-symmetric solution (1) $E_0(r, z)$ and a small perturbation $E_1(r, \varphi, z)$ with strongly differing space scales $r_0 \gg r_1$. Linearizing (1) with respect to E_1, we obtain the linear homogeneous equation

$$2ik\partial E_1/\partial z=\Delta_{\perp}E_1+k^2(\varepsilon_2/\varepsilon_0)[2|E_0|^2E_1+E_0^2E_1^*]. \tag{2}$$

Assuming that the wave E_0 is plane at $z=0$, we can put for the distances $z \ll L_c$ of interest to us, where L_c is the self-focusing length of the beam as a whole (see also[10]),

$$E_0=A_0(r)e^{-iB(r)z/2k},\quad E_1=A_1(r,\varphi,z)e^{-iB(r)z/2k},\quad B(r)=k^2(\varepsilon_2/\varepsilon_0)A_0^2(r). \tag{3}$$

Introducing the large parameter

$$M=B_0^{1/2}r_0\sim r_0/r_1,\ \text{where}\quad B_0=\max_r B(r),$$

we arrive at the truncated equation (see Appendix, Sec. 1)

$$2ik\frac{\partial A_1}{\partial z}=\frac{\partial^2 A_1}{\partial r^2}+\frac{1}{r}\frac{\partial A_1}{\partial r}+\frac{1}{r^2}\frac{\partial^2 A_1}{\partial\varphi^2}+B(r)(A_1+A_1^*). \tag{4}$$

Separating the variables

$$A_1=\psi(r)\exp\{\beta z/2k+im\varphi\}+\chi^*(r)\exp\{\beta^* z/2k-im\varphi\},\quad m=0,1,2,\ldots, \tag{5}$$

we obtain a system of radial equations

$$\begin{aligned}&\psi''+r^{-1}\psi'-m^2r^{-2}\psi-i\beta\psi+B\psi+B\chi=0,\\&\chi''+r^{-1}\chi'-m^2r^{-2}\chi+i\beta\chi+B\psi+B\chi=0.\end{aligned} \tag{6}$$

In the system (6), the quantity β (the real part of which determines the growth rate of the perturbations) plays the role of the eigenvalue, and ψ and χ are the eigenfunctions. If we expand in terms of these eigenfunctions the arbitrary initial perturbation $E_1(r, \varphi, 0)$, then the perturbation in any section $z \ll L_c$ is determined with the aid of (5). If the unperturbed field is an infinite plane wave $B=\text{const}$, then the spectrum is continuous,

$$\psi=C_1Z_m(\varkappa r),\quad \chi=C_2Z_m(\varkappa r). \tag{7}$$

Here Z_m is a Bessel function of order m, and in accordance with the result of Bespalov and Talanov[1] we have $\beta^2=\varkappa^2(2B-\varkappa^2)$, the maximum growth rate $\beta_{max}=B$ corresponding to $\varkappa^2=B$.

In the case of a bounded beam, the continuous spectrum corresponds to perturbations that do not increase (relative to z), for which β is pure imaginary. Under definite conditions there exists also a discrete spectrum of the system (6) supplemented by the conditions that the equations be finite at $r=0$ and that they decrease sufficiently rapidly as $r\to\infty$. The arbitrary initial perturbation $E_1(r, \varphi, 0)$ can be expanded in terms of the functions of the discrete and continuous spectra. It is precisely the discrete spectrum which is of interest, since it corresponds to perturbations that increase with increasing z.

We note that the system (6) is equivalent to the fourth-order equation

$$\begin{aligned}&y^{(4)}+\frac{2}{r}y^{(3)}+\left(2B-\frac{2m^2+1}{r^2}\right)y''+\left(\frac{2B}{r}+\frac{2m^2+1}{r^3}\right)y'\\&\quad+\left(\frac{m^4-4m^2}{r^4}-2B\frac{m^2}{r^2}+\beta^2\right)y=0,\\&y=i(\psi-\chi),\quad \psi+\chi=-\frac{1}{\beta}\left(y''+\frac{1}{r}y'-\frac{m^2}{r^2}y\right).\end{aligned} \tag{8}$$

In the approximation wherein the transverse intensity of the unperturbed beam $B(r)$ varies slowly, we can obtain a system of two uncoupled second-order equations for the functions

$$u=B\psi+C^{(+)}\chi,\quad v=B\psi+C^{(-)}\chi; \tag{9}$$

$$u''+r^{-1}u'+k_1^2u=0,\quad v''+r^{-1}v'+k_2^2v=0;$$

$$k_{1,2}^2=B-m^2r^{-2}\pm(B^2-\beta^2)^{1/2},\quad C^{(\pm)}=i\beta\pm(B^2-\beta^2)^{1/2}. \tag{10}$$

Equations (9) are valid everywhere except in narrow vicinities $|r-r_b|\sim B_0^{1/2}$ of the points at which $B(r_b)=\beta$, which we[8] shall call branch points. In the vicinities of the branch points, where $|B-\beta|\ll\beta$, we can separate waves that travel in opposite directions (along r). At $m=0$ we obtain the equation

$$y=A_1\exp(i\beta^{1/2}r)+A_2\exp(-i\beta^{1/2}r), \tag{11}$$

$$A_p=g_p\exp\left\{\frac{i}{2\beta^{1/2}}\int^r(B-\beta)\,dr\right\},\quad g_p''+\frac{1}{2}(B-\beta)g_p=0,\quad p=1,2.$$

