[3)]If the number of layers $N_\perp$ is large, then the cancellation of the linear terms is complete at $qb \ll 1$ (q is the momentum perpendicular to the layers and b is the distance between the layers). On the other hand if $N_\perp \sim 2$ or 3, then there is no complete cancellation and the dipole forces should stabilize the long-range order, but an investigation of this question is beyond the scope of the present article.

[4)]Thus, the magnetic interaction between two uniformly magnetized closely-located monatomic layers with sides l and m along the axes x and y is described by the formula

$$H_d=2(g\mu)^2(v_2^2 d)^{-1}\{(m^2+ld)S_x^{(1)}S_x^{(2)}+(l^2+md)S_y^{(1)}S_y^{(2)}-d(l+m+d)S_z^{(1)}S_z^{(2)}\},$$

where $d^2=l^2+m^2$, v_2 is the "volume" of the planar unit cell, and $\mathbf{S}^{(1,2)}$ is the average spin of the atom in the layer.

[5)]It is interesting to note that if we consider the region of the crystal near the boundary, then the summation over ρ has a finite limit on one side, and as a result the linear terms are not completely cancelled out in the expression for Q_p. The terms linear in $p_\parallel$ also remain if $p_\perp \sim a_3^{-1}$.

[1]L. D. Landau and E. M. Lifshitz, Statisticheskaya fiziak (Statistical Physics), Nauka, 1964 [Pergamon, 1971].

[2]A. Z. Patashinskiĭ and V. L. Pokrovskiĭ, Fluktuatsionnaya teoriya fazovykh perekhodov (Fluctuation Theory of Phase Transition), Nauka, 1975.

[3]A. I. Akhiezer, V. G. Bar'yakhtar, and S. V. Peletminskiĭ, Spinovye volny (Spin Waves), Nauka, 1967 [Pergamon, 1971].

[4]V. L. Berezinskii and A. Ya. Blank, Zh. Eksp. Teor. Fiz. **64**, 725 (1973) [Sov. Phys. JETP **37**, 369 (1973)].

[5]H. E. Stanley and T. A. Kaplan, Phys. Rev. Lett. **17**, 913 (1966).

[6]L. J. de Jongh and A. R. Miedema, Adv. Phys. **23**, 1 (1974).

[7]J. H. P. Colpa, Physica **56**, 185 (1971).

[8]J. H. P. Colpa, *ibid.* 205,

[9]H. Benson and D. L. Mills, Phys. Rev. **178**, 839 (1969).

[10]I. S. Gradshteĭn and I. M. Ryzhik, Tablitsy integralov, summ, ryadov i proizvedeniĭ (Tables of Integrals, Sums, Series, and Products), Fizmatgiz, 1962 [Academic, 1966].

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Contribution to the theory of liquid ^{3}He

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We show that the anomalous properties of liquid ^{3}He can be explained by assuming that the fluctuation spectrum of its spin density has a deep roton minimum. This means that ^{3}He is close to a phase transition to an antiferromagnetic state. Comparison between theory and experiment confirms this assumption. We find the spin-roton parameters: $\Delta = 0.09$ K, $k_0 = 0.7\ p_F$, $M = 0.06\ m$. We determine the temperature dependence of the specific heat up to $T \approx 1.5$ K. For $0.05 < T < 0.5$ K the main term in the specific heat is $\propto \sqrt{T}$. A temperature of $T \approx 0.5$ K has the meaning of the degeneracy temperature of the spin-roton gas. We find the quasi-particle spectrum and the Landau Fermi-liquid theory parameters. We determine the wavevector dependence of the magnetic susceptibility χ. At $k = k_0$ the value of χ is 50 times larger than the susceptibility of a perfect Fermi gas of the same density.

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1. PHYSICAL PICTURE

Liquid ^{3}He can be satisfactorily described by Landau theory only for $T < 0.1$ K. At $T > 0.1$ K the specific heat, viscosity, and other physical characteristics of ^{3}He have a different order of magnitude than the values predicted by Landau and Pomeranchuk.[1–3] The strong difference between the properties of ^{3}He and those of a gas of quasi-particles can be explained if we assume that the liquid is close to a phase transition. Four types of instability are possible in a Fermi liquid which are connected with the two forms of its excitations—zero and spin sound. The first kind of instability is connected with long-wavelength density fluctuations. This instability arises when the velocity of the virtual zero sound is much smaller than the quasi-particle velocity on the Fermi surface. It is clear that such an instability can not be realized in ^{3}He as real zero sound can propagate in it, which has been observed experimentally.

The second type of instability is connected with short-wavelength zero sound which has a roton gap Δ for $k \approx 2p_F$. Such an instability is destroyed when the liquid goes over into the solid state. As the zero sound spectrum is unknown for $k \approx 2p_F$ it is necessary to consider the theoretical arguments for and against the existence of soft zero sound rotons in ^{3}He. The arguments for such a possibility are based upon the fact that ^{3}He under pressure becomes a solid. Moreover, in liquid ^{4}He, which differs from ^{3}He only by the statistics of the particles and an unimportant difference in the mass of the atoms, there are sound excitations with a roton gap. The argument against consists of the fact that a phase transition into the solid state is always a first order one and it seems doubtful that it takes place when $\Delta \ll \varepsilon_F$, when the rotons strongly affect the properties of the liquid phase. In particular, ^{4}He which in the solid state is very much like ^{3}He undergoes a transition into the solid state before the roton gap decreases so much that it becomes necessary to take the effect of the rotons on the nature of the transition into account. If, nevertheless, it turns out that when the density changes Δ becomes much less than ε_F prior to the occurrence of the phase transition, the exchange scattering amplitude of the quasi-particles Γ must have a positive sign. This statement follows from the symmetry properties of the amplitude, according to which a strong attraction connected with the exchange of a soft roton in the direct channel leads to a strong repulsion in the exchange channel. Experiments indicate the opposite: the scattering amplitude in the exchange channel is strongly attractive. Therefore, if there is in ^{3}He also a zero-sound excitation with a roton gap Δ, then $\Delta \sim \varepsilon_F$, and not $\Delta \ll \varepsilon_F$.

The third type of instability means that the liquid is close to the ferromagnetic phase transition and it is connected with the long-wavelength spin density fluctuations. As the static magnetic susceptibility of ^{3}He is an order of magnitude larger than the susceptibility of a perfect gas of the same density it would be natural that just this type of instability can be realized in ^{3}He. However, although there is a strong argument in favor of this possibility, it is logically contradictory. Indeed, there is an exact relation between the magnetic susceptibility χ and the specific heat $C(T)$[2]:

$$\chi/\chi_0=(1+Z_0)^{-1}m^*/m;\quad C(T)={}^1/_3 p_F m^* T,$$

as $T\to 0$. This relation differs from the usual gas formula by the factor $(1+Z_0)^{-1}$ which characterizes the exchange amplification of the quasi-particle scattering amplitude. In an almost ferromagnetic liquid Z_0 is close to -1 and the quantity $(1+Z_0)^{-1}$ must be very sensitive to a change in the ^{3}He parameters, e.g., its density. Experimentally, however, $1+Z_0$ hardly changes when the density varies from 0 to 29 atm. Yet another argument against this kind of instability is based upon the fact that the long-wavelength fluctuations have a small phase volume and affect the properties of the liquid when not only $1+Z_0\ll 1$, but also $\ln(1+Z_0)^{-1}\gg 1$. For ^{3}He we have $1+Z_0\approx 0.3$ and we could not explain the anomalous properties of ^{3}He when we have such a "weak" small parameter.

The fourth type of instability is connected with the short-wavelength spin density fluctuations and means that the liquid is close to an antiferromagnetic phase transition. The magnetic susceptibility of such a liquid has a maximum at $k=k_0$, where $k_0\sim p_F$, while the spin fluctuation spectrum has a roton gap $\Delta\ll\varepsilon_F$. In the present paper we construct a quantitative theory of liquid ^{3}He based upon the assumption that just this instability is realized in ^{3}He.

We describe the spin fluctuation spectrum for $k\approx k_0$ by three parameters: Δ, k_0, and an effective roton mass M, and assume that $\Delta\ll\varepsilon_F$. Two of these parameters are determined from the requirement that for $T\ll\Delta$ our theory must go over into the Landau theory which is exact as $T\to 0$. The third parameter is determined by comparing the theoretical and experimental temperature dependence of the specific heat for $T\sim\Delta$.

