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A theory of direct four-fermion interactions

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A solution has been obtained for the "parquet" equations for the vertex $\Gamma(p_1,p_2;p_3,p_4)$ of the direct four-fermion interaction in a space of dimension $d=2+\epsilon$. For the existence of such a solution it is necessary that the interaction have a symmetry of the type of $SU(2)$-invariance, and that the coupling constant G be positive. For high energies $Gp^2 \gg 1$ this solution is scale-invariant and corresponds to a stable fixed point of the Gell-Mann–Low equations. It is shown that a similar solution approximately satisfies the system of equations in four-dimensional space $d=4$, where all the integrals in the equations turn out to be convergent. With the help of this solution the contribution of the so-called "non-parquet" terms is estimated, terms which have not been taken into account in the equations. It is shown that these terms are numerically small. The solution can be used as a zeroth approximation of an iterative method of solution of the exact equations.

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1. INTRODUCTION

The direct point interaction of fermions

$$H=G_0(\bar{\nu}O_\alpha e)(\bar{e}O_\alpha \nu) \qquad (1)$$

for $O_\alpha=\gamma_\alpha(1+\gamma_5)/2$ (the $V-A$ variant) describes well in the Born approximation all weak interaction processes at low energies. However, since the cross section for this interaction increases with energy, $\sigma \sim G^2E^2$, and only the S-wave participates in scattering, at an energy $E \sim 10^3$ GeV the growth of the cross sections runs into contradiction with unitarity, and it becomes necessary to take into account terms of higher order in the coupling constant.[1]

In order to determine such higher-order contributions one cannot make use of perturbation theory, since the interaction (1) is not renormalizable in the usual sense. For this reason the renormalizable Weinberg–Salam scheme[2] for the weak interactions has acquired popularity in recent years. Unfortunately, this scheme requires the introduction of a series of new particles and is not quite simple. Also, the strong interaction scheme which is based on the intermediate nonabelian gauge vector fields is not simple. All other types of renormalizable interactions (e.g., the Yukawa $\pi N\bar{N}$ interaction, or meson self-interactions of the type $\lambda\varphi^4$, as well as the electromagnetic $\gamma\bar{e}e$ interaction) lead to the well known problem of "vanishing charge,"[3] i.e., the vanishing of the physical coupling constant in these theories in the limit of a point interaction, i.e., in the local limit. This manifests itself also in the fact that the effective coupling constant $g^2(p^2)$, which characterizes the interaction at a momentum p^2, increases with p^2, in distinction from the asymptotically free gauge theories, where it decreases. Theories are possible where $g^2(p^2) \to g_1^2=\text{const}$ for $p^2 \to \infty$, the so-called theories with a "fixed point." This is the kind of possibility that will be explored for the four-fermion interaction in this paper.

The weak interaction has been investigated in a number of papers by means of dispersion relations. This has allowed one to find,[4] up to arbitrary constants related to the cross sections of physical processes, the corrections of order G^2 and G^3 to the Born approximation amplitudes at energies smaller than 10^3 GeV, and to obtain a restriction[5] on the weak interaction cross sections at high energies.

In the present paper the weak interaction is investigated by means of the so-called "parquet" integral equations for the vertex function, formulated in 1958 by one of the authors[6] (for the case of boson fields, cf. also[7]). The iteration of these equations with the zeroth approximation in the form of the Born term leads to the usual divergences of perturbation theory, however, outside the framework of perturbation theory the equations may have finite solutions.[8,9]

In the paper by Abrikosov *et al.*[10] the two-limit technique has been used to obtain a solution which in the local limit leads to a vanishing coupling constant of the four-fermion interaction. However, the two-limit technique does violence to the symmetry of the problem (the antisymmetry of fermion vertices). This is particularly manifest with the Thirring model as an example.[11] We consider below the possibility that the integral equations have in reality a solution of a different kind, which decreases so fast with the increase of momenta that the integrals are convergent. In this case the dependence of the solution on the cutoff momentum disappears completely, and instead of the vanishing charge there appears the situation corresponding to the fixed point.

In Sec. 2 and Appendix 1 we write out the integral equations for the $V-A$ interaction. In Sec. 3 we give general arguments[12–14] in favor of the fact that these equations have finite solutions which exhibit asymptotic scale invariance.

In a space of dimension $d=2+\varepsilon$ ($\varepsilon<1$) such a solution is obtained in Sec. 4. In order to obtain it the spin matrices and integrals are continued to an odd number of dimensions.[15,16] To first order in ε the solution is obtained both by summing the highest order diagrams by means of the renormalization group,[17] and by solving the parquet equations. It is shown that for $\varepsilon>0$ the integrals in the parquet equations for this solution converge. Although this solution cannot be extended to $\varepsilon=0$, it serves as an illustration of the basic idea of this paper, which consists in the fact that the Hamiltonian renormalized in the usual sense may have a renormalizable finite solution.

In Secs. 5 and 7 we discuss the possibility of constructing a solution of the parquet equations in real four-dimensional space $d=4$. In Sec. 6 it is shown that for the existence of such a solution it is necessary that the number of interacting fields be larger than two, that the Hamiltonian contain neutral currents and that the weak interaction be constructed in an $SU(2)$-invariant manner with $G>0$. In Sec. 7 it is shown that a finite solution is obtained only for a definite relation between the coupling constants of the neutral currents of the type $(\bar{\nu}_\mu\nu_\mu)(\bar{e}e)$ and $(\bar{\mu}\mu)(\bar{e}e)$. The asymptotic values of the effective coupling constants are small, which allows one to neglect in the first approximation the so-called "non-parquet" terms.

Section 8 is devoted to an investigation of the equations for the fermion propagator, which must be solved together with the parquet equations.

The physical consequences of the obtained results are discussed in Sec. 9.

2. THE PARQUET EQUATIONS

We consider the Hamiltonian of the $SU(2)$-invariant interaction

$$H={}^1/_4F_0J_0J_0+{}^1/_4G_0\mathbf{J}\mathbf{J}, \tag{2}$$

where

$$J_0=\sum_{i=1}^{N}(\bar{\Psi}_i\Psi_i),\quad \mathbf{J}=\sum_{i=1}^{N}(\bar{\Psi}_i\boldsymbol{\tau}\Psi_i)$$

are respectively the isoscalar and isovector currents. Here $\boldsymbol{\tau}$ are the Pauli matrices acting on the doublets

$$\Psi_i=\begin{pmatrix}\nu_i\\ e_i\end{pmatrix},\quad i=1,\dots,N.$$

For brevity, the spin matrices O_α have been omitted, i.e., in fact

$$J_0=J_{0\alpha}=\sum_{i=1}^{N}(\bar{\Psi}_iO_\alpha\Psi_i)$$

etc. We show that the integral equations for an interaction of a simpler form (e.g., with $G_0=0$ or with $N=1$[1)]) do not have finite solutions.

The currents J_0 and $\mathbf{J}$ can also be written in the form $J_0=\bar{\Psi}\Psi$, $J=\bar{\Psi}\boldsymbol{\tau}\Psi$, where

$$\Psi=\begin{pmatrix}\nu\\ e\end{pmatrix},\quad \nu=\begin{pmatrix}\nu_1\\ \vdots\\ \nu_N\end{pmatrix},\quad e=\begin{pmatrix}e_1\\ \vdots\\ e_N\end{pmatrix}.$$

This is the form of the equations which we will use below.