Let us investigate the character of the solutions in the vicinity $0<r<r_b$, where r_b is the branch point. A good approximation in this region is the solutions of (9). Equations of this type are investigated in the theory of elastic atomic collisions.[11] The most exact approximation was obtained by Rosen and Yennie[12] and is given by

$$\binom{u}{v} = \left[\frac{Q_{(\pm)}}{(Q'_{(\pm)}r)}\right]^{1/2} J_m(Q_{(\pm)}), \tag{12}$$

where J_m is a Bessel function of order m, and $Q_{(\pm)}(r)$ is determined from the equations

$$[Q_{(\pm)}^2-m^2]^{1/2}-m\arccos\frac{m}{Q_{(\pm)}}=\int_{r_t}^{r}\left[B-\frac{m^2}{r^2}\pm(B^2-\beta^2)^{1/2}\right]^{1/2}dr,\quad Q_{(\pm)}>m,$$
$$[m^2-Q_{(\pm)}^2]^{1/2}-m\ln\frac{m\pm[m^2-Q_{(\pm)}^2]^{1/2}}{Q_{(\pm)}}=\int_{r_t}^{r}\left[\frac{m^2}{r^2}-B\mp(B^2-\beta^2)^{1/2}\right]^{1/2}dr,\quad Q_{(\pm)}<m, \tag{13}$$

r_t is the turning point closest to zero. In this case, in contrast to the characteristic problems of the theory of elastic collisions, this approximation is valid for all m, including $m=0$.

We shall need subsequently the asymptotic forms of the functions u and v far beyond the first turning point. In this case $Q_{(\pm)}\gg m$ and it follows from (12) and (13) that

$$\binom{u}{v}\sim r^{-1/2}\left[B-\frac{m^2}{r^2}\pm(B_2-\beta^2)^{1/2}\right]^{-1/4}\sin\left\{\frac{\pi}{4}+\int_{r_t}^{r}\left[B-\frac{m^2}{r^2}\pm(B^2-\beta^2)^{1/2}\right]^{1/2}dr\right\}. \tag{14}$$

The asymptotic form (14) is valid up to the vicinities of the next turning point of branch point. Far from the turning and branch points, as seen from (9), there are four solutions containing exponentials of the type (see, e.g.,[13])

$$\exp\left[\pm i\int^{r}k_p\,dr\right],\quad p=1,2.$$

In accordance with (14), we have taken into account here a distingiushing feature of the use of the WKB method for a radial equation.[1)] As $r\to\infty$ we have $B\to 0$ and the asymptotic form of these solutions is

$$\exp\left[(\pm1\pm i)(\beta/2)^{1/2}r\right]. \tag{15}$$

The problem of the bound states corresponds to positive eigenvalues $\beta>0$. To obtain the discrete spectrum it is necessary to choose from among the four solutions (15) the two decreasing ones.

2. DISCRETE SPECTRUM AND PERTURBATION GROWTH RATES

The results of the preceding section enable us to determine the eigenvalues β and the corresponding growth rates of the perturbations for arbitrary sufficiently smooth intensity profiles. The discrete-spectrum functions are obtained by joining together the solutions (14) and the WKB solutions with allowance for the condition that they decrease as $r\to\infty$. To this end it is necessary

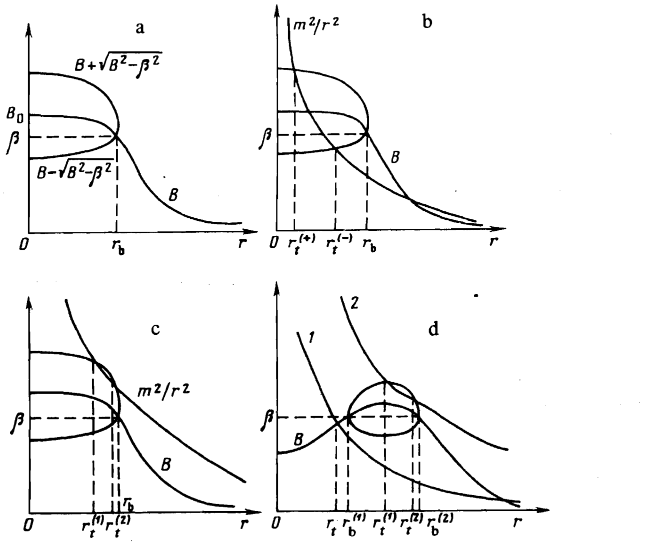

FIG. 1. Arrangement of the turning points and branch points for different azimuthal numbers and different laser-beam profiles: a) $m=0$, dispersion equation (19); b) Eq. (23); c) Eq. (25); d) curve 1—plot of m^2/r^2 at small m, Eq. (29); curve 2—large m.

to examine in greater detail the vicinities of the turning points and the branch points. For the ordinary turning points at which $k_p=0$, there are the well known Jeffreys formulas[15] connecting the asymptotic forms of the functions on the left and on the right of the turning points (see, e.g.,[6]). Using (11), we can show that the Jeffreys coupling formulas as valid for branch points.