Once the parameters have been determined the theory allows us to find: the specific heat for $T<1.5$ K, the unknown Landau-theory parameters, the quasi-partical spectrum, the magnetic-susceptibility dispersion for $k\approx k_0$, and so on. Moreover, the theory shows a connection between the various physical characteristics of ^{3}He. For instance, the specific heat for $T>0.5$ K is connected with the sound speed and with the jump in the Fermi occupation of the particles. Using these connections we can find the phenomenological parameters Δ, k_0, and M from three experimental points on the $C(T)$ curve and afterwards establish the whole $C=C(T)$ curve, determine the sound velocity, the magnetic susceptibility, and many other quantities. We emphasize that in such an approach the requirement that our theory agrees with Landau's theory is automatically satisfied.

The theory which is exact as $\Delta\to 0$ is based upon the fact that the contribution of the spin fluctuations to the self-energy part Σ, to the scattering amplitude Γ, and to the specific heat C is a non-analytical function of Δ and when $\Delta\ll\varepsilon_F$ it is larger than the contribution from the other degrees of freedom, such as the total density fluctuations. As $\Delta\to 0$, the self-energy $\Sigma(\varepsilon)$ has a square-root singularity at $\varepsilon=0$: $\Sigma\propto\sqrt{\varepsilon}$. This leads to a strong amplification of the effective mass $m^*\gg m$ and a small jump in the Fermi occupation of the particles: $a\ll 1$. When $\Delta=0$ there is, in general, no Fermi surface and the quasi-particle pole in the particle Green function becomes pure imaginary and corresponds to a single-particle virtual excitation with a dispersion law which is quadratic in $p-p_F$. The specific heat also has a singularity at $T=0$ as $\Delta\to 0$: $C\propto\sqrt{T}$, which limits the applicability of the Landau theory to the region $T<\Delta$. In the quasi-particle scattering amplitude Γ the exchange of a single roton leads as $\Delta\to 0$ to a δ-function-like dependence of Γ on the scattering angle, and the exchange of two rotons to a threshold singularity.

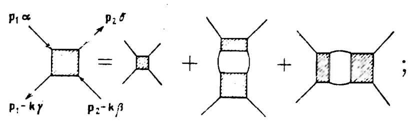
The physical reason for such a strong effect of the rotons on the properties of the liquid is due to the fact that spin fluctuations for $k_0\neq 0$ have a large phase volume. As $\Delta\to 0$ the whole region of the spectrum near the roton minimum is equivalent to a single state with $k=k_0$ and $\omega=\Delta$ and a quantum-mechanical situation is realized with two close-lying levels: the "quasi-particle" and the "quasi-particle + roton" states combine strongly. It is very natural that such a situation can be described exactly as one can neglect transitions to the remaining states.

We make more exact the term "theory which is exact as $\Delta\to 0$." The fact is that the transition to an antiferromagnetic state is a first order one, i.e., it proceeds when $\Delta\neq 0$ and it is not clear that the condition $\varepsilon_F\gg\Delta$ is satisfied. Quantitatively this problem can be considered only for the simplest models which show that one can allow $\Delta>\Delta_c$, where $\Delta\sim\varepsilon_F(k_0/2p_F)^9$. When $\Delta>\Delta_c$ the strong roton-roton interaction begins to play a role and this increases as $\Delta^{-1/2}$ when Δ decreases. For ^{3}He the ratio $\Delta/\varepsilon_F\sim 0.02$ and $(k_0/2p_F)^9\sim 10^{-4}$ so that we have grounds for neglecting the anharmonicity of the spin rotons.

The impossibility to determine exactly the region of applicability is characteristic of any phenomenological theory. For instance, as $T\to 0$ the Landau theory ceases to operate for $T \sim T_c$, where T_c is the temperature of the transition to the superfluid state. The Landau theory does not predict the quantity T_c and in principle a region $T_c \ll T \ll \epsilon_F$ might not exist. In our case we shall assume that in ^{3}He the condition $\Delta_c \ll \Delta \ll \epsilon_F$ is satisfied. In that case, the rotons affect, on the one hand, the properties of the liquid strongly and it is necessary to take them exactly into account while, on the other hand, effects connected with their interaction are still small. Apparently when the pressure increases ^{3}He solidifies before Δ reaches the dangerous value Δ_c. In the present paper we restrict our considerations to ^{3}He at a low pressure of $P = 0.28$ atm.

2. PARAMETRIZATION OF THE SCATTERING AMPLITUDE

We consider the static quasi-particle scattering amplitude $\Gamma(\mathbf{k}, \mathbf{p}_1, \mathbf{p}_2)$ for $p_1, p_2 \sim k_F$ and $k < 2p_F$. From Landau's equations for Γ it follows that there are two particle-hole channels for it, with a momentum transfer $\mathbf{k}$ for one of them and $\mathbf{p}_1 - \mathbf{p}_2$ for the other one.



$$\Gamma = \Gamma_0 + \Gamma_1 + \Gamma_2. \quad (1)$$

It follows from (1) that Γ_2 depends strongly on $\mathbf{k}$ and Γ_1 on $\mathbf{p}_1 - \mathbf{p}_2$, and in first approximation we may assume that Γ_2 is a function of k only and Γ_1 of $\mathbf{p}_1 - \mathbf{p}_2$ only. Indeed, the k-dependence of Γ_2 is determined by ladder diagrams for which the momentum $\mathbf{k}$ is transferred in the direction of the minimum number of integrations over the intermediate momentum, while the dependence of Γ_2 on $\mathbf{p}_1 - \mathbf{p}_2$ is connected with a large number of integrations and is thus weaker than the k-dependence. This fact is just the basis of Landau's theory. No graph contributing to Γ can depend strongly both on $\mathbf{k}$ and on $\mathbf{p}_1 - \mathbf{p}_2$ so that it is possible to separate the singularities in each channel.

We have thus an approximate expression for Γ:

$$\Gamma = \Gamma_0 + \Gamma_1(\mathbf{p}_1 - \mathbf{p}_2) + \Gamma_2(\mathbf{k}),$$

where Γ_0 is a slowly varying function of $\mathbf{k}$ and $\mathbf{p}_1 - \mathbf{p}_2$. We shall assume that Γ_2 has a steep maximum at $k = k_0$ which corresponds to the exchange of a spin roton. We introduce the dependence of Γ_2 on $\mathbf{k}$ and Γ_1 on $\mathbf{p}_1 - \mathbf{p}_2$ taking into account that Γ changes sign under an exchange of particles:

$$\nu\Gamma_{\alpha,\beta,\gamma,\delta} = \nu\Gamma_0 + \boldsymbol{\sigma}_{1\alpha\delta}\boldsymbol{\sigma}_{2\beta\gamma} D(\mathbf{p}_1 - \mathbf{p}_2) - \boldsymbol{\sigma}_{1\alpha\gamma}\boldsymbol{\sigma}_{2\beta\delta} D(\mathbf{k}). \quad (2)$$

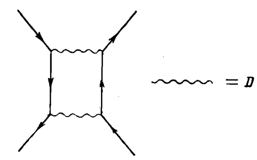
Here D has the meaning of the paramagnon distribution function in the static limit

$$D^{-1}(k^2) = \xi^2 + \gamma^2(k^2/k_0^2 - 1)^2 + \eta^2(k^2/k_0^2 - 1)^3 + \ldots, \quad (3)$$

$\nu = a^2 p_F^2/\pi^2 V$ is the density of quasi-particle states. As $k \to 0$ the quantity Γ becomes Γ^k of Landau's theory. The parameter ξ^2 in (3) is a measure of how close the liquid is to the phase transition to the antiferromagnetic state; it will be connected with the roton gap in the spin fluctuation spectrum. The coefficients γ^2 and η^2 in (3) determine the magnetic-susceptibility dispersion.

Writing Γ in the form (2) corresponds to the usual division of the scattering amplitude into a potential and a resonant part which is used in quantum mechanics[4]; Γ_0 corresponds to the potential and D to the resonance scattering. The next approximation consists in neglecting Γ_0. This approximation is justified by the smallness of the factor ν in (2). Below we determine the jump a and show that ν is two orders of magnitude smaller than ν_0 of a perfect gas at the same density. In the resonance part of Γ this smallness is compensated by the closeness to the phase transition point.