The parquet equations have been derived in Ref. 6 in the following manner. We define the fermion-fermion scattering amplitude $\hat{T}$ as a sum of contributions of all connected Feynman graphs with four external lines. We split this amplitude into the parts $\hat{R}$, $\hat{\Phi}^s$, $\hat{\Phi}$, $\hat{\tilde{\Phi}}$ (Fig. 1):

$$\hat{T}=\hat{R}+\hat{\Phi}^s+\hat{\Phi}-\hat{\tilde{\Phi}}. \tag{3}$$

Here $\hat{\Phi}^s$ is the total contribution of all diagrams consisting of two blocks connected along the s-channel by two fermion lines (carrying the total momentum $p_s=p_1+p_2$). It satisfies an integral equation similar to the

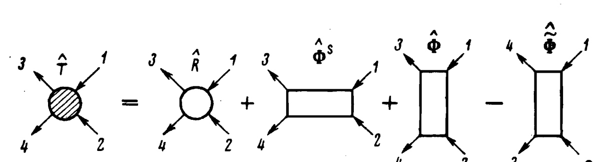


FIG. 1.

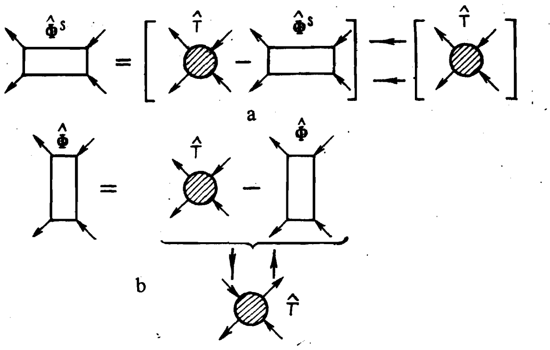


FIG. 2.

Bethe–Salpeter equation (Fig. 2a)

$$\Phi^s=\frac{1}{2}\int(\hat{T}-\Phi^s)S(k)S(p_s-k)\hat{T}\frac{d^4k}{(2\pi)^4i}, \tag{4}$$

where

$$S(p)=S_\nu(p)=S_e(p)=\beta(p^2)/\not{p}$$

is the Green's function (the Green's functions are the same for electrons and neutrinos, since we neglect the electron mass), and the factor $\frac{1}{2}$ arose out of the identity of the fermions.

In the same manner, denoting by $p_t=p_1-p_3$ the total t-channel momentum, we obtain (cf. Fig. 2b)

$$\Phi=-\int(\hat{T}-\Phi)S(k)\hat{T}S(k-p_t)\frac{d^4k}{(2\pi)^4i} \tag{5}$$

and $\hat{\bar{\Phi}}=(\hat{\Phi})_{1\rightleftarrows2}$ is the total contribution of the graphs consisting of two parts connected by two lines (a fermion and antifermion) respectively along the t- and u-channels.

Here $\hat{R}$ denotes the sum of all graphs which cannot be split into parts connected by only two particle lines. The simplest graphs of this type are shown in Fig. 3.

With the help of the identity $O_\alpha\gamma_\mu O_\beta=\chi_{\alpha\mu\beta\nu}O_\nu$, where

$$\chi_{\alpha\mu\beta\nu}=g_{\alpha\mu}g_{\beta\nu}+g_{\alpha\nu}g_{\beta\mu}-g_{\nu\mu}g_{\beta\alpha}-ie_{\alpha\mu\beta\nu}, \tag{6}$$

the contribution to $\hat{T}$ or $\hat{\Phi}^s$, $\hat{\Phi}$ of any graph can be written in the form[2)]

$$\begin{aligned}
\hat{T}&=-{}^1/_2\{\Gamma_{f\alpha\beta}(O_\alpha)_{31}(O_\beta)_{42}+\Gamma_{g\alpha\beta}(\tau O_\alpha)_{31}(\tau O_\beta)_{42}-(2\rightleftarrows1)\},\\
\hat{\Phi}^s&=-{}^1/_2\{\Phi^s_{f\alpha\beta}(O_\alpha)_{31}(O_\beta)_{42}+\Phi^s_{g\alpha\beta}(\tau O_\alpha)_{31}(\tau O_\beta)_{42}-(2\rightleftarrows1)\},\\
\hat{\Phi}&=-{}^1/_2\{\Phi^t_{f\alpha\beta}(O_\alpha)_{31}(O_\beta)_{42}-\Phi^u_{f\alpha\beta}(O_\alpha)_{41}(O_\beta)_{32}\\
&\quad+\Phi^t_{g\alpha\beta}(\tau O_\alpha)_{31}(\tau O_\beta)_{42}-\Phi^u_{g\alpha\beta}(\tau O_\alpha)_{41}(\tau O_\beta)_{32}\}.
\end{aligned} \tag{7}$$

Then Eq. (4) can be rewritten in the form of the following system of equations

$$\begin{aligned}
\Phi^s_{f\alpha\beta}&=-\frac{1}{2}\int\{[\Gamma_{f\gamma\delta}-\Phi^s_{f\gamma\delta}]\Gamma_{f\sigma\rho}+3[\Gamma_{g\gamma\delta}-\Phi^s_{g\gamma\delta}]\Gamma_{g\sigma\rho}\}\\
&\quad\times\chi_{\gamma\mu\sigma\alpha}\chi_{\delta\nu\rho\beta}\frac{k_\mu(p_s-k)_\nu}{k^2(p_s-k)^2}\beta(k^2)\beta((p_s-k)^2)\frac{d^4k}{(2\pi)^4i},\\
\Phi^s_{g\alpha\beta}&=-\frac{1}{2}\int\{[\Gamma_{f\gamma\delta}-\Phi^s_{f\gamma\delta}]\Gamma_{g\sigma\rho}+[\Gamma_{g\gamma\delta}-\Phi^s_{g\gamma\delta}]\Gamma_{f\sigma\rho}\\
&\quad-2[\Gamma_{g\gamma\delta}-\Phi^s_{g\gamma\delta}]\Gamma_{g\sigma\rho}\}\chi_{\gamma\mu\sigma\alpha}\chi_{\delta\nu\rho\beta}\frac{k_\mu(p_s-k)_\nu}{k^2(p_s-k)^2}\beta(k^2)\beta((p_s-k)^2)\frac{d^4k}{(2\pi)^4i}.
\end{aligned} \tag{8}$$

Similarly we obtain four other equations for Φ^t_f, Φ^u_f, Φ^t_g and Φ^u_g.

We shall look for an approximate solution of these equations, such that the integrals converge, but converge slowly, and in the integrals large integration momenta are important. In this case one may consider that $\Gamma_{f\alpha\beta}=\Gamma_f g_{\alpha\beta}$, where Γ_f is a function of a scalar argument (and similarly for $\Gamma_{g\alpha\beta}$ and all the quantities $\Phi_{\alpha\beta}$), since the integrals which are the coefficients of terms of the form $p_\alpha p_\beta$ converge faster and are consequently numerically small (cf. infra).