We note immediately the following distinguishing feature of bounded laser beams. First, the eigenvalues β cannot exceed a maximum value $\beta_{max}<B_0$. Indeed, in the opposite case it would follow from (14) that the solutions are not bounded as $r\to\infty$. Further, in contrast to the case of an unperturbed beam in the form of an infinite plane wave, when the growth increment β does not depend on m, this dependence is essential for a bounded beam. In addition, β depends on the transverse intensity profile of the unperturbed beam.

We consider a typical case of laser beams with bell-shaped intensity profile (Fig. 1(a)), when $B_0=B(0)$. At $m=0$ there are no real turning points and the only cause of the formation bound states is the presence of branch points. Investigating the behavior of the functions k_p(10) as $r\to\infty$, we can verify that the decreasing solutions take the form (the signs are in agreement)

$$\exp\left\{\pm i\int^{r}[B\pm(B^2-\beta^2)^{1/2}]^{1/2}dr\right\}. \tag{16}$$

In the vicinity of the branch point r_b, at $x=r-r_b>0$, expression (16) takes the form of a decreasing exponential ($B'<0$):

$$\exp\left[\pm i\beta^{1/2}x-{}^{2}/_{3}(|B'|/2)^{1/2}x^{3/2}\right]. \tag{17}$$

Using Jeffreys coupling formulas we obtain from (17) at $x<0$

$$\exp(\pm i\beta^{1/2}x)\sin(^2/_3\sqrt{|B'|/2}|x|^{3/2}+{}^1/_4\pi). \quad (18)$$

Matching this to the solution (14), we obtain an equation for the eigenvalue β:

$$\int_0^{r_t}\{[B+(B^2-\beta^2)^{1/2}]^{1/2}-[B-(B^2-\beta^2)^{1/2}]^{1/2}\}dr=\frac{\pi}{2}+n\pi, \quad n=0,1,2,\ldots. \quad (19)$$

The number of discrete eigenvalues can be estimated by putting in (19) $\beta=0$ and $r_b=0$. For a Gaussian beam $B=B_0\exp(-r^2/r_0^2)$ we have then

$$n_{max}=\frac{\sqrt{2}}{\pi}\int_0^\infty B^{1/2}(r)\,dr-\frac{1}{2}=\left(\frac{B_0r_0^2}{\pi}\right)^{1/2}-\frac{1}{2}. \quad (20)$$

Greatest interest attaches to the maximum values of β corresponding to the fastest-growing perturbations. These perturbations are concentrated in the region of maximum intensity, that is, in the vicinity of $r=0$. We can then put $B(r)=B_0(1-r^2/r_0^2)$, $B_0-\beta\ll B_0$. Equation (19) yields then

$$\beta/B_0=1-\sqrt{2}M^{-1}(1+2n). \quad (21)$$

The rapidly-growing perturbations propagate in the central region of the band $0<r<r_b$, inside which they are modulated at a period r_1, where

$$r_b^2=r_0^2 2^{1/2}M^{-1}(1+2n)\ll r_0^2, \quad r_1=2\pi B_0^{-1/2}\ll r_b. \quad (22)$$

Thus, we have succeeded in obtaining the bound states, despite the absence of turning points. We return now to the determination of the dependence of β on m. From (10), as well as from Figs. 1(b) and 1(c), it is seen that at $m>0$ but somewhat smaller than a certain number m_1 determined from the condition that the $2B(r)$ and m^2/r^2 curves be tangent, there exist β for which equations (9) have real turning points. We consider first the case when each equation has one turning point, $r_t^{(+)}$ and $r_t^{(-)}$ (Fig. 1(b)). In analogy with the case $m=0$, we can obtain the following equation for the eigenvalues:

$$\int_{r_t^{(+)}}^{r_b}\left[B-\frac{m^2}{r^2}+(B^2-\beta^2)^{1/2}\right]^{1/2}dr-\int_{r_t^{(-)}}^{r_b}\left[B-\frac{m^2}{r^2}-(B^2-\beta^2)^{1/2}\right]^{1/2}dr=\frac{\pi}{2}+n\pi, \quad n=0,1,2,\ldots. \quad (23)$$

This case does not differ in principle from the case $m=0$, but yields the dependence of β_{max} on m. The geometrical determination of this dependence with allowance for $M\gg1$ reduces to finding the intersection point of the $B(r)$ and m^2/r^2 curves (Fig. 1(b)).