As $\xi^2 \to 0$ there occurs in Γ_0 a singularity of the kind $\nu\Gamma_0 \sim (k_0/2p_F)^4/\xi\gamma$ which is weaker than the singularity in D and contains an additional small factor $(k_0/2p_F)^4$. This singularity is connected with the anharmonicity of the rotons and arises when we take into account graphs of the form:



(4)

The small factor $(k_0/2p_F)^4$ determines the magnitude of the roton phase volume. When $\xi\gamma > (k_0/2p_F)^4$ the neglect of $\nu\Gamma_0$ is thus justified not only as compared to D, but also as compared to terms ~ 1.

Writing Γ in the form (2) is the usual Hartree–Fock approximation for Γ but the expansion is not in terms of the bare interaction between the ^{3}He atoms, but in the parameter ξ^2 which characterizes the closeness to the transition point. We can improve this approximation by evaluating graphs such as (4). It turns out that it is more important to express the zeroth approximation for D correctly, i.e., to take into account the next terms in the expansion of D^{-1} in powers of $k^2/k_0^2 - 1$. This is connected with the fact that they give a correction to the small quantity ξ^2, while the graphs (4) correct quantities of order unity.

We note that the graphs (4), which are small when $\xi\gamma > (k_0/2p_F)^4$, change the character of the phase transition to the antiferromagnetic state from second to first order. This statement is proved in [5], where I have considered a similar situation which arises during the condensation of mesons in nuclear matter. For ^{3}He the strong inequality $\xi\gamma \gg (k_0/2p_F)^4$ is satisfied so that we shall neglect the contribution from graphs such as (4).

3. CONNECTION WITH LANDAU'S THEORY

In the limit $k \to 0$ and $p_1 = p_2 = p_F$ the scattering amplitude Γ depends solely on the angle between $\mathbf{p}_1$ and $\mathbf{p}_2$ and can be written in the form

$$\nu\Gamma^k = A(x) - \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2(^1/_3 A(x) + D(0)), \quad x = \mathbf{p}_1\mathbf{p}_2/p_F^2. \quad (5)$$

The function $A(x)$ is connected with D and has a Lorentzian form:

$$A(x)=\frac{3}{2}D(k^2=2p_F^2(1-x))\approx\frac{2}{\pi}A_{00}\frac{\varkappa}{\varkappa^2+(x-x_0)^2}. \quad (6)$$

The quantities $\varkappa$, A_{00}, and x_0 are connected with ξ, γ, and k_0:

$$\varkappa=\frac{\xi k_0^2}{2p_F^2\gamma},\quad A_{00}=\frac{3k_0^2\pi}{8p_F^2\xi\gamma},\quad x_0=1-\frac{k_0^2}{2p_F^2}=\cos\theta_0. \quad (7)$$

According to the Landau theory the physical characteristics of a Fermi liquid can at $T=0$ be expressed in terms of the coefficients of the expansion of Γ^ω in spherical harmonics:

$$\nu\Gamma^\omega=\Phi(x)+\sigma_1\sigma_2Z(x),$$
$$\Phi(x)=\Sigma\Phi_lP_l(x),\quad Z(x)=\Sigma Z_lP_l(x).$$

The quantity Γ^k can also be expanded in the P_l functions:

$$\nu\Gamma^k=A(x)+\sigma_1\sigma_2B(x),$$
$$A(x)=\Sigma A_lP_l(x),\quad B(x)=\Sigma B_lP_l(x).$$

In our case $A(x)$ is determined by Eq. (6) and $B(x)=-\frac{1}{3}A(x)-D(0)$. There exists a connection between Φ_l, Z_l and A_l, B_l which was found by Landau:

$$A_l=\Phi_l\left(1+\frac{\Phi_l}{2l+1}\right)^{-1},\quad B_l=Z_l\left(1+\frac{Z_l}{2l+1}\right)^{-1}. \quad (8)$$

The first two harmonics of Φ are well known from experimental data on the sound velocity and the specific heat: $\Phi_0=10.77$; $\Phi_1=6.25$. We show that the large magnitude of Φ_0 and Φ_1 follows from the theory. In the next section of this paper we shall express the auxiliary quantities ξ and γ in terms of the "physical" parameters Δ and M (see (26)). There follows thus from (26) and (7) a connection between $\varkappa$, A_{00} and Δ, M:

$$\varkappa=2\frac{k_0^2}{p_F^2}\left(\frac{\Delta M}{2k_0^2}\right)^{1/2},\quad A_{00}=\frac{3}{1+\Phi_1/3}\frac{k_0^2}{p_F^2}\left(\frac{\varepsilon_FM}{\Delta m}\right)^{1/2}. \quad (9)$$

In deriving (9) we used the relation $m^*/m=1+\Phi_1/3$. We determine the first harmonic Φ_1 from (6), (8). Up to terms $\propto\varkappa$ the harmonic Φ_1 has the form

$$\Phi_1=9x_0\frac{k_0^2}{p_F^2}\left(\frac{\varepsilon_FM}{\Delta m}\right)^{1/2}.$$

It follows from this relation that Φ_1 increases as $\Delta^{-1/2}$ as $\Delta\to0$. To show that also $\Phi_0\gg1$ as $\Delta\to0$ we turn to the relation between the sound speed and Φ_0 or Φ_1[2]:

$$c^2=\frac{p_F^2}{3m^2}\frac{1+\Phi_0}{1+\Phi_1/3}.$$

We shall show below that the sound speed is an analytic function of Δ (see (37)) so that it follows from the relation $\Phi_1\propto\Delta^{-1/2}$ as $\Delta\to0$ that also $\Phi_0\propto\Delta^{-1/2}$. We emphasize that the large magnitude of Φ_0 and Φ_1 is a direct indication that ^{3}He is close to a phase transition. In a normal Fermi liquid Φ_0, $\Phi_1\sim1$. The sound speed in ^{3}He which differs only by a factor two from its value for a non-interacting Fermi gas is in the Landau theory expressed in terms of the anomalously large quantities Φ_0 and Φ_1. The analytical dependence of c on Δ is thus the result of the cancelling of the large quantities Φ_0 and Φ_1.

To determine the parameters ξ, γ, and k_0 from a comparison with experiments we need three conditions. Two conditions follow from the fact that the first two harmonics in $A(x)$ calculated from Eq. (6) and their experimental values should be the same. The third condition arises from the requirement that the calculated and experimental T-dependence of the specific heat at $T\sim\Delta$ should be the same, (see (57)). Using the calculated values of Φ_0 and Φ_1 we get from (6), (8), and (57)

$$k_0\approx0.7p_F,\quad x_0\approx0.74,\quad \theta_0\approx42°,$$
$$\xi^2\approx0.063,\quad \gamma^2\approx7.5,\quad \varkappa\approx0.025.$$

Since $\varkappa\ll1$, we shall put $\varkappa=0$ in all quantities which are insensitive to the exact values of ξ, γ, and k_0: $A(x)=A_0\delta(x-x_0)$.

Because of the δ-function-like x-dependence of A the expansion of Φ and Z in spherical harmonics will converge slowly. The number of harmonics which we must take into account is $\propto\varkappa^{-1}\approx40$. The magnitude of each harmonic in Φ and Z, taken separately, is therefore of no interest. We separate from Φ and Z the part which depends strongly on x and which is the result of the coherent addition of different harmonics, while we expand their slow part in a fast converging series in the P_l. We turn to the equation which connects A with Φ.[2]

$$\Phi(\mathbf{n}_1,\mathbf{n}_2)=A(\mathbf{n}_1,\mathbf{n}_2)+\int A(\mathbf{n}_1,\mathbf{n}_3)\Phi(\mathbf{n}_3,\mathbf{n}_2)\frac{d\Omega_{\mathbf{n}_3}}{4\pi}\equiv A+(A\Phi). \quad (10)$$

We look for Φ in the form $\Phi=A+(AA)+\tilde{\Phi}$. The function $\tilde{\Phi}$ is a solution of the equation

$$\tilde{\Phi}=((AA)A)+(A\tilde{\Phi}). \quad (11)$$

We have separated from Φ the contribution connected with the exchange of one or two rotons. We note that taking the two-roton exchange into account in Γ^ω we do not exceed the accuracy when we neglected them in Γ^k. The two-roton graph in Γ^k contains an integration over d^4q, and in Γ^ω over $d\Omega$. The two-roton contribution to Γ^ω does therefore not contain the small parameter $(k_0/2p_F)^4\approx0.02$. We consider the quantity (AA) as $\varkappa\to0$, when up to terms $\propto\varkappa$ the quantity $A=2A_0\delta(x-x_0)$:

$$(AA)=\frac{2A_0^2}{\pi}\frac{\theta(1+x-2x_0^2)}{[(1-x)(1+x-2x_0^2)]^{1/2}}. \quad (12)$$

The square root singularity of (AA) for $x=1$, i.e., for $\theta=0$, is connected with the fact that as $x\to1$ two δ-function-like maxima in (AA) merge and for $x=1$ the square of a δ-function is integrated.