For the scalar functions Γ and Φ one can write the equations

$$\begin{aligned}
\Phi_f^s&=2\int\{[\Gamma_f-\Phi_f^s]\Gamma_f+3[\Gamma_g-\Phi_g^s]\Gamma_g\}\frac{\beta(k^2)\beta((p_s-k)^2)}{k^2(p_s-k)^2}\frac{d^4k}{(2\pi)^4i},\\
\Phi_g^s&=2\int\{[\Gamma_g-\Phi_g^s]\Gamma_f+[\Gamma_f-\Phi_f^s]\Gamma_g-\\
&\quad-2[\Gamma_g-\Phi_g^s]\Gamma_g\}\frac{\beta(k^2)\beta((p_s-k)^2)}{k^2(p_s-k)^2}\frac{d^4k}{(2\pi)^4i}.
\end{aligned} \tag{9}$$

The complete set of equations is listed in Appendix 1.

We shall show below that the nonparquet graphs of Fig. 3b, c give a small contribution, and can be neglected in the first approximation. Thus only the simplest graph of Fig. 3a contributes to $\hat{R}$:

$$\begin{aligned}
\hat{R}\approx\hat{R}_0=-{}^1/_2\{&F_0[(O_\alpha)_{31}(O_\alpha)_{42}-(O_\alpha)_{32}(O_\alpha)_{41}]\\
&-G_0[(\tau O_\alpha)_{31}(\tau O_\alpha)_{42}-(\tau O_\alpha)_{32}(\tau O_\alpha)_{41}]\}.
\end{aligned}$$

The equations (9) for Φ with $\hat{R}\approx\hat{R}_0$ are called the "parquet" equations. They form a closed system and for a given Green's function $S(p)=\beta(p^2)/\not{p}$ allow one to construct the vertices $\hat{T}$, i.e., the functions

$$\Gamma_f=F_0+\Phi_f^s+\Phi_f^t+\Phi_f^u,\quad \Gamma_g=G_0+\Phi_g^s+\Phi_g^t+\Phi_g^u. \tag{10}$$

In order to determine the functions $\beta(p^2)$ it is necessary to compete the system (9)–(10) with the Dyson–Schwinger equation (or the unitarity equation for $S^{-1}(p)$).

The main attention here is devoted to the analysis of the parquet equations (9)–(10), which contain the main difficulties. In the equation for β we restrict ourselves to simple estimates.

3. THE SCALE-INVARIANT SOLUTION AND EFFECTIVE COUPLING CONSTANTS

If the vertex Γ decreases in the region of large momenta, for $\bar{p}_t^2\to-p_t^2\to\infty$, guaranteeing the convergence of the integrals in (4), (5), then the bare values of the charges F_0 and G_0 vanish (otherwise Eqs. (10) would be violated as $\bar{p}_t^2\to\infty$). According to the general arguments of dimensional analysis[12–14] the system of equations

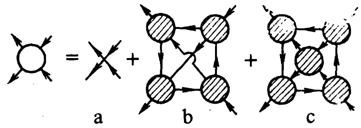


FIG. 3.

for Γ and β in the high-momentum region will under these conditions have a scale-invariant solution of the form

$$\Gamma_i(\tilde p_1^2, \tilde p_2^2, \tilde p_3^2, \tilde p_4^2, \tilde p_s^2, \tilde p_t^2) = (\tilde p_1^2)^{-\gamma}\lambda^{4\Delta} F_i(p_2^2/p_1^2, \dots, p_t^2/p_1^2),$$
$$\beta(\tilde p^2) = (\tilde p^2/\lambda^2)^{\Delta}, \tag{11}$$

where

$$\gamma = d/2 - 1 + 2\Delta, \quad i = f, g. \tag{12}$$

Here $d = 4$ is the dimensionality of space, Δ is a positive number by virtue of Lehmann's theorem, and λ^2 is a normalization momentum.

The strong coupling condition (12) reflects the fact that in the Feynman graphs to each new isoscalar and isovector vertex there corresponds a factor

$$f(\tilde p^2) = \Gamma_f(\tilde p^2)\beta^2(\tilde p^2)(\tilde p^2)^{d/2-1}, \qquad g(\tilde p^2) = \Gamma_g(\tilde p^2)\beta^2(\tilde p^2)(\tilde p^2)^{d/2-1}, \tag{13}$$

respectively, and called effective coupling constants. Here $\Gamma_i(\tilde p^2) = \Gamma_i(\tilde p^2, \dots, \tilde p^2)$ and the factor $(\tilde p^2)^{d/2-1}$ is a phase-space factor. The condition (12) denotes that the effective coupling constants for the scale-invariant solution are constants:

$$f = f_1 = F_f(1, \dots 1), \quad g = g_1 = F_g(1, \dots 1),$$

i.e., the contributions of all the graphs are quantities of the same order as $\tilde p_i^2 \to \infty$.

4. FERMION INTERACTIONS IN A SPACE OF DIMENSION $2+\varepsilon$

A. The method of analytic continuation in d

In a space of dimension $d = 2$ perturbation theory leads to logarithmic divergences, and the problem of the four-fermion interaction can be solved by means of summing the leading logarithmic terms. Therefore, in order to understand the general properties of the solution of the above equations in a real four-dimensional space we first consider the case of a nonintegral dimensionality $d = 2 + \varepsilon$. Here for small $\varepsilon \ll 1$ the solution can be obtained in closed form by summing the highest terms in ε in perturbation theory.

In second order of perturbation theory the amplitude for fermion interactions is determined by the graphs of Fig. 4. Their contributions contain products of the γ-matrices ordered according to the continuous lines in Fig. 4. In analytically continuing these products there appears a difficulty[15] related to the fact that the quantity $e_{\alpha\beta\gamma\delta}$ cannot be continued to nonintegral d. However the contributions of the graphs contain only products of pairs of the quantities $e_{\alpha\beta\gamma\delta}$ which are tensors (and not pseudotensors as are the $e_{\alpha\beta\gamma\delta}$ for $d = 4$) which by virtue of their antisymmetry and covariance can be represented in the form

$$e_{\alpha\mu\beta\rho}e_{\alpha\nu\beta\sigma} = -f(d)(d-2)(d-3)[g_{\mu\nu}g_{\rho\sigma} - g_{\rho\nu}g_{\mu\sigma}],$$

where $f(4) = 1$ and $f(2) \neq \infty$ (the latter owing to the fact that in two-dimensional space $e_{\alpha\beta\gamma\delta} = 0$). In the contributions of the graphs of Fig. 4 this quantity enters multiplied by terms of the order $\varepsilon = d - 2$ or even of order ε^2. Therefore in first order in ε the tensor does not contribute and in the correction of the next order depends on the choice of the function $f(d)$.

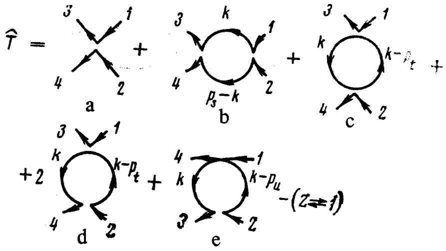


FIG. 4.