We turn now to the second possibility, when the equation in (9) for the function u has two real turning points (Figs. 1(c)). In the region behind the second turning point ($r>r_t^{(2)}$) we have $B-m^2/r^2<0$. Proceeding in analogy with the case $m=0$, we obtain the following two decreasing solutions

$$\exp\left\{i\int^r\left[B-\frac{m^2}{r^2}\pm i(B^2-\beta^2)^{1/2}\right]^{1/2}dr\right\}=\exp\left\{-\int^r\left[\left|B-\frac{m^2}{r^2}\right|\mp i(\beta^2-B^2)^{1/2}\right]^{1/2}dr\right\}. \quad (24)$$

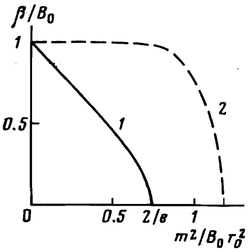

FIG. 2. Dependence of maximum growth rate of the perturbations on the azimuthal number.

As seen from (24), in this case solutions that decrease to the branch point will decrease also as $r\to\infty$. Since the equation in (9) for v has no turning points, it follows that $v(r)$ is a monotonic function of r and the solution that decreases with increasing r will diverge at $r=0$. Consequently, the only solution of (8) is a function of u (Eq. (9)) that decreases after the second turning point.

Thus, the presence of a branch point turns out to inessential here and the equation for the eigenvalues assumes the well known form (the Bohr–Sommerfeld quantization condition)

$$\int_{r_t^{(1)}}^{r_t^{(2)}}k_1\,dr=\int_{r_t^{(1)}}^{r_t^{(2)}}\left[B-\frac{m^2}{r^2}+(B^2-\beta^2)^{1/2}\right]^{1/2}dr=\frac{\pi}{2}+n\pi, \quad n=0,1,2,\ldots \quad (25)$$

Since the main interest attached to the maximum values of β, the most important case is $n=0$. Although in this case the WKB approximation is not suitable between the turning point, formula (25), as shown by Heading,[6] is valid. In view of the fact that $M\gg1$, the turning points $r_t^{(1)}$ and $r_t^{(2)}$ are quite close. Therefore the maximum value of β can be obtained geometrically (see Fig. 1(c)) from the condition of tangency of the curves m^2/r^2 and $B+(B^2-\beta^2)^{1/2}$. For a Gaussian beam, the result is shown in Fig. 2 (curve 1).

In the general case $m\neq0$, the maximum values of β correspond either to the case of two turning points or to the case of one turning and a branch point. The second case is typical of m smaller than a certain value m_0, and the first is typical of $m>m_0$. Estimates show that $m_0\sim M^{3/4}$.

Finally, at very large $m>m_1$, there are no real turning points for Eqs. (9). All the solutions of (8) are then monotonic and the boundary conditions at $r=0$ and $r=\infty$ cannot be satisfied simultaneously. Therefore there is no discrete spectrum. Geometrically, m_1 is determined from the condition that the curves $2B(r)$ and m^2/r^2 be tangent. For a Gaussian profile $m_1=(2/e)^{1/2}M$. For $m>m_1$ the perturbations do not increase with increasing z. The conclusion that the perturbations do not increase at large m is general for bounded laser beams, whereas for the unperturbed beam in the form of an unbounded plane wave there is no dependence of β on m. We note that the dispersion equations in the case (23) when one turning point exists for each of Eqs. (9) and when two turning points exist, the equations for u (25) go over into each other when the branch point coincides with the second turning point.

We consider now the case when the intensity profile has a maximum at $r=r_{max}\gg r_1$. We assume first that m is not large enough to bring the intersection point r_t of curves $B(r)$ and m^2/r^2 close to the maximum of $B(r)$ (Fig. 1(d)). In this case the mixing of the WKB approximations at the point $r_b^{(1)}$ raises the difficulties typical of the WKB method. Indeed, on going from $r>r_b^{(1)}$ to $r<r_b^{(1)}$ it becomes necessary to separate the decreasing solution against the background of the increasing one. If it is assumed that near the point $r=r_b^{(1)}$ the solution is well described by the Airy equation then, as shown by Heading, [6] we can use the following coupling formula:

$$A\exp\left[i\int\limits_{r_b^{(1)}}^{r}\frac{(B^2-\beta^2)^{1/2}}{2\alpha^{1/2}}dr\right]+B\exp\left[-i\int\limits_{r_b^{(1)}}^{r}\frac{(B^2-\beta^2)^{1/2}}{2\alpha^{1/2}}dr\right]$$
$$\to(A+iB)\exp\left[-\int\limits_{r_b^{(1)}}^{r}\frac{(\beta^2-B^2)^{1/2}}{2\alpha^{1/2}}dr\right]+\left[\frac{i}{2}(A-iB)+k(A+iB)\right]$$
$$\times\exp\left[\int\limits_{r_b^{(1)}}^{r}\frac{(\beta^2-B^2)^{1/2}}{2\alpha^{1/2}}dr\right],\qquad(26)$$

where k is a real and generally speaking interminate number.

Joining the solutions at the branch points $r_b^{(1,2)}$, we obtain

$$e^{-\Gamma}[{}^1/_4(1-\cos\Delta)+k^2(1+\cos\Delta+\sin\Delta)]=1+\cos\Delta,\qquad(27)$$

$$\Delta=\int\limits_{r_b^{(1)}}^{r_b^{(2)}}\{[\alpha+(B^2-\beta^2)^{1/2}]^{1/2}-[\alpha-(B^2-\beta^2)^{1/2}]^{1/2}\}dr,$$
$$\Gamma=4\,\mathrm{Im}\int\limits_{r_{(+)}}^{r_b^{(1)}}[\alpha+i(\beta^2-B^2)^{1/2}]^{1/2}dr,\qquad\alpha=B-\frac{m^2}{r^2}.\qquad(28)$$

The integration in the expression for Γ is carried out from the complex turning point $r_{(+)}$ to the first branch point.