The singularity at $x=2x_0^2-1$, i.e., at $\theta=2\theta_0$, has a threshold. When $\theta>2\theta_0$ the momentum conservation law does not allow quasi-particles to exchange two rotons at once during scattering. If we take into account that $\varkappa\neq0$, but $\varkappa\ll1$, for $|1-x|>\varkappa$ and $|1+x-2x_0^2|$

$> \varkappa$ Eq. (12) remains the same as for $\varkappa=0$. As $x\to 1$ the quantity (AA) tends to a finite limit $\propto\varkappa^{-1}$, and as $x\to 2x_0^2-1$ to a value $\propto\varkappa^{-1/2}$. The x-dependence of the quantity $\tilde\Phi$ is not sensitive to the value of $\varkappa$ as many-roton exchange does not lead to strong threshold singularities. Only the derivatives of $\tilde\Phi$ have discontinuities when $\varkappa=0$. The expansion of $\tilde\Phi$ in terms of P_l converges fast and we cut it off at the third term:

$$\tilde\Phi=\tilde\Phi_0+\tilde\Phi_1 x+\tilde\Phi_2{}^1/_2(3x^2-1). \tag{13}$$

The coefficients $\tilde\Phi_l$ are connected with the A_l and Φ_l:

$$\tilde\Phi_l=\frac{A_l{}^3}{(2l+1)(2l+1-A_l)}=\frac{\Phi_l{}^3}{(1+2l+\Phi_l)^2}.$$

Since the zeroth harmonic A_0 is close to unity as $\Delta\to 0$: $1-A_0\propto\Delta^{1/2}$, while the first harmonic is close to three: $3-A_1\propto\Delta^{1/2}$, writing A in the form (6) leads to corrections $\propto\varkappa\propto\Delta^{1/2}$ to the terms of the same order as $1-A_0$ and $3-A_1$. We must thus find the first two harmonics $\tilde\Phi_l$ using the experimental values of A_0 and A_1 rather than starting from the approximate expansion of D in powers of k^2/k_0^2-1. Experimentally $1-A_0\approx 0.1$ and $2l+1-A_1\approx 1$ so that we can replace A by $2A_0\delta(x-x_0)$. The zeroth harmonic of A is then the same as its true value and the difference between the first harmonic and its experimental value leads to corrections $\sim\varkappa$ besides terms $\sim 2l+1-A_1$. The correct way of taking the limit $\varkappa\to 0$ consists by the same token in replacing A_{00} in (6) by A_0. For the quantity (AA) of (12) the difference between A_{00} and A_0 is unimportant. Using the numerical values of A_0 and x_0 we get from (12) and (13)

$$\Phi(x)=1.83\delta(x-x_0)+0.53\frac{\theta(x-0.1)}{[(1-x)(x-0.1)]^{1/2}}+9+2.85x+0.09(3x^2-1). \tag{14}$$

The last term in (14) gives a 2% correction. We can also perform a similar calculation for the function $Z(x)$:

$$Z(x)=-0.61\delta(x-x_0)+0.059\frac{\theta(x-0.1)}{[(1-x)(x-0.1)]^{1/2}}-0.45-0.013x. \tag{15}$$

The expansion of Z in terms of the P_l can already be cut off at the second term which gives a 3% correction.

Taking into account the fact that $\varkappa$ is finite changes Eqs. (14) and (15) in a narrow region near $x=1$ and $x=0.1$. The width of that region is $\propto\varkappa\ll 1$. As all observable quantities can be expressed in terms of integrals of Φ and Z and as their singularities are only square root ones, we can put $\varkappa=0$.

Relation (14) gives us the possibility to determine the frequencies of the higher zero sounds in ^{3}He; it explains the good agreement of experiment with a theory in which only two harmonics of Φ are taken into account. Although the convergence of the expansion of Φ in terms of the P_l is slow, the total contribution from the higher harmonics is small on the background of the zeroth harmonic $\Phi_0\approx 11$. Taking these harmonics into account therefore makes little change in the value of the zero sound velocity determined by Abrikosov and Khalatnikov[6] in a theory with the two harmonics Φ_0 and Φ_1. When considering the higher zero sounds it is necessary to take into account the next harmonics as in the equation for determining their velocity the large zeroth harmonic does not occur.

We note that Eq. (5) is in agreement with the sum rule

$$\sum_l (A_l+B_l)=0,$$

which is connected with the symmetry of the scattering amplitude. To prove this we rewrite (5) in the form

$$\nu\Gamma^k(x)={}^3/_2D(\mathbf{p}_1-\mathbf{p}_2)-\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2({}^1/_2D(\mathbf{p}_1-\mathbf{p}_2)+D(0)),\quad |\mathbf{p}_1-\mathbf{p}_2|=p_F[2(1-x)]^{1/2}. \tag{16}$$

It follows from (16) that $A+B=D(\mathbf{p}_1-\mathbf{p}_2)-D(0)$ and

$$\sum_l (A_l+B_l)=\sum_l (A_l+B_l)P_l(1)=D(0)-D(0)=0.$$

We can enhance the accuracy of Eqs. (14) and (15) if we give up the expansion of $\tilde\Phi$ in terms of the P_l and determine Φ from (11) which after integration over the angle φ becomes

$$\Phi(x)=A(x)+\frac{A_0}{\pi}\int_{b(x)}^{a(x)}\frac{\Phi(x')dx'}{[(a(x)-x')(x'-b(x))]^{1/2}},\quad a(x)=xx_0+(1-x^2)^{1/2}(1-x_0^2)^{1/2},\quad b(x)=xx_0-(1-x^2)^{1/2}(1-x_0^2)^{1/2}. \tag{17}$$

We could not solve Eq. (17) so that we restricted ourselves to writing Φ and Z in the form (14), (15).

In concluding this section we note that to determine the parameters of the Landau theory exact values of ξ^2 and γ^2 are not important. Only the width of the δ-function-like maximum of the function $A(x)$ depends on ξ^2 and γ^2. As with good accuracy we can put $\varkappa=0$ the Landau theory parameters can be expressed merely in terms of the two numbers A_0 and x_0 from (7).

4. SPIN FLUCTUATION SPECTRUM

1. We determine the dependence of the amplitude Γ on the frequency ω. To do this we consider Γ for $p_1=p_2=p_F$ and $\omega\ll kV$. In that case Γ depends on ω, k and the angle between $\mathbf{p}_1$ and $\mathbf{p}_2$. There is for Γ a representation similar to Eq. (5) for Γ^k:

$$\nu\Gamma(x,k,\omega)=A(x,k,\omega)+\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2B(x,k,\omega). \tag{18}$$

$\Gamma(\omega)$ is connected with its static value $\Gamma(0)$ through the Dyson equation

$$A(\mathbf{n}_1,\mathbf{n}_2,\omega)=A(\mathbf{n}_1,\mathbf{n}_2,0)+\int A(\mathbf{n}_1,\mathbf{n}_3,0)A(\mathbf{n}_3,\mathbf{n}_2,\omega)\frac{d\Omega_{\mathbf{n}_3}}{4\pi}\delta\Pi(\omega),\quad B(\mathbf{n}_1,\mathbf{n}_2,\omega)=B(\mathbf{n}_1,\mathbf{n}_2,0)+\int B(\mathbf{n}_1,\mathbf{n}_3,0)B(\mathbf{n}_3,\mathbf{n}_2,\omega)\frac{d\Omega_{\mathbf{n}_3}}{4\pi}\delta\Pi(\omega). \tag{19}$$

$\delta\Pi(\omega)$ in (19) is the correction to the static value of the polarization operator: $\delta\Pi=\Pi(\omega)-\Pi(0)$. The main term in $\delta\Pi$ which is linear in ω has the form

$$\delta\Pi(\omega)=-i\frac{\pi|\omega|}{2kV}\theta(2p_F-k), \quad \omega\ll kV. \tag{20}$$

One can show that Eq. (20) is valid for all k and $\omega \ll kV$ and is not connected with the requirement $k\to 0$, $\omega\to 0$; when that requirement is met $\delta\Pi$ can be evaluated exactly in the Landau theory.