B. Perturbation theory and the Gell-Mann-Low equations

For the construction of the amplitudes Γ_f and Γ_g for $\varepsilon < 1$ we use the method of ε-expansion[18] and the technique of the renormalization group. We consider the interaction (2) with $G_0 > 0$. The corresponding two-dimensional model has an asymptotically free solution.[19] As will be seen, in a space of $d = 2 + \varepsilon$ dimensions it leads to a theory with a "fixed point" with numerically small effective coupling constants $g_1 \sim \varepsilon$, $f_1 \sim \varepsilon^2$. All quantities can be found in the form of a series in powers of ε.

We first consider the amplitudes Γ_f and Γ_g in perturbation theory. In second order in the coupling constants F_0 and G_0 for $\varepsilon \to 0$ the most important are the diagrams of Fig. 4b and e, since they contain ε in the denominator, in distinction of the graphs of Fig. 4c, d, where ε cancels in their denominators (cf. infra).

For $d = 2$, i.e., $\varepsilon = 0$, the contributions of Fig. 4b and e to Γ_f differ only by sign and by the fact that the first depends on p_s^2 whereas the second depends on p_u^2. This leads to a cancellation of divergences in their sum, analogous to the cancellation in the Thirring model.[3)] The corresponding two-dimensional model (with $G_0 = 0$) is scale-invariant for arbitrary F_0. It cannot be investigated by means of the ε-expansion method used in this section.

We consider the contribution of the diagram of Fig. 4b:

$$\Gamma^{(b)}_{f\alpha\beta} = \frac{F_0^2 + 3G_0^2}{2}\chi_{\alpha\mu\beta\lambda}\chi_{\alpha\nu\beta\rho}\int \frac{k_\mu(k-p_s)_\nu}{k^2(k-p_s)^2}\frac{d^dk}{(2\pi)^d i}, \tag{14}$$

with its contribution to Γ_g (i.e., $\Gamma_g^{(b)}$) differing only by the substitution of $G_0^2 - G_0F_0$ for $(F_0^2 + 3G_0^2)/2$. The contribution of the diagram of Fig. 4e to Γ_f and Γ_g is given by the same expression with p_s replaced by p_u and the appropriate sign change, or F_0 replaced by $-F_0$ (the isospin coefficients are given in Appendix 1). The tensor $\chi_{\alpha\mu\beta\lambda}$ is defined in (6), and for nonintegral d the normalization of the Feynman integrals is chosen such that

$$\int \varphi(-k^2)\frac{d^dk}{(2\pi)^d i} = \int_0^{\Lambda^2} \varphi(\bar k^2)\frac{(\bar k^2)^{d/2-1}d\bar k^2}{(4\pi)^{d/2}\Gamma(d/2)},$$

where $\Lambda^2 \to \infty$ is the cutoff momentum.

Calculating the integral by means of the standard method[15,16] we obtain

$$\Gamma^{(b)}_{f\alpha\beta} = \Gamma^{(b)}_{f} g_{\alpha\beta}, \quad \Gamma^{(b)}_{g\alpha\beta} = \Gamma^{(b)}_{g} g_{\alpha\beta},$$

$$\Gamma^{(b)}_{f} = \frac{F_0^2 + 3G_0^2}{4\pi} \frac{(\tilde{p}_s^2)^{\varepsilon/2} - \Lambda^\varepsilon}{\varepsilon/2} + O(\varepsilon^2),$$

where the contributions $\Gamma^{(b)}_{g}$, $\Gamma^{(e)}_{f}$ and $\Gamma^{(e)}_{g}$ are obtained from here by means of the replacement of the isospin structure. Here $O(\varepsilon^2)$ denotes the contributions of the higher-order terms in ε (some of them are related to the contribution of the tensor $e_{\alpha\mu\beta\rho}e_{\alpha\nu\beta\sigma}$). Therefore, not taking into account the contributions of the graphs of Fig. 4c, d, we obtain in highest order in

$$\begin{aligned}
\Gamma_f &= F_0 + \frac{1}{4\pi}(F_0^2 + 3G_0^2)\frac{(\tilde{p}_s^2)^{\varepsilon/2} - (\tilde{p}_u^2)^{\varepsilon/2}}{\varepsilon/2}, \\
\Gamma_g &= G_0 - \frac{(G_0^2 - G_0F_0)}{2\pi}\frac{(\tilde{p}_s^2)^{\varepsilon/2} - \Lambda^\varepsilon}{\varepsilon/2} - \frac{(G_0^2 + F_0G_0)}{2\pi}\frac{(\tilde{p}_u^2)^{\varepsilon/2} - \Lambda^\varepsilon}{\varepsilon/2}
\end{aligned} \tag{15}$$

Let us show that the contributions of the graphs of Fig. 4c, d are indeed small. For the graph of Fig. 4c we have

$$\begin{aligned}
\Gamma^{(c)}_{f\alpha\beta} &= F_0^2 N \operatorname{Tr}(O_\alpha\gamma_\mu O_\beta\gamma_\nu)\int \frac{k_\mu(k-p_t)_\nu}{k^2(k-p_t)^2}\frac{d^dk}{(2\pi)^d i} \\
&= -F_0^2(N\operatorname{Tr}1)\frac{2\Gamma(2-d/2)\Gamma^2(d/2)}{(4\pi)^{d/2}\Gamma(d)}(\tilde{p}_t^2)^{\varepsilon/2}\left(g_{\alpha\beta} - \frac{p_\alpha^t p_\beta^t}{p_t^2}\right),
\end{aligned}$$

where $p^t = p_1 - p_3$ and $\operatorname{Tr} O_\mu\gamma_\mu = g_{\mu\nu}\operatorname{Tr}1$, where $\operatorname{Tr}1 = 2$ for $d = 4$. Here a cancellation of singularities has occurred for $d = 2$ (i.e., the quantities ε in the denominator cancel) and the matrix element has become transverse.[15] A similar cancellation occurs in $\Gamma^{(c)}_{g\alpha\beta}$ (which differs from $\Gamma^{(c)}_{f\alpha\beta}$ by the substitution $F_0^2 \to G_0^2$) and also in the graph of Fig. 4d. Its contribution is

$$\Gamma^{(d)}_{f\alpha\beta} = \left(1 + \frac{3G_0}{F_0}\right)\frac{d-2}{2N\operatorname{Tr}1}\Gamma^{(c)}_{f\alpha\beta},$$

i.e., is of even higher order in ε. The same is true of the graph $\Gamma^{(d)}_{g\alpha\beta}$.

For the sequel it is convenient to introduce the dimensionless coupling constant: bare

$$f_0 = \Lambda^\varepsilon F_0/(4\pi)^{d/2}\Gamma(d/2), \quad g_0 = \Lambda^\varepsilon G_0/(4\pi)^{d/2}\Gamma(d/2)$$

and renormalized

$$f_c = \lambda^\varepsilon F_c/(4\pi)^{d/2}\Gamma(d/2), \quad g_c = \lambda^\varepsilon G_c/(4\pi)^{d/2}\Gamma(d/2);$$

here λ, a quantity of the order of the particle masses, is the normalization momentum. The coupling constants are equal to the values (13) of the effective coupling constants for $\tilde{p}^2 = \Lambda^2$ or $\tilde{p}^2 = \lambda^2$, respectively.