In the situation of practical interest, when the point of intersection r_t is not too close to the branch point $r_b^{(1)}$, we have $\Gamma\gg1$ and the left-hand side of (27) can be neglected. To determine β we then have the equation

$$\int\limits_{r_b^{(1)}}^{r_b^{(2)}}\left\{\left[B-\frac{m^2}{r^2}+(B^2-\beta^2)\right]^{1/2}-\left[B-\frac{m^2}{r^2}-(B^2-\beta^2)^{1/2}\right]^{1/2}\right\}dr=\pi(1+2n)\qquad(29)$$

For the maximum growth rates of interest to us we have $B^2-\beta^2\ll\alpha^2$. Then Eq. (29) becomes simpler:

$$\int\limits_{r_b}^{r_b^{(2)}}\left[\frac{(B^2-\beta^2)}{(B-m^2/r^2)}\right]^{1/2}dr=\pi(1+2n).\qquad(30)$$

We note that at $m=0$ this condition can be obtained from (11). The latter describes correctly also the case of two close branch points. [6] By estimates similar to the foregoing ones it can be shown that the maximum values of β is very close to B_0. Therefore β_{max} does not depend on m at $m^2<B_0r_{max}^2$.

The form of the dispersion equation is significantly altered when the branch points $r_b^{(1)}$ and points r_t come closer together. Comparing the conditions for one (23) and for (29) branch points, we see that in the latter case the spacing between the spectral line is twice as large. When $r_b^{(1)}$ and r_t are closed, the form of the dispersion equation is quite complicated (see, e.g., (27)). These regions, however, are very narrow, and will therefore not be considered. Excluding correspondingly the region $m^2\approx B_0r_{max}^2$, we proceed to the case of large m (but $m<m_1$ where, as before, m_1 is determined from the condition that $2B(r)$ and m^2/r^2 be tangent). Equation (9) for u now has two turning points located between two branch points (Fig. 1(d)). As shown above, the decreasing solutions continue to decrease on going through the branch point. In this case, therefore, the dispersion equation coincides with (25). The dependence of β_{max} on m is shown qualitatively in Fig. 2 (curve 2).

3. MEDIUM WITH QUADRATIC NONLINEARITY

We investigate the decay of a pump beam of frequency ω_3, in a quadratic medium, into weak fields with frequencies ω_1 and ω_2 ($\omega_1+\omega_2=\omega_3$). We assume the initial notation and the system of parabolic equations used in the paper of Karamzin and Sukhorukhov, [16] confining ourselves to the given-field approximation ($\gamma_3=0$) and to the absence of wave detuning ($\Delta=0$):

$$\Delta_\perp A_1-2ik_1\partial A_1/\partial z+2k_1\gamma_1A_3A_2^*=0,$$
$$\Delta_\perp A_2^*+2ik_2\partial A_2^*/\partial z+2k_2\gamma_2A_3A_1=0.\qquad(31)$$

The analysis is similar to a considerable degree to that carried out above for a cubic medium. We assume that the pump wave has a plane phase front and is sufficiently broad (the characteristic transverse scales are $r_3\gg r_{1,2}$), so that at distances $z\ll r_3^2/\lambda$ the diffraction spreading of the pump beam can be neglected. Then the variables in Eqs. (31) for the weak-field amplitudes become separated:

$$A_1=\psi(r)e^{\alpha z+im\varphi},\quad A_2^*=\chi(r)e^{\alpha z+im\varphi};\qquad(32)$$

$$\psi''+r^{-1}\psi'-m^2r^{-2}\psi-i\beta_1\psi+B_1\chi=0,$$
$$\chi''+r^{-1}\chi'-m^2r^{-2}\chi-i\beta_2\chi+B_2\psi=0,\qquad(33)$$

$$\beta_p=2k_p\alpha,\quad B_p(r)=2k_p\gamma_pA_3(r),\quad p=1,2.\qquad(34)$$

The system (33) is similar to the system of the radial equations (6) in a medium with cubic nonlinearity. The quantity α has the meaning of an eigenvalue, while ψ and χ are the radial functions. For an unbounded plane pump wave $A_3=\mathrm{const}$ the spectrum is continuous, and ψ and χ are expressed in terms of a Bessel function of order m. We consider first the discrete spectrum of a bounded pump beam for the degenerate case $k_1=k_2$, $\gamma_1=\gamma_2$, so that $\beta_1=\beta_2\equiv\beta$, $B_1=B_2\equiv B$. We introduce the linear combinations

$$u=B\psi+[i\beta+(B^2-\beta^2)^{1/2}]\chi,\quad v=B\psi+[i\beta-(B^2-\beta^2)^{1/2}]\chi.\qquad(35)$$