It follows from (19) that

$$A_l(\omega)=\frac{A_l}{1-\delta\Pi A_l/(2l+1)}, \quad B_l(\omega)=\frac{B_l}{1-\delta\Pi B_l/(2l+1)}. \tag{21}$$

We shall take the ω-dependence into account only for the zeroth harmonics of A and B. The fact is that in all observable quantities the combination $M_l=3B_l+A_l$ enters. If $l=0$, this combination has a finite limit as $\omega\to 0$, equal to $-3D$:

$$M_0=\frac{A_0}{1-A_0\delta\Pi}-3\frac{D+{}^1/_3A_0}{1+(D+{}^1/_3A_0)\delta\Pi}. \tag{22}$$

The ω-dependence of M_0 occurs already when $D\delta\Pi\sim 1$, i.e., when $\omega > kV\xi^2$, if $k\approx k_0$. The quantities $M_{l\neq 0}$ start to depend strongly on ω only when $\omega\sim kV$, and the limit $M_{l\neq 0}(\omega\to 0)=0$

$$M_{l\neq 0}=\frac{A_l}{1-\delta\Pi A_l/(2l+1)}-\frac{A_l}{1+\delta\Pi A_l/(3(2l+1))}. \tag{23}$$

It will become clear in what follows that the contribution from M_0 to the self-energy Σ and the specific heat C is non-analytical as $\xi^2\to 0$, so that it is necessary to take the ω-dependence of M_0 exactly into account. The contribution from $M_{l\neq 0}$ is independent of ξ^2 and leads to corrections in Σ and C of the form

$$\varepsilon\left(\frac{\varepsilon}{\varepsilon_F}\right)^2\ln\frac{\varepsilon_F^2}{\varepsilon^2}, \quad T\left(\frac{T}{\varepsilon_F}\right)^2\ln\frac{\varepsilon_F^2}{T^2}.$$

We shall not take these corrections correctly into account, remaining within the framework of the linear expansion of $\delta\Pi$ in ω. Because of this it is necessary to renormalize also the zeroth harmonic M_0. In Σ and C there occurs the quantity

$$\tilde{M}_0=-3(D+{}^1/_3A_0)[1+\delta\Pi(D+{}^1/_3A_0)]^{-1}+A_0(1+\delta\Pi A_0/3)^{-1}. \tag{24}$$

2. The poles of $B(k,\omega)$ correspond to particle-hole excitations with spin one. We can find them by analytically continuing $B(k,\omega)$ onto the second sheet of the ω-plane. This continuation must proceed from the right-hand semi-axis $\mathrm{Im}\,\omega=0$ into the lower half-plane and from the left-hand semi-axis $\mathrm{Im}\,\omega=0$ into the upper ω-halfplane:

$$\omega=\pm i\cdot 2\pi^{-1}kVB_0^{-1}(k,0), \quad B_0(k,0)=-D(k^2)-A_0/3. \tag{25}$$

In accordance with the expansion (3) for $k\approx k_0$ the frequency ω depends quadratically on $k-k_0$:

$$|\omega|=\Delta+(k-k_0)^2/2M,$$
$$\Delta=2k_0V\xi^2/\pi\approx 0.094\mathrm{K}, \quad M=m^*\pi k_0/16p_F\gamma^2\approx 0.06m. \tag{26}$$

As $k\to 0$ we can also find the k-dependence of ω as the value of $B_0(0,0)$ is known from data on the magnetic susceptibility of ^{3}He for $k=0$:

$$|\omega|=kv, \quad v=-\frac{2V}{\pi}B_0^{-1}(0,0),$$
$$B_0(0,0)=\frac{Z_0}{1+Z_0}\approx -2. \tag{27}$$

The spin fluctuation spectrum thus corresponds to virtual spin-sound excitations and consists of phonons and rotons.

The form of the spectrum $\omega(k)$ near the maximum, where the phonon part of the spectrum goes over into the roton part, is not known. We give an interpolation formula for ω which is exact as $k\to 0$ and for $k\approx k_0$. It is convenient to write ω in the form

$$|\omega|=\omega_0 f(x), \quad \omega_0=2k_0V/\pi;$$
$$x=k/k_0,$$
$$f(x)=x\frac{D^{-1}(x^2)}{1+{}^1/_3A_0D^{-1}(x^2)} \tag{28}$$

In the expansion of D in powers of x^2-1 where $x=k/k_0$ we retained two terms:

$$D(x^2)=\xi^2+\gamma^2(x^2-1)^2+\eta^2(x^2-1)^3$$

and we determined η^2 from the requirement that the spin-sound velocity, found from (28), be the same as the exact value from (25): $\eta^2=7$.

We give a diagram of the function $f(x)$ in Fig. 1.

5. THE SINGLE-PARTICLE EXCITATION SPECTRUM

1. We split off the roton contribution in $\Sigma(p^2,\varepsilon)$. We write Σ in the form

$$\Sigma=\{\Sigma(p^2,0)-\Sigma(p_F^2,0)\}+\{\Sigma(p^2,\varepsilon)-\Sigma(p^2,0)\} \tag{29}$$

The first difference in (29) can be expressed in terms of integrals over a wide range of intermediate momenta and the region near the Fermi surface where the form (2) of Γ is valid is not at all distinguished. We expand $\Sigma(p^2,0)-\Sigma(p_F^2,0)$ in a series in $p^2-p_F^2$. The series converges well as this quantity is an analytic function of Δ:

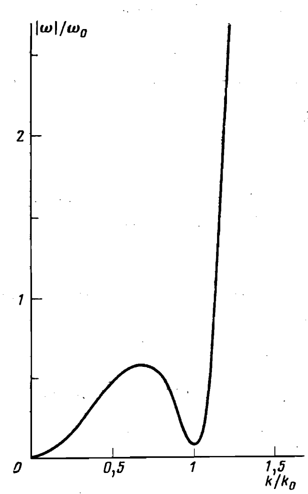


FIG. 1. The spin fluctuation spectrum. The frequency ω is in units $\omega_0=1.5$ K, the momentum in units $k_0=0.7\,p_F$.

$$\Sigma(p^2,0)-\Sigma(p_F^2,0)\approx(p^2-p_F^2)\left.\frac{\partial\Sigma}{\partial p^2}\right|_{p^2=p_F^2}. \tag{30}$$

The contribution from the rotons, non-analytical in Δ, to the second difference in (29) is given by the expression

$$\Sigma(\varepsilon)=\frac{1}{\nu}\int \operatorname{Im} G_{\mathbf{p}-\mathbf{k},\varepsilon-\omega}\mathcal{M}_0(k,\omega)\frac{d\mathbf{k}\,d\omega}{(2\pi)^4}. \tag{31}$$

This expression corresponds to the Hartree–Fock approximation in terms of the effective interaction $\Gamma(k,0)$. Corrections to this approximation contain the small parameter $(k_0/2p_F)^4$ which is the analog of ω_D/ε_F for the solid state.