We shall assume that f_0, $g_0(f_c g_c) \ll 1$, but that

$$f_0\frac{(\tilde{p}^2/\Lambda^2)^{\varepsilon/2} - 1}{\varepsilon/2}, \quad g_0\frac{(\tilde{p}^2/\Lambda^2)^{\varepsilon/2} - 1}{\varepsilon/2} \sim 1$$

(or that $2f_c[(\tilde{p}^2/\lambda^2)^{\varepsilon/2} - 1]/\varepsilon \sim 1$, $2g_c[(\tilde{p}^2/\lambda^2)^{\varepsilon/2} - 1]/\varepsilon \sim 1$) and sum the leading terms by means of the renormalization group.[17] Since in the approximation under discussion the renormalization of the wave function is absent (it is determined by the diagrams of Fig. 5 and higher-order diagrams which yield a contribution of order $g_1^2 \sim \varepsilon^2$), i.e., $\beta = 1$, the effective coupling constants are expressed in terms of the vertices Γ_f and Γ_g. Making use of their values (15) for $\tilde{p}_s^2 = \tilde{p}_u^2 = \tilde{p}^2$, we obtain, differentiating (13) with respect to $\ln(\tilde{p}^2/\lambda^2)$ the following system of Gell-Mann–Low equations[17]:

$$\frac{df}{d\ln(\tilde{p}^2/\lambda^2)} = \frac{\varepsilon}{2}f + \frac{3}{2}\varepsilon g^2, \quad \frac{dg}{d\ln(\tilde{p}^2/\lambda^2)} = \frac{\varepsilon}{2}g - 4g^2, \tag{16}$$

where $(\frac{3}{2})\varepsilon g^2$ is a correction of order ε^3 corresponding to the higher order terms which have not been written in (15)[4)] (in ε), terms which have the form

$$\frac{3}{8\pi}G_0^2\varepsilon\frac{(\tilde{p}^2)^{\varepsilon/2} - \Lambda^\varepsilon}{\varepsilon/2},$$

for $\tilde{p}_s^2 = \tilde{p}_u^2 = \tilde{p}^2$; they are obtained for $f(d) \equiv 1$.

The solution of the second equation is

$$g(\tilde{p}^2) = g_c\left(\frac{\tilde{p}^2}{\lambda^2}\right)^{\varepsilon/2} \Big/ \left\{1 + \frac{g_c}{g_1}\left[\left(\frac{\tilde{p}^2}{\lambda^2}\right)^{\varepsilon/2} - 1\right]\right\}. \tag{17}$$

where $g_1 = \varepsilon/8$ is the limit to which $g(\tilde{p}^2)$ converges for $\tilde{p}^2 \to \infty$ if $g_c > 0$. With the choice $g_c = g_1$, or for any other $g_c > 0$, when

$$\frac{g_c}{g_1}\left(\frac{\tilde{p}^2}{\lambda^2}\right)^{\varepsilon/2} \gg 1,$$

the renormalization group equations have a scale-invariant solution.

The solution of the equation for f

$$f(\tilde{p}^2) = {}^3/_8\varepsilon g(\tilde{p}^2) + (f_c - {}^3/_8\varepsilon g_1)(\tilde{p}^2/\lambda^2)^{\varepsilon/2}$$

converges for $\tilde{p}^2 \to \infty$ to the constant value $f \to (\frac{3}{8})\varepsilon g_1$ (i.e., is stable) only in the case when the second term in it vanishes, i.e., when the physical values of the coupling constants are related by $F_c = (\frac{3}{8})\varepsilon G_c$. We shall discuss the meaning of this relation in Sec. 6.

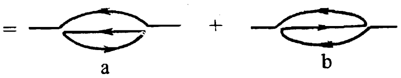
$\hat{\Sigma} =$ 

FIG. 5.

C. Anomalous dimensions

We now determine the dimensions Δ of the field Ψ in the scale-invariant theory. The second order diagrams for $\Sigma = (1 - \beta^{-1}(p^2))\not{p}$ are represented in Fig. 5. Calculating according to the method described above we find that the contribution of the graph of Fig. 5b does not contain terms of the order ε, and the contribution of the graph of Fig. 5a is[5)]

$$\hat{\Sigma} = 2(f_c^2 + 3g_c^2)\frac{(\tilde{p}^2/\lambda^2)^\varepsilon - 1}{\varepsilon}n\not{p}, \quad n = N\frac{\operatorname{Tr}1}{2}.$$

The renormalization group equations[17] for β have the form

$$\frac{d\ln\beta}{d\ln(\tilde{p}^2/\lambda^2)}=2(f^2+3g^2)n.$$

According to what was said above, for large $\tilde{p}^2$ one may assume that $g^2=g_1^2$, $f^2=f_1^2$. Then

$$\beta(\tilde{p}^2)=(\tilde{p}^2/\lambda^2)^\Delta,\quad \Delta={}^3/_{32}n\varepsilon^2,$$

where $\Delta>0$, in agreement with Lehmann's theorem. Thus, in units of length, the dimension of the field is

$$[\Psi(x)]=-({}^1/_2+{}^1/_2\varepsilon+{}^3/_{32}n\varepsilon^2).$$

The strong coupling condition (12) to order ε^2 yields for the index γ of the vertex the equation

$$\gamma={}^1/_2\varepsilon+{}^3/_{16}\varepsilon^2 n.$$

D. The solution of the parquet equations

We find the solution of the system of parquet equations for Γ_f and Γ_g in a $d=2+\varepsilon$-dimensional space. For equal and large momenta $\tilde{p}_i^2=\tilde{p}^2$ this solution must correspond to Eqs. (17) and (13) for $\beta=1$, i.e., it must yield

$$\Gamma_f=f_1/(\tilde{p}^2)^{\varepsilon/2},\quad \Gamma_g=g_1/(\tilde{p}^2)^{\varepsilon/2},\quad g_1=\varepsilon/8,\quad f_1=3(\varepsilon/8)^2.$$

To first order in ε the system of equations for Γ_f and Γ_g becomes substantially simpler and can be solved by methods applicable in the logarithmic case.[7,9,20] The first of the simplification consists in the fact that Φ^s depends under these conditions only on three variables

$$\xi=\max\{\tilde{p}_1^2,\ \tilde{p}_2^2,\ \tilde{p}_s^2\},\quad \zeta=\max\{\tilde{p}_3^2,\ \tilde{p}_4^2,\ \tilde{p}_s^2\},\quad \eta=\tilde{p}_s^2. \tag{18}$$

One may consider, without loss of generality, that the momenta p_1 and p_2 are large, i.e., $\xi>\zeta\gtrsim\eta$. If the momenta p_1 and p_3 (or p_1 and p_4) are large, then $\xi\sim\zeta\sim\eta$ and Φ^s will depend on one variable, which is confirmed by perturbation theory calculations.