In analogy with the conclusion (9) for a slowly varying pump-beam amplitude profile $B(r)$, we obtain

$$u''+r^{-1}u'-m^2r^{-2}u+(B^2-\beta^2)^{1/2}u=0,\qquad(36)$$
$$v''+r^{-1}v'-m^2r^{-2}v-(B^2-\beta^2)^{1/2}v=0.\qquad(37)$$

Equation (36) at $m\neq0$ and $\beta<B_0$ has two turning points, r_1 and r_2, so that the dispersion equation takes the usual form

$$\int_{r_1}^{r_2}[-m^2/r^2+(B^2-\beta^2)^{1/2}]^{1/2}\,dr=\frac{\pi}{2}+n\pi. \qquad (38)$$

This case is similar to the case of two turning points, which was considered above for a cubic medium. Asymptotically ($B_0r_0^2\gg 1$), the maximum perturbation growth rates for a given m correspond to tangency of the curves m^2/r^2 and $(B^2-\beta^2)^{1/2}$. The dependence of the maximum increment on m for a bell-shaped intensity profile of the pump beam agrees qualitatively with that given in Fig. 2. Equation (37) has solutions that are finite at $r=0$ and $r\to\infty$.

At $m=0$, Eq. (36) has only one turning point r_t, at which all four roots of the initial system (33) vanish. This case calls for a special analysis, which is given in Sec. 2 of the Appendix, where coupling formulas are given for a solution that decreases as $r\to\infty$. Those solutions of (37) which are finite at $r=0$ take the form (12) with Bessel functions of imaginary argument. Their asymptotic expression increases exponentially to the turning point r_t. Then, according to Appendix (2), such solutions of (37) increase also beyond the turning point. Therefore bound states can be obtained only from Eq. (36) for u. Using the asymptotic forms

$$u\sim\sin\left[\frac{\pi}{4}+\int_0^r(B^2-\beta^2)^{1/4}\,dr\right] \qquad (39)$$

and (9), we arrive at the dispersion equation

$$\int_0^{r_t}(B^2-\beta^2)^{1/4}\,dr=\frac{\pi}{2}+n\pi. \qquad (40)$$

In analogy with (20), we obtain an estimate for the number of discrete states:

$$n_{max}=\frac{1}{\pi}\int_0^\infty B^{1/2}(r)\,dr-\frac{1}{2}=\left(\frac{B_0r_0^2}{\pi}\right)^{1/2}-\frac{1}{2}. \qquad (41)$$

The last equality is valid for a Gaussian beam $B=B_0\times\exp\{-r^2/(2r_0^2)\}$.

In the nondegenerate case we have, in place of (36) and (37), equations with a complex potential

$$\left\{\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}+\left[-\frac{m^2}{r^2}+i\frac{\beta_2-\beta_1}{2}\pm\left(B_1B_2-\left(\frac{\beta_1+\beta_2}{2}\right)^2\right)^{1/2}\right]\right\}\binom{u}{v}=0. \qquad (42)$$

The deviation from the degenerate case will be regarded as small, assuming $k_1=k-\Delta k$, $k_2=k+\Delta k$, $\Delta k/k\ll 1$. Starting from a condition analogous to (38), we obtain in the lowest-order approximation the pure imaginary correction $\Delta\alpha$:

$$\frac{\Delta\alpha}{\alpha_0}\sim i\frac{\Delta k}{k}[C-1]^{-1/2},\quad C\sim\left[k\gamma\Big/\frac{d^2A_s^2}{dr^2}\Big|_{r=0}\right]^{1/4}A_s^2(0)\,m^{-1/2}. \qquad (43)$$

We note that the constant C should be large under conditions when the quasiclassical approximation is valid. Then, as follows from (43), the results of the analysis of the degenerate case will be valid even if $\Delta k/k$ is not too small.

According to (43) the correction decreases with decreasing m. However, the estimate (43) itself can be used only under the additional condition $\alpha_0\Delta k\ll m^2/\bar{r}^2$, where the mean value $\bar{r}$ is determined from the indicated tangency conditions. The case of small m is nevertheless of considerable interest, since it corresponds to maximum growth rates. It can be shown that in the limiting case $m=0$ the maximum eigenvalue for given k_1 and k_2 is

$$\alpha=(\gamma_1\gamma_2)^{1/2}A_s(0). \qquad (44)$$

Since $\gamma_1+\gamma_2=\text{const}$, [16] it follows from (43) that the growth rate is maximal in the degenerate case, in agreement with the analysis of the unbounded pump beam. [5] Just as for media with cubic nonlinearity, the fastest to grow are the axially-symmetrical ($m=0$) perturbations that propagate in the region of the maximum beam intensity.

CONCLUSION

Thus, the main feature of high-power bounded beams propagating in nonlinear media is the presence of a discrete spectrum of perturbations whose growth can be interpreted as the decay of the initial pump beam. With increasing beam width, the number of such discrete states increases, and the spectrum itself becomes denser, and it is in this manner that transition to the case of an unbounded pump beam is attained.