We can in (31) integrate over the angle between **k** and **p**, using the method applied by Migdal[7] to evaluate the phonon contribution to the Σ of the electrons

$$\Sigma(\varepsilon)=\frac{1}{16p_F^2a}\int_{-\varepsilon}^{\varepsilon}d\omega\int_0^{4p_F^2}dk^2\mathcal{M}_0(k,\omega). \tag{32}$$

Using Eq. (24) we can easily integrate over $d\omega$:

$$\Sigma(\varepsilon)=-i\frac{3\omega_0\operatorname{sign}\varepsilon}{8ap_F^2}\int_0^{4p_F^2}dk^2\ln\frac{1-B_0(k)\delta\Pi(k,\varepsilon)}{1+{}^1/_3A_0\delta\Pi(k,\varepsilon)}. \tag{33}$$

We consider the limit as $\varepsilon\to 0$

$$\Sigma(\varepsilon)=-\varepsilon\frac{3}{8ap_F^2}\int_0^{4p_F^2}D(k^2)\,dk^2=-\varepsilon\frac{A_0}{a}. \tag{34}$$

In obtaining (34) we used the connection between D and A (see (5), (6)):

$$D(k^2)={}^2/_3A(x),\quad k^2=2p_F^2(1-x). \tag{35}$$

The jump in the particle momentum distribution is determined by the relation

$$a^{-1}=1-\left.\frac{\partial\Sigma}{\partial\varepsilon}\right|_{\varepsilon=0}. \tag{36}$$

From (34) and (36) we get the very important connection between the zeroth harmonic of A and the jump a:

$$a=1-A_0.$$

As $A_0=\Phi_0(1+\Phi_0)^{-1}$ and Φ_0 is connected with the sound velocity [2] we can directly express a in terms of the sound velocity:

$$c^2=p_F^2/3mm_0^*,\quad m_0^*=am^*,\quad a=(1+\Phi_0)^{-1}=0.085. \tag{37}$$

We shall make clear below the physical meaning of the effective mass m_0^*. It turns out that m_0^* determines the T-dependence of the specific heat for large T. The theory therefore allows us to connect the sound velocity and the specific heat with the jump a.

2. We evaluate Σ for $\varepsilon<\varepsilon_F$. It is convenient, when integrating over dk^2 in (33), to change to the dimensionless variable $x^2=k^2/k_0^2$ and split Σ into its real and imaginary parts:

$$\operatorname{Re}\Sigma=-\frac{3k_0^2}{8p_F^2a}\omega_0\int dx^2\,x\operatorname{arctg}\frac{\varepsilon}{\tilde\omega(x,\varepsilon)}. \tag{38}$$

The quantity $\tilde\omega(x,\varepsilon)$ has the meaning of the renormalized spin fluctuation spectrum; $\tilde\omega$ depends on ε:

$$\tilde\omega(x,\varepsilon)=x\omega_0\left\{D^{-1}(x^2)\left(1+\frac{A_0^2\varepsilon^2}{9\omega_0^2x^2}\right)+\frac{1}{3}A_0\frac{\varepsilon^2}{\omega_0^2x^2}\right\}. \tag{39}$$

For small $\varepsilon<\varepsilon_F$ the integral comes mainly from the roton region $x\approx 1$. The contribution from the phonon region $x\ll 1$ is small because of the small phase volume of the phonons. We can therefore put $x=1$ in all quantities which are slowly varying for $x\approx 1$. We restrict ourselves to the approximation $D=\xi^2+\gamma^2(x^2-1)^2$. Then

$$\tilde\omega\approx\omega_0\{\xi^2(\varepsilon)+\gamma^2(\varepsilon)\,(x^2-1)^2\}\equiv\tilde\omega_1,$$
$$\xi^2(\varepsilon)=\xi^2\left(1+\frac{A_0^2\varepsilon^2}{9\omega_0^2}\right)+\frac{1}{3}\frac{\varepsilon^2}{\omega_0^2},\qquad \gamma^2(\varepsilon)=\gamma^2\left(1+\frac{1}{9}\frac{A_0^2\varepsilon^2}{\omega_0^2}\right). \tag{40}$$

Using the form (40) for $\tilde\omega$ we can use Eq. (38) for $\operatorname{Re}\Sigma$:

$$\operatorname{Re}\Sigma=-\frac{1-a}{a}\varepsilon\left(\frac{2}{1+\varepsilon^4/\varepsilon_2^2}\right)^{1/2}\left[1+\frac{\varepsilon^2}{\varepsilon_1^2}+\left(\frac{\varepsilon^2}{\Delta^2}+\left(1+\frac{\varepsilon^2}{\varepsilon_1^2}\right)\right)^2\right)^{1/2}\Bigg]^{-1/2}, \tag{41}$$
$$\varepsilon_2=\frac{3}{A_0}\omega_0=4.9\,\mathrm{K},\qquad \varepsilon_1=\omega_0\xi\left(\frac{3}{A_0}\right)^{1/2}=0.68\,\mathrm{K}.$$

For small ε Eq. (41) simplifies:

$$\operatorname{Re}\Sigma=-\sqrt{2}\frac{1-a}{a}\varepsilon[1+(1+\varepsilon^2/\Delta^2)^{1/2}]^{-1/2}. \tag{42}$$

For large ε the quantity $\operatorname{Re}\Sigma$ depnnds only on the ratio of ε to ε_2:

$$\operatorname{Re}\Sigma\approx-\frac{1-a}{a}\operatorname{sign}\varepsilon\left(\frac{2\varepsilon_2\Delta}{1+\varepsilon^2/\varepsilon_2^2}\right)^{1/2}[1+(1+\varepsilon_2^2/\varepsilon^2)^{1/2}]^{-1/2}. \tag{43}$$

When $\Delta<\varepsilon<\varepsilon_F$ Eqs. (42) and (43) are the same:

$$\operatorname{Re}\Sigma=-\frac{1-a}{a}\operatorname{sign}\varepsilon\,(2|\varepsilon|\Delta)^{1/2}. \tag{44}$$

As $\Delta\to 0$ the self-energy Σ has thus a square-root branch point on the Fermi surface. We note that when the phonon velocity v decreases the quantity Σ also has a singularity on the Fermi surface, but this singularity is weaker than (44):

$$\Sigma\sim\varepsilon\ln\frac{\varepsilon_F^2}{\varepsilon^2+\varepsilon_F^2v^3/V^3},\qquad v/V\to 0. \tag{45}$$

We can evaluate the imaginary part of Σ similarly to $\operatorname{Re}\Sigma$. For small ε

$$\operatorname{Im}\Sigma=2\frac{1-a}{a}\Delta\frac{|\varepsilon|}{\varepsilon}\left\{1-\left[\frac{1+(1+\varepsilon^2/\Delta^2)^{1/2}}{2}\right]^{1/2}\right\}. \tag{46}$$

For large ε the quantity $\operatorname{Im}\Sigma$ also depends solely on $\varepsilon/\varepsilon_2$:

$$\operatorname{Im}\Sigma=-\frac{1-a}{a}\operatorname{sign}\varepsilon\,(\Delta|\varepsilon|)^{1/2}\left[\frac{(1+\varepsilon^2/\varepsilon_2^2)^{1/2}+1}{1+\varepsilon^2/\varepsilon_2^2}\right]^{1/2}. \tag{47}$$

When $\varepsilon\ll\varepsilon_F$ we get from (42) and (43) an expression for $\Sigma=\operatorname{Re}\Sigma+i\operatorname{Im}\Sigma$:

$$\Sigma = -\frac{1-a}{a}\frac{2\varepsilon}{(1-i|\varepsilon|/\Delta)^{1/2}+1}. \quad (48)$$

3. The single-particle excitation spectrum is the same as the poles of the Green function $G(p^2, \varepsilon)$. In our case for $\varepsilon \ll \varepsilon_F$

$$G^{-1}(p^2, \varepsilon) = \varepsilon - \frac{p^2-p_F^2}{2m_0^*} + \frac{1-a}{a}\frac{2\varepsilon}{(1-i|\varepsilon|/\Delta)^{1/2}+1} \quad (49)$$

The quantity m_0^* is determined by the relation $m_0^* = am^*$ and connected with the expansion (30):

$$\frac{m}{m_0^*} = 1 + 2m\frac{\partial\Sigma}{\partial p^2}\bigg|_{p^2=p_F^2}, \quad m_0^* \approx 0.25\, m.$$

The poles of $G(p^2, \varepsilon)$ lie on the non-physical sheet of the ε-plane for $\varepsilon = E_p$; $E_p = \varepsilon_p - i\gamma_p$:

$$\varepsilon_p = \frac{p^2-p_F^2}{2m_0^*}\left\{1-(1-a)\left[\frac{2}{1+\sqrt{1+\lambda^2}}\right]^{1/2}\right\},$$
$$\gamma_p = (1-a)\frac{(p^2-p_F^2)|p^2-p_F^2|}{4(m^*)^2\Delta}\frac{2}{\lambda^2}\left\{\left[\frac{1+(1+\lambda^2)^{1/2}}{2}\right]^{1/2}-1\right\}. \quad (50)$$

The parameter λ is connected with Δ by the relation $\lambda = (p^2 - p_F^2)a/2m^*\Delta$.