The system of equations for spacelike values of the momenta is closed and does not require knowledge of the solution in other regions for its solution. We carry out a Wick rotation $k_0\to ik_0$[8] thus passing to an Euclidean metric. We consider the vertex for $\xi\gtrsim\eta\sim\zeta$ since such a region will suffice for our purposes.[7] We introduce the notation

$$x=\xi^{\varepsilon/2},\quad y=\eta^{\varepsilon/2},\quad z=(k^2)^{\varepsilon/2}.$$

Then the system of parquet equations can be written in the form

$$\begin{aligned}\Phi_f^s(x,y)=-\frac{2}{\varepsilon}\int_y^x\{[\Gamma_f(x)-\Phi_f^s(x)]\Gamma_f^s(z,y)\\+3[\Gamma_g(x)-\Phi_g^s(x)]\Gamma_g^s(z,y)\}dz-\frac{2}{\varepsilon}\int_x^\infty\{[\Gamma_f(z)-\Phi_f^s(z)]\Gamma_f^s(z,y)\\+3[\Gamma_g(z)-\Phi_g^s(z)]\Gamma_g^s(z,y)\}\,dz.\end{aligned} \tag{19}$$

The lower limit has been chosen in such a manner since only this region contributes to the first order in ε. In this order there are similar simplifications in the other equations (cf. Appendix 1).

The solving of a system of the type (19) is difficult,[17] owing to the fact that the function $\Phi(x)=\Phi(x,x)$ can only be defined if the quantity $\Phi(x,y)$ is known, and the latter depends on two variables. A further simplification occurs if in place of this system one solves the system of Sudakov equations,[7,20,14] which is equivalent to (19). The Sudakov equations are obtained by cutting the reducible graphs for T, i.e., graphs consisting of two parts connected by two particle lines only (cf. Sec. 2) along the smallest momentum. For our case these equations have the following form:

$$\begin{aligned}\Phi_f^s(x,y)&=-\frac{6}{\varepsilon}\int_y^x\Gamma_g^s(x,z)\Gamma_g(z)dz-\frac{6}{\varepsilon}\int_x^\infty\Gamma_g^2(z)dz,\\ \Phi_g^s(x,y)&=-\frac{2}{\varepsilon}\int_y^x[\Gamma_f^s(x,z)-2\Gamma_g^s(x,z)]\Gamma_g(z)dz+\frac{4}{\varepsilon}\int_x^\infty\Gamma_g^2(z)dz,\\ \Phi_f^u(x^u,y^u)&=-\frac{6}{\varepsilon}\int_{y^u}^{x^u}\Gamma_g^u(x^u,z)\Gamma_g(z)dz-\frac{6}{\varepsilon}\int_{x^u}^\infty\Gamma_g^2(z)dz,\\ \Phi_g^u(x^u,y^u)&=-\frac{2}{\varepsilon}\int_{y^u}^{x^u}[\Gamma_f^u(x^u,z)+2\Gamma_g^u(x^u,z)]\Gamma_g(z)dz-\frac{4}{\varepsilon}\int_{x^u}^\infty\Gamma_g^2(z)dz.\end{aligned} \tag{20}$$

Here $\Gamma_f^s(x,y)$ or $\Gamma_g^u(x^u,y^u)$ are the vertices when the momenta p_1 and p_2, or p_1 and p_4, respectively, are large. They are defined as follows (cf. for more details see Appendix 1):

$$\Gamma_f^s(x,y)=\Phi_f^s(x,y)+\Phi_f^u(x),\quad \Gamma_g^s(x,y)=g_0/\Lambda^\varepsilon+\Phi_g^s(x,y)+\Phi_g^u(x),$$
$$\Gamma_f^u(x^u,y^u)=\Phi_f^s(x^u)+\Phi_f^u(x^u,y^u),$$
$$\Gamma_g^u(x^u,y^u)=g_0/\Lambda^\varepsilon+\Phi_g^s(x)+\Phi_g^u(x^u,y^u),$$

where

$$x^u=(\xi^u)^{\varepsilon/2},\quad \xi^u=\max\{\tilde{p}_1^2,\ \tilde{p}_4^2,\ \tilde{p}_u^2\},\quad y^u=(\tilde{p}_u^2)^{\varepsilon/2}$$

are quantities analogous to (18) for the u-channel. We consider that $f_0=0$, since it was shown that $f_0=O(\varepsilon^2)$.

Adding, we obtain for $\Gamma_f^s(x,y)$ and $\Gamma_g^s(x,y)$ the equations

$$\begin{aligned}\Gamma_f^s(x,y)&=-\frac{6}{\varepsilon}\int_y^x\Gamma_g^s(x,z)\Gamma_g(z)dz,\\ \Gamma_g^s(x,y)&=-\frac{2}{\varepsilon}\int_y^x[\Gamma_f^s(x,z)-2\Gamma_g^s(x,z)]\Gamma_g(z)dz+\frac{8}{\varepsilon}\int_x^\infty\Gamma_g^2(z)dz.\end{aligned} \tag{21}$$

Let us first find the solution for $x=y$. Differentiating with respect to x we obtain $\Gamma_g(x)=g_1/x_1$, $\Gamma_f(x)=0$. This solution coincides of course, with the answer one gets from the use of the renormalization group, (17). Differentiating (21) with respect to y we obtain

$$\begin{aligned}\Gamma_f^s(x,y)&=-{}^3/_4g_1[x^{-1/4}y^{-3/4}-y^{1/4}x^{-5/4}],\\ \Gamma_g^s(x,y)&={}^3/_4g_1x^{-1/4}y^{-3/4}+{}^1/_4g_1y^{1/4}x^{-5/4}.\end{aligned} \tag{22}$$

Similarly one can determine the vertex in the other region of momentum values, when p_1 and p_4 are large; here (cf. (22))

$$\Gamma_g^u = \Gamma_g^s(x^u, y^u), \quad \Gamma_f^u = -\Gamma_f^s(x^u, y^u).$$

We note that

$$\Gamma_f^s - 3\Gamma_g^s = -3g_1 x^{-1/4} y^{-3/4}, \quad \Gamma_f^s + \Gamma_g^s = g_1 y^{1/4} x^{-5/4}.$$

these combinations correspond to the state when the particles have in the s-channel a definite total isospin: 0 or 1 (for more details, cf. Sec. 7).

Finally, we determine the functions (x) which we shall also need in the sequel. Substituting $\Gamma_g(x) = g_1/x$ into (20) we obtain

$$\Phi_f^s(x) = -3g_1/4x, \quad \Phi_g^s(x) = g_1/2x. \tag{23}$$

E. Convergence of the integrals in the parquet equations for $\varepsilon > 0$

We substitute the solution (22), (23) obtained by means of the Sudakov equations into the system of parquet equations (19) and show that it guarantees the convergence of the integrals. In distinction from the Sudakov equations, which are valid only to first order in ε, the equations (19) may serve for $\varepsilon \to 2$ as a good model of the real four-dimensional equations. We show that if the vertex decreases as a function of the largest momentum, as in (22), then all integrals turn out to be convergent. For the four-dimensional space such a discussion is carried out in Sec. 7.

Substituting the solutions (22), (23) into the right-hand side of (19) yields

$$\Phi_f^s(x) = -\frac{3g_1}{2^5} \int_x^\infty \left\{ \frac{3}{x^{1/4} z^{3/4}} + \frac{5x^{1/4}}{z^{5/4}} \right\} \frac{dz}{z} = -\frac{3}{4} \frac{g_1}{x}. \tag{24}$$

As can be seen, the integrals converge in the region of large momenta $z \to \infty$, and the expression obtained coincides with Φ_f^s from (23). Similar considerations apply also to the other equations of the system (19).