Let us trace this transition for a cubic medium in the case of axially-symmetrical perturbations ($m=0$). The eigenfunctions of the discrete spectrum are high-frequency oscillations with a characteristic period r_1, filling the region near the beam axis $\sim r_b\gg r_1$. The period r_1 (22) for the "lowest" discrete levels corresponds to the characteristic transverse scale of the fastest-growing perturbations for an unbounded plane wave. [1] At the same time, the scale r_b is possessed precisely by bounded beams. If r_1 determines the distance between the filaments into which the laser beam breaks up, then r_b can be connected with the dimension of the "filament-formation region." With increasing beam width ($r_0\to\infty$) we get in accordance with (22) r_b.

For bounded beams, only the discrete spectrum corresponds to positive growth increments. A discrete spectrum exists when the beam power exceeds a certain critical value, the estimate of which can be obtained from (20) by putting $n_{max}=1$. The critical value agrees with that obtained by Lyakhov. [2] Under conditions typical of high-power solid-state amplifiers (nonlinearity $n_2\sim 2.5\times10^{-13}$ cgs esu, light-energy density ~ 1 J/cm^2 at a pulse duration 10^{-10} sec, beam radius ~ 1 cm) the basic parameter of our paper is $B_0r_0^2\sim 10^4\gg 1$. The number of discrete states is then $n_{max}=56$.

Owing to the distortions of the pump beam as a whole (characteristic length L_s), the description in the language of eigenfunctions is approximate. The approximation is justified precisely for high-power beams, where the characteristic growth length of the perturbations is $l_s\ll L_s$, and is valid in the region $z\ll L_s$. We note that in contrast to these conditions, the procedure employed in [2], based on expanding the solutions of (1) in a Taylor series, is justified only in the immediate vicinity of the boundary of the nonlinear medium $z=0$. Thus, perturbations whose amplitude decrease at small z can increase at larger distances $\sim l_s$. Such a situation is typical of perturbations whose scale is noticeably smaller than r_1. Physically this means that these perturbations are initially spread out, on account of diffraction, to optimal dimensions, after which they begin to increase (see, e.g., curve 5 on Fig. 4 of[3]). Therefore the concept of "minimum width of growing perturbations"[2] does not have a clear-cut physical meaning.

For a medium with quadratic nonlinearity, the estimate that follows from (41) for the critical power required for the existence of a discrete spectrum, and the characteristic dimension r_t of the lower oscillation mode of the perturbation, agree with the quantities $S_{cr\,2}$ and a_p introduced by Karamzin and Sukhorukov.[16] We note that for a laser beam power greatly exceeding the critical value, it is necessary to take into account also higher discrete levels.

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APPENDIX

1. Derivation of Eq. (4)

After substituting (3) in (2) we obtain

$$2ik\frac{\partial E_1}{\partial z}=\frac{\partial^2 E_1}{\partial r^2}+\frac{1}{r}\frac{\partial E_1}{\partial r}+\frac{1}{r^2}\frac{\partial^2 E_1}{\partial \varphi^2}+B(E_1+E_1^*)+R, \tag{A.1}$$

$$R=-i\frac{1}{k}z\frac{dB}{dr}\frac{\partial E_1}{\partial r}-i\frac{z}{2k}\frac{d^2B}{dr^2}E_1-\frac{z^2}{4k^2}\left(\frac{dB}{dr}\right)^2E_1-i\frac{z}{2kr}\frac{dB}{dr}E_1. \tag{A.2}$$

We shall show now that under the indicated conditions the quantity R in (A.1) can be neglected. We introduce to this end the characteristic parameters of the diffraction length (L_d, l_d) and the soft-focusing lengths (L_s, l_s) of the unperturbed beam and of the perturbation:

$$L_d=r_0^2/\lambda,\quad l_d=r_1^2/\lambda,\quad L_s=r_0/\lambda B_0^{1/2}=r_0^2/\lambda M,$$
$$l_s=1/\lambda B_0=r_0^2/\lambda M^2,\quad r_1\sim B_0^{-1/2}=r_0/M.$$

On the basis of the estimates

$$\frac{dB}{dr}\sim\frac{B_0}{r_0},\quad \frac{d^2B}{dr^2}\sim\frac{B_0}{r_0^2},\quad \frac{\partial E_1}{\partial r}\sim\frac{E_1}{r_1}$$

we obtain from a comparison of the different terms of (A.1) and (A.2) that in order for (4) to be valid we must have

$$z\ll L_s. \tag{A.3}$$

Since $L_s \ll L_d$ at $M\gg 1$, the diffraction spreading of the unperturbed beam can also be neglected when the condition (A.3) is satisfied.