For $\lambda^2 < 1$ Eq. (50) can be simplified:

$$E_p = V(p-p_F)\{1 - iV|p-p_F|/4\Delta\}. \quad (51)$$

As $\Delta \to 0$ the pole of G lies on the imaginary axis of the ε-plane and corresponds to a virtual single-particle state. In ^{3}He at a pressure of 0.28 atm the roton gap is no longer sufficiently small to distort the spectrum so much that $\gamma_p \gg \varepsilon_p$. Using the numerical values of Δ, a, and m_0^* we can get from (50) an expression for ε_p and γ_p as functions of $y = (p - p_F)/p_F$ which are valid for $\lambda^2 < 1$:

$$\varepsilon_p/\varepsilon_F = 0.65y(1+11.9y^2), \quad \gamma_p/\varepsilon_F = 5y^2(1-2.7y^2). \quad (52)$$

The quantities ε_p and γ_p become comparable for $x = 0.15$ and for large x are approximately the same. When x increases further, when $\lambda^2 > 1$, the spectrum is linear: $\varepsilon_p = V_0(p - p_F)$, $V_0 = p_F/m_0^*$, and the damping is small compared to ε_p: $\gamma_p \propto |p - p_F|^{1/2}$.

The single-particle spectrum of ^{3}He is thus similar to the electron spectrum in a solid. Far away from and close to the Fermi surface there are quasi-particles and in the intermediate region where there is a strong interaction with rotons the damping γ_p is larger than or of the order of ε_p. The spectrum for large $\varepsilon \sim \varepsilon_F$ is not given by the theory as for such ε the asymmetry of particles and holes and the cubic terms in ε and Σ are important, and we have neglected those.

6. PARAMAGNON CONTRIBUTION TO THE SPECIFIC HEAT OF ^{3}He

The calculation of the paramagnon contribution to the free energy of ^{3}He, which is non-analytical in Δ, proceeds similarly to that of Σ. This contribution is given by ring diagrams in which $\Gamma(k, 0)$ occurs as the interaction and $\delta\Pi$ from (20) as the particle-hole loop. For an almost ferromagnetic Fermi liquid such diagrams were summed by Larkin and Mel'nikov.[8] Up to terms $\propto T^2(T^2/\varepsilon_F^2)$ the paramagnetic contribution to the free energy of ^{3}He is given by the expression

$$\delta F = \frac{3}{2}\int\frac{d\mathbf{k}}{(2\pi)^3}T\sum_{\omega_n}\ln\frac{1+(D(\mathbf{k})+{}^1/_3A_0)\pi|\omega_n|/2kV}{1+{}^1/_3A_0\pi|\omega_n|/2kV}. \quad (53)$$

Formula (53) corresponds to the Hartree–Fock approximation with the effective interaction $\Gamma(k, 0)$. Larkin and Mel'nikov's summation of dangerous diagrams led to the Hartree approximation. As $T \to 0$ the difference between these two approximations vanishes.

The summation over the frequencies ω_n in (53) can be reduced to an integration over a contour enclosing the imaginary axis of the ω-plane. After differentiation with respect to T and introducing a new integration variable $k^2/k_0^2 = x^2$ we get an expression for the paramagnetic contribution to the specific heat $C(T)$:

$$\delta C(T) = \frac{3k_0^3}{16\pi^3}\int_0^\infty\frac{z^2\,dz}{\mathrm{sh}^2(z/2)}\frac{\partial}{\partial z}\int dx^2 x \operatorname{arctg}\frac{zT}{\tilde\omega(x, zT)}. \quad (54)$$

In (54), as in (38), the renormalized spectrum $\tilde\omega(x, \varepsilon = zT)$, given by Eq. (40), occurs. As $T \to 0$ the addition $\delta C(T)$ can be evaluated exactly; as in Σ the addition $\delta C(T)$ is connected as $T \to 0$ with the zeroth harmonic A_0 and the jump a:

$$\delta C = {}^1/_3 p_F m^* T(1-a), \quad C = C_0 + \delta C = {}^1/_3 p_F m^* T, \quad a = 1 - A_0. \quad (55)$$

It follows from (55) that $C_0 = \frac{1}{3}p_F m_0^* T$. This expression for C_0 clearly follows from the form (49) for G. If we neglect the magnon contribution to Σ, G has the form $G^{-1} = \varepsilon - (p^2 - p_F^2)/2m_0^*$. Such a Green function describes non-interacting particles with mass m_0^*; the specific heat is thus connected with m_0^* by the usual gas formula, if we neglect magnons. As C_0 is an analytic function of Δ the linear T-dependence of C_0 is broken only when $T \sim \varepsilon_F$.

When using (54) to evaluate δC we can approximately replace the function $z^2/4\sinh^2(z/2)$ by unity for small z and put it equal to zero for large z. Physically such a cut-off means that the contributions to the specific heat from degrees of freedom with energies $\varepsilon < T$ are approximately the same, while those with energies $\varepsilon > T$ give an exponentially small contribution. The cut-off parameter z_c follows from the requirement that as $T \to 0$ the specific heat must be the same as (55): $z_c = \pi^2/3$. After this we integrate over z by parts and the integral over dx^2 can be approximately evaluated by the same method as for $\Sigma(\varepsilon)$ in (38) (we also neglect the phonon contribution):

$$C = \frac{1}{3}p_F m^* T\left\{a + (1-a)\left(\frac{2}{1+T^2/T_2^2}\right)^{1/2}\left[1 + \frac{T^2}{T_1^2} + \left(\frac{T^2}{T_0^2} + \left(1 + \frac{T^2}{T_1^2}\right)^2\right)^{1/2}\right]^{-1/2}\right\}; \quad (56)$$
$$T_2 = z_c^{-1}\varepsilon_2 \approx 1.5\text{ K}, \quad T_1 = z_c^{-1}\varepsilon_1 \approx 0.2\text{ K}, \quad T_0 = \Delta z_c^{-1} \approx 0.028\text{ K}.$$

The T-dependence of δC therefore differs from the ε-dependence of $\mathrm{Re}\Sigma$ only in a change in scalelength ε

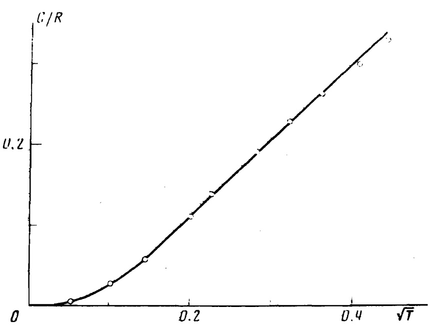


FIG. 2. The specific heat C, divided by the gas constant R as function of $T^{1/2}$ for $T < 0.2$ K.

$\to z_c T$.

The quantity ξ^2 is determined by comparing Eq. (56) with experiment for $T \sim \Delta$, where (56) has the form

$$C = \frac{1}{3} p_F m^* T \left\{ a + (1-a) \left[\frac{2}{1+(1+T^2/T_0^2)^{1/2}} \right]^{1/2} \right\}. \quad (57)$$

For $T < T_0$ the paramagnons give a correction to the linear law $C \propto T$ of the form $T(T^2/T_0^2)$. When $T > T_0$ the specific heat like Σ depends on $T^{1/2}$:

$$C \approx {}^1/_3 p_F m^* \{ aT + (1-a)(2T_0 T)^{1/2} \}. \quad (58)$$

We show in Fig. 2 the $T^{1/2}$-dependence of C, corresponding to Eq. (57) with $\xi^2 = 0.063$. The points correspond to the experimental data.[9–11] The data on $C(T)$ for $T \sim \Delta$, taken from different papers, differ from one another by 10%. From comparing Eq. (57) with experiment we derive ξ rather than ξ^2. The quantity ξ^2, and then also Δ, is thus determined with an accuracy of 20%. This inaccuracy by far exceeds the relative contribution of the spin phonons to C. In the present stage, therefore, when we restrict ourselves to a qualitative consideration, we cannot find the phonon dispersion from a comparison with experiment. For a more precise determination of the spin fluctuation spectrum it is necessary to evaluate the double integral in (54) without applying the cut-off procedure in z.

Having determined the parameter ξ^2 in the small T region where the rotons give the main contribution to $C(T)$ we can determine the T-dependence of C for large T from (56). This dependence is given by curve 1 in Fig. 3. Up to $T \approx 0.4$ K the agreement with experiment is good, while for larger T the T-dependence of C is determined correctly: $C \approx c_1 + c_2 T$, but the constant c_1 is overestimated by 25%. This overestimate arises due to the inapplicability of the theory for $T > 0.4$ K, and is the result of approximately replacing the spectrum $\tilde{\omega}$ by $\tilde{\omega}_1$, i.e., by changing from Eq. (39) to (40).