5. SOLUTION OF THE FOUR-DIMENSIONAL EQUATIONS BY MEANS OF THE ε-EXPANSION

In this and the following two sections we shall find an approximate solution of the system (9) of four-dimensional equations. We are looking for a solution for which

a) the integrals converge slowly, and large integration momenta are important in them,

b) in the zeroth approximation the Φ^s depend on the variables ξ, η, ζ (cf. (18)), but not on ratios of the type p_1^2/p_2^2.

As we have seen in Sec. 4, both these properties are true in a space dimension $d = 2 + \varepsilon$ in first order in ε. Therefore, if a solution of this type exists, it can be found by means of the method of ε-expansion in the version described below.

The transition to a nonintegral dimension d occurs, as we have seen, in the system of integral equations (A.1), written out in Appendix 1. At $d = 4$ these equations coincide with the system (9) and at $d \neq 4$ they differ from it, first, by the factors $a(d)$, $b(d)$, $c(d)$ appearing in the right-hand sides of (A.1) as a result of the elimination of the spin structure, and, second, by the fact that the scalar integrals are defined in a space of $d = 2 + \varepsilon$ dimensions. In the two-dimensional case, when $d = 2$ and $a(2) = b(2) = 1$, $c(2) = 0$, the equations (A.1) coincide with the system (19) considered above.

We consider the system (A.1) for the real four-dimensional case $d = 4$, when $a(4) = 2$, $b(4) = c(4) = \frac{1}{2}$, but we leave the scalar integrals defined for nonintegral d. Let us attempt to solve it in the following manner. We formally find its solution by means of the ε-expansion,[6] but make use of it for $\varepsilon = 2$. Substituting this solution into the right-hand side of the equations, we show that owing to the small numerical factors large momenta are important in the integrals even for $\varepsilon = 2$, i.e., the property (a) is satisfied. Therefore the solution will approximately satisfy[7] the system (A.1) also in the realistic four-dimensional case.

In order to determine in first order in ε the solutions of the system (A.1) with $a = 2$, $b = c = \frac{1}{2}$ and with scalar integrals defined for $d = 2 + \varepsilon$ it is again convenient to go over to the appropriate equations of the Sudakov type. This yields (in place of (21)):

$$\begin{aligned} \Gamma_f^s(x, y) &= \frac{f_1}{\lambda^\varepsilon} - \frac{4}{\varepsilon} \int_y^x [\Gamma_f^s(x, z)\Gamma_f(z) + 3\Gamma_g^s(x, z)\Gamma_g(z)]\,dz \\ &+ \frac{1}{\varepsilon} \int_x^\infty [(2N-1)\Gamma_f^2(z) + 6\Gamma_f(z)\Gamma_g(z) - 9\Gamma_g^2(z)]\,dz, \\ \Gamma_g^s(x, y) &= \frac{g_1}{\lambda^\varepsilon} - \frac{4}{\varepsilon} \int_y^x [\Gamma_f^s(x, y)\Gamma_g(z) + \Gamma_g^s(x, z)\Gamma_f(z) \\ &- 2\Gamma_g^s(x, y)\Gamma_g(z)]\,dz + \frac{2}{\varepsilon} \int_x^\infty [(N+4)\Gamma_g^2(z) - 2\Gamma_f(z)\Gamma_g(z)]\,dz, \end{aligned} \tag{25}$$

where N is the number of doublets

$$\Psi_i = \begin{pmatrix} \nu_i \\ e_i \end{pmatrix}$$

of fermion fields and $x = \xi^{\varepsilon/2}$, $y = \eta^{\varepsilon/2}$ and $z = (k^2)^{\varepsilon/2}$ are the same variables as in (19).

In the equations (25) the Green's functions have not been taken into account (cf. (11)), i.e., one assumes that the terms of order Δ may be neglected in the zeroth approximation. However, taking Δ into account leads only to the substitution $\varepsilon \to \varepsilon + 4\Delta$ in the equations (25). For more details, cf. Sec. 8.

Let us first determine the solution of the system (25) first for $x = y$. Differentiating, we obtain for the effective coupling constants $f(x) = \Gamma_f(x)x$ and $g(x) = x\Gamma_g(x)$ the system of Gell-Mann–Low equations

$$\begin{aligned} \frac{df}{d\ln(\xi/\lambda^2)} &= \frac{\varepsilon}{2} f - f^2\left(N - \frac{1}{2}\right) + \frac{9}{2} g^2 - 3fg, \\ \frac{dg}{d\ln(\xi/\lambda^2)} &= \frac{\varepsilon}{2} g - g^2(N+4) + 2fg. \end{aligned} \tag{26}$$

We discuss the solution of this system in the next section.

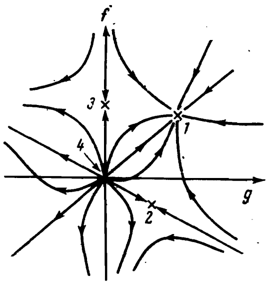


FIG. 6.

6. STABILITY AND SYMMETRY

The phase plane of the system of Gell-Mann–Low equations (26) is represented in Fig. 6. The arrows indicate the "flow" of the effective coupling constants as the momentum increases. The system has "fixed" points situated on the singular solutions $f = a^{(1,2)}g$ and $g = 0$, where

$$a^{(1,2)} = \frac{N+1 \pm (N^2+20N+28)^{1/2}}{2N+3}.$$

Their coordinates are

$$g_1^{(1,2)} = \varepsilon/2(N+4-2a^{(1,2)}),$$
$$f_1^{(1,2)} = a^{(1,2)} g_1^{(1,2)},$$
$$g_1^{(3)} = 0, \quad f_1^{(3)} = \varepsilon/(2N-1),$$
$$g_1^{(4)} = f_1^{(4)} = 0.$$

The first and fourth are respectively a stable and unstable node. The two others are saddle-points and are stable, in the sense that if the physical values of the coupling constants are situated on the singular solution for $g > 0$ for the second point and $f > 0$ for the third, then with the increase of momentum the effective coupling constants will tend to constant values.

The presence of the saddle points in the phase plane allows one to fix the form of the interaction. If one breaks the $SU(2)$-symmetry of the Hamiltonian (2) by setting

$$H = {}^1/_4 F_0 J_0 J_0 + {}^1/_4 G_{z0} J_z J_z + G_0 J_+ J_-,$$

then in the GG_z-plane there will appear a singular point of the saddle type. The corresponding singular solution is $G = G_z$, i.e., corresponds to the $SU(2)$-symmetric form of the Hamiltonian H (cf. Appendix 3). Similarly, an attempt to write the isoscalar current in the nonuniversal form, e.g.,

$$J_0 = \sum_{i=1}^{N} c_i \bar{\Psi}_i \Psi_i$$

with $c_i \neq 1$, also does not lead to a stable solution.