2. Coupling formulas in the degenerate case

The fourth-order equation corresponding to (33) takes in the vicinity of the turning point the form ($x=r-r_t$, $m=0$)

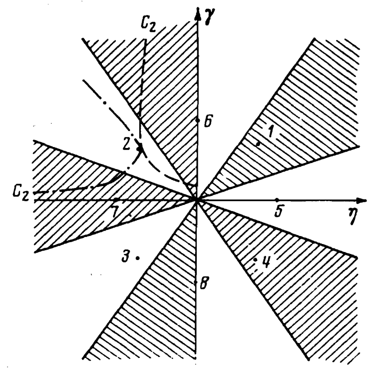

FIG. 3.

$$y^{(4)}+2\beta|B'|xy=0. \tag{A.4}$$

We obtain coupling formulas for a solution of (A.4) that decreases as $x\to+\infty$ and oscillates as $x\to-\infty$. We note that the formal solution of (A.4), obtained with the aid of a Laplace transform, is given in[17]. It is more convenient for us to use the method of Laplace contour integrals,[18] according to which

$$y=\text{const}\cdot\int_C \exp\left[xt+\frac{t^5}{10\beta|B'|}\right]dt. \tag{A.5}$$

The integration contour C in the complex variable plane must be chosen such that the values of the integrand on its ends are equal. Since the integrand has no singularities at finite t, the asymptotic form of the solutions at large $|x|$ can be determined by using the saddle-point method. Figure 3 indicates the directions (shaded sectors) in which the integrand tends to zero as $|t|\to\infty$. The figure shows also the saddle points t_q at $x\to+\infty$ ($q=1, 2, 3, 4$) and at $x\to-\infty$ ($q=5, 6, 7, 8$).

The solutions that decrease as $x\to+\infty$ correspond to the points t_2 and t_3. The equation for the steepest-descent lines are

$$\operatorname{Im}\varphi=\text{const}=A \tag{A.6}$$

or

$$\eta^2=\gamma^2\pm[{}^4/_5\gamma^4+A/5c\gamma-x/5c]^{1/2}. \tag{A.7}$$

Here $t=\eta+i\gamma$, $c=1/10\beta|B'|$, and φ is the argument of the exponential in (A.5). Consider, for example, a concontour passing through the point t_2 (Fig. 3). At this point, the radicand in (A.7) vanishes, whence

$$A=2^{-1/2}W,\quad W={}^1/_5 2^{9/4}(\beta|B'|)^{1/4}|x|^{5/4}.$$

Figure 3 shows two contours passing through t_2 and touching each other at this point; these contours correspond to the two signs of the root in (A.7). The integration contour C_2 is chosen in the form of two branches that go off to infinity in the shaded regions. Analogously, in the third quadrant we choose the contour C_3, which is obtained by specular reflection of C_2 about the axis $\gamma=0$.

As $x\to-\infty$, the same shaded regions contain the saddle points t_6, t_7, and t_8, the point t_7 corresponding to increasing solutions. It is therefore necessary to take a linear combination of the solutions, which corresponds to successive integration along the contours C_2 and C_3. Deforming these contours and taking into account the analyticity of the integrand, we find that the integration can be carried out along the axis $\eta = 0$, and the main contribution to the integral is made by the vicinities of the points t_6 and t_8. Accurate to constant factors we obtain as $x \to -\infty$

$$y \sim t_6^{-1/4} e^{-iW} + t_8^{-1/4} e^{iW} \sim \cos(-\pi/4 + W). \quad (A.8)$$

For a solution that decreases as $x \to -\infty$, we must construct contour that passes through the point t_5. As $x \to +\infty$, the main contribution to the integration along such a contour is made by the vicinities of the points t_1 and t_4, which lead to increasing (as $x \to +\infty$) solutions.

1)Generally speaking, it will be possible to conclude from (9) that the WKB solution takes the form

$$\exp\left[\pm i \int^r (k_p^2 + 1/4r^2)^{1/2} dr\right].$$

As shown by Langer[14] (see, e.g., [6]), a consistent allowance for the singularity in the vicinity of $r = 0$ leads to the expressions.

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Experimental investigation of spontaneous emission by neon in the presence of a strong monochromatic field

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Quantitative measurements of the spontaneous emission spectra of atoms in the presence of a laser field are performed. Spontaneous emission is recorded for the $3s_2$-$2p_4$ (0.6328 μ) and $3s_2$ -$2p_{10}$ (0.5434 μ) transitions in neon. The laser field is at resonance with the $3s_2$–$2p_4$ transition. The line broadening and shifts due to atom-atom and atom-electron collisions are measured. The relative amplitude of the broad spectral component of the nonlinear resonance is determined. The experimental results are used to verify the nonlinear-resonance theory.

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1. INTRODUCTION

The first experimental[1–4] and theoretical[5–8] studies of the spectroscopy of spontaneous emission of a gas in the presence of a monochromatic laser field were published relatively long ago. These papers were devoted to the modification of the spontaneous-emission spectrum by the laser field as a result of saturation effects, field splitting, and nonlinear interference effects. These changes manifest themselves as relatively sharp spectral structures (resonances) superimposed on inhomogeneously broadened lines. The shapes of the resonances depend, generally speaking, on the observation direction, while their widths are as a rule much smaller than the Doppler width.

Field-induced changes in the spectra can be registered by measuring the absorption of a probing laser field, or by observing the spontaneous emission from the perturbed levels. The difficulties connected with the low brightness of the spontaneous emission and with the need for using high-resolution spectral apparatus hinder the progress of research on spontaneous emission. The available data[1,3] are by way of qualitative reports of the phenomena and cannot claim to offer a quantitative check on the theory. At the same time, the