To get a more exact form of $C(T)$ for large T we must take into account that the spin fluctuation spectrum is asymmetric relative to $k = k_0$. The part of the spectrum ω for $k < k_0$ saturates appreciably earlier than the degrees of freedom connected with the region $k > k_0$. It is clear that for $T > \omega_{max}$, where ω_{max} is the maximum value of ω in the interval $0 < k < k_0$, the contribution from this integral to $C(T)$ will be approximately constant. It is thus reasonable to take the contribution from the rotons for $k > k_0$ into account as before and to use in the region $k < k_0$ the expansion for D^{-1} up to the term $\propto \eta^2$. For large T it follows from (53) that

$$\operatorname{arctg} \frac{z_c T}{\tilde{\omega}} \approx \frac{\pi}{2} - \frac{\tilde{\omega}}{z_0 T}$$

and the fact that the spectrum is not determined for $k < k_0$ affects only the correction to the specified heat. The contribution from this "saturated" part of the spectrum to C is given by the expression

$$C_{k<k_0} = \frac{1}{2} p_F m^* T_3 \frac{k_0^2}{p_F^2} \left\{ \frac{2}{3} \operatorname{arctg} \frac{1}{\alpha} - \frac{\alpha}{3} - \frac{1}{3\alpha} A_0 \int_0^1 \frac{x^2\, dx^2}{D + {}^1/_3 A_0} \right\}; \quad (59)$$
$$T_3 = \omega_0 z_c^{-1} = 0.45 \text{ K}, \quad \alpha \equiv T/T_2.$$

The total specific heat has the form

$$\frac{C}{R} = 0.32\alpha + \frac{0.39}{(1+\alpha^2)^{1/2}[1+(1+\alpha^{-2})^{1/2}]^{1/2}} + 0.175 \operatorname{arctg} \frac{1}{\alpha} - 0.02 \frac{1}{\alpha}. \quad (60)$$

In the region $0.3 < \alpha < 1$ the specific heat C is with great accuracy a linear function of $\alpha(T)$:

$$C/R = 0.32\alpha + 0.33 = 0.21\, T + 0.33.$$

According to Roberts and Sydoriak [10] in the same T region the experimental results agree with a dependence of the form

$$C/R = 0.29 + 0.2\, T + 0.03\, T^3.$$

Curve 2 in Fig. 3 corresponds to Eq. (60) which determines the specific heat at high temperatures more exactly. Apparently, the agreement with experiment can be improved if we do not make simplifications of a computational nature when determining $C(T)$ from Eq. (54).

7. THE MAGNETIC SUSCEPTIBILITY

The theory presented above depends on three phenomenological parameters which parametrize the spin fluctuation spectrum and are determined from a comparison between theory and experiment. The number of physical quantities determined by the theory is much

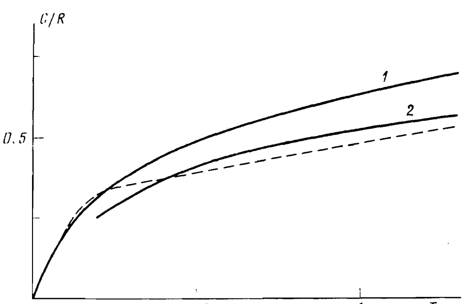


FIG. 3. T-dependence of C for $T < 1.5$ K. Curve (1) corresponds to Eq. (56), curve 2 to Eq. (60). The dashed curve is drawn through the experimental points from the data of[9,10]. For $T > 0.5$ K we have split off from C a term $\propto T^3$.[10]

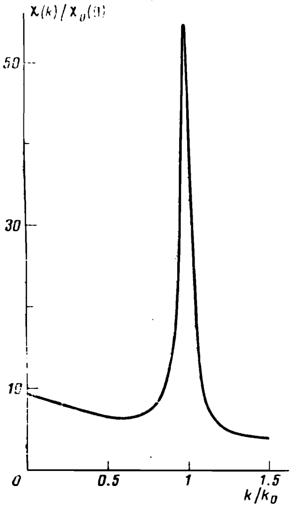


FIG. 4. The magnetic susceptibility $\chi(k)$ in units $\chi_0(0)$ as a function of k/k_0.

larger than the number of these parameters. At present there is no direct verification of the results of the present paper as the magnetic susceptibility of ^{3}He for $k \neq 0$ is not known. The experimental difficulties of measuring χ are connected with the fact that ^{3}He, in contrast to ^{4}He, absorbs neutrons strongly. The usual method of probing the liquid with a neutron flux which enabled one to determine the excitation spectrum of ^{4}He in detail is unsuitable for ^{3}He.

We give an approximate expression for χ which one can obtain by using the fact that k_0 is small compared to $2p_F$:

$$\frac{\chi(k)}{\chi_0(0)} \approx \left(1+\frac{1}{3}A_0+D(k)\right)\frac{m^*}{m}, \quad (61)$$

where χ_0 is the static susceptibility of a perfect gas. To obtain (61) up to terms $\propto k_0^2/4p_F^2$ we neglected the k^2 dependence of the polarization operator Π. As $k \to 0$ Eq. (61) changes to the exact Landau formula:

$$\frac{\chi}{\chi_0} = \frac{1}{1+Z_0}\frac{m^*}{m}.$$

The quantity Z_0 is connected with D and A_0 through Eq. (27). The k/k_0-dependence of χ is shown in Fig. 4. We have used the form (3) for D.

We emphasize that we do not know the dispersion of χ, like that of ω. We must thus consider Eq. (61) as exact for $k \to 0$ and $k \approx k_0$, but for $k < k_0$ as an interpolation formula. We believe a theoretical determination of the spectrum ω near the maximum of ω and χ near its minimum is impossible. This is connected with the fact that the spectrum ω integrated over a wide k-interval enters into all physical quantities except χ. The k-region where the spectrum is unknown has a small phase volume. As the theory is phenomenological and its accuracy depends on the accuracy of the experiments from which the parameters Δ, k_0, and M are determined it is very difficult to separate the theoretical and the experimental errors from one another. In particular, χ may not have a minimum at all, but monotonically increase up to $k = k_0$ when k increases. Such a possibility would be most favorable as an experimental measurement of χ in the small k range would enable us to verify with confidence that χ has a maximum and thus the spectrum ω a roton minimum. Our confidence that there are, indeed, spin excitations in ^{3}He with a roton gap is based upon the agreement between theory and experiment which is achieved by a small number of phenomenological parameters.

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[1] L. D. Landau, Zh. Eksp. Teor. Fiz. 30, 1058 (1956) [Sov. Phys. JETP 3, 920 (1956)].
[2] L. D. Landau, Zh. Eksp. Teor. Fiz. 32, 59 (1957) [Sov. Phys. JETP 5, 101 (1957)].
[3] I. Ya. Pomeranchuk, Zh. Eksp. Teor. Fiz. 20, 919 (1950).
[4] L. D. Landau and E. M. Lifshitz, Kvantovaya mekhanika (Quantum Mechanics) Nauka, 1974, p. 637 [translation published by Pergamon Press, Oxford].
[5] A. M. Dyugaev, Pis'ma Zh. Eksp. Teor. Fiz. 22, 181 (1975) [JETP Lett. 22, 83 (1975)].
[6] A. A. Abrikosov and I. M. Khalatnikov, Zh. Eksp. Teor. Fiz. 13, 110 (1957) [Sov. Phys. JETP 6, 84 (1958)].
[7] A. B. Migdal, Zh. Eksp. Teor. Fiz. 34, 1438 (1958) [Sov. Phys. JETP 7, 996 (1958)].
[8] A. I. Larkin and V. I. Mel'nikov, Pis'ma Zh. Eksp. Teor. Fiz. 20, 386 (1974) [JETP Lett. 20, 173 (1974)].
[9] A. S. Mota, Phys. Rev. 177, 266 (1969).
[10] T. R. Roberts and S. G. Sydoriak, Phys. Rev. 98, 1672 (1955).
[11] A. C. Anderson, W. Reese, and J. C. Wheatley, Phys. Rev. 130, 195 (1963).

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