7. THE DEPENDENCE OF THE VERTEX ON MOMENTA OF DIFFERENT ORDERS

From (25) we determine the function $\Gamma_i(x, y)$. Knowing that for $x = y$

$$\Gamma_f(x) = f_1/x, \quad \Gamma_g = g_1/x,$$

where f_1 and g_1 are the asymptotic values of the effective coupling constants, we obtain

$$\begin{aligned} \Gamma_f^s(x, y) &= -{}^1/_{16}\varepsilon[(1-\delta_0^s)x^{-\delta_0^s}y^{-1+\delta_0^s} + 3(1-\delta_1^s)x^{-\delta_1^s}y^{-1+\delta_1^s}], \\ \Gamma_g^s(x, y) &= {}^1/_{16}\varepsilon[(1-\delta_0^s)x^{-\delta_0^s}y^{-1+\delta_0^s} - (1-\delta_1^s)x^{-\delta_1^s}y^{-1+\delta_1^s}]. \end{aligned} \tag{27}$$

Here $\delta_0^s = 1 + 4(f_1 - 3g_1)/\varepsilon$ and $\delta_1^s = 1 + 4(f_1 + g_1)/\varepsilon$ are numbers. Substituting the quantities f_1 and g_1 obtained above leads to values which do not depend on ε [8]:

$$\delta_0^s = \frac{N-2}{N+4-2a}, \quad \delta_1^s = \frac{N+6}{N+4-2a}. \tag{28}$$

Here, e.g., for the second fixed point

$$a = \frac{N+1-(N^2+20N+28)^{1/2}}{2N+3}.$$

We note that for $N > 2$ both quantities δ_0^s and δ_1^s are positive.

In Appendix 2 one can find the vertex in the case when both momenta p_1 and p_3 (or p_1 and p_4) are large. For $\Phi^s(x)$ we obtain

$$\Phi_f^s(x) = -\frac{4}{\varepsilon x}(f_1^2 + 3g_1^2), \quad \Phi_g^s(x) = \frac{8}{\varepsilon x}(g_1 - f_1)g_1. \tag{29}$$

We now substitute the quantities (27), (28) into the system of parquet equations, similar to what was done in Sec. 4 (subsection E) and convince ourselves that all integrals converge and that the solution can be extended to $\varepsilon = 2$ in the four-dimensional space. At the same time we show that the solution (27), (29) obtained by means of the Sudakov equation satisfies the system (A.1).

The system (A.1) has the property that it relates the vertices Γ_f and Γ_g (which have definite t-channel isospin). This substantially complicates the solution. In order to determine Γ^s and Γ^u it is more convenient to consider vertices which have definite isospins respectively in the s- and u-channels. They are related to the vertices Γ_f and Γ_g by the following relations:

$$\begin{aligned} \Gamma_0^s &= \Gamma_f^s - 3\Gamma_g^s, \quad \Gamma_0^u = \Gamma_f^u + 3\Gamma_g^u, \\ \Gamma_1^s &= \Gamma_f^s + \Gamma_g^s, \quad \Gamma_1^u = \Gamma_f^u - \Gamma_g^u. \end{aligned} \tag{30}$$

Since the interaction (2) conserves isospin these vertices are expressed in the parquet equations only in terms of themselves. For the t-channel a similar property is exhibited by

$$\Gamma_0^t = 2\Gamma_f^t + \Gamma_0^u/N, \quad \Gamma_1^t = 2\Gamma_g^t + \Gamma_1^u/N. \tag{30'}$$

We introduce the notation Γ_T^i, where $T = 0$, 1 is the value of the isospin in channel i, $i = s$, t, u. Defining Γ_T^i with the help of equations analogous to (30), (30′), after some algebraic transformations the system (A.1) can be rewritten in the form

$$\begin{aligned} \Phi_T^i(x, y) = \frac{2c^i}{\varepsilon}\Big\{ &\int_y^x [\Gamma_T^i(x) - \Phi_T^i(x)]\Gamma_T^i(z, y)\,dz \\ &+ \int_x^\infty [\Gamma_T^i(z) - \Phi_T^i(z)]\Gamma_T^i(z, y)\,dz \Big\}. \end{aligned} \tag{31}$$

The values of the coefficients $c^s=-2$, $c^u=\frac{1}{2}$, $c^t=N/2$ correspond to the transformations of the spin structures respectively in the diagrams b, e, c of Fig. 4.

The solution (27), (29), (A.2), (A.4) obtained by means of the Sudakov equations can be written in a uniform way

$$\Gamma_T{}^i(x,y)=\frac{\varepsilon}{2c^i}(1-\delta_T{}^i)x^{-\delta_T{}^i}y^{\delta_T{}^i-1},\qquad \Phi_T{}^i(x)=\frac{\varepsilon}{2c_i}(1-\delta_T{}^i)^2\frac{1}{x},\tag{32}$$

where δ_T^i are expressed by the Eqs. (28) and (A.3). Substitution of this solution into (31) yields

$$\Phi_T{}^i(x,y)=\frac{\varepsilon}{2c^i}(1-\delta_T{}^i)^2\delta_T{}^i\left\{\frac{1}{x}\int_y^x dz\, z^{-\delta_T{}^i}y^{-(1-\delta_T{}^i)}+\int_x^\infty dz\, z^{-(1+\delta_T{}^i)}y^{-(1-\delta_T{}^i)}\right\}.\tag{33}$$

If $\delta_T^i>0$ the integral converges. It can be seen from Eqs. (28), (A.3) that all $\delta_T^i>0$, except

$$\delta_0{}^t=(2N+5)(1-a)/2(N+4-2a);$$

the latter is positive in the second fixed point and is negative in the other two, since in them $a>1$. Therefore only the second point corresponds to a finite scale-invariant solution. As has been seen in Sec. 6, this is possible only if the physical values of the coupling constants are related by $F_c=aG_c$. It is this solution which will be considered below.

Numerically the quantity δ_0^s turns out to be small. For example, for $N=3$, or $N=4$ δ_0^s is respectively 0.12 or 0.22. This allows one to extend the solution to $\varepsilon=2$, i.e., to the real four-dimensional space. Indeed the contribution to Φ_0^s from the region of small integration momenta $k^2<x$, where it is essential to assume that the solution has the property (b) (cf. Sec. 5), contains the small quantity δ_0^s, since this contribution is of order of unity, whereas the contribution from the region $k^2>x$ is of the order $1/\delta_0^s$. The contribution from the region $k^2<x$ to the other Φ_T^i is not small compared to the contribution from the region $k^2>x$), but therefore the Φ_T^i themselves are small compared to Φ_0^s (e.g., the functions $\Phi_T^i(x)$ which determine f_1 and g_1 contain the small quantity[9] $(1-\delta_T^i)^2$ and all δ_T^i except δ_0^s, are close to unity for $N=3$ or 4).

Thus we have shown that in the four-dimensional space the vertex $\Gamma_0^s(p_1^2, p_2^2, p_3^2, p_4^2, p_s^2, p_t^2)$ in the region $p_1^2=p_2^2\gg p_s^2$, $p_3^2\sim p_4^2$ has the following scale-invariant form:

$$\Gamma_0{}^s=-{}^1/{}_2(p_1{}^2)^{-\delta_0{}^s}(p_s{}^2)^{-(1-\delta_0{}^s)}\gamma_0(p_s{}^2/p_3{}^2,\ p_4{}^2/p_3{}^2).\tag{34}$$

If in the zeroth approximation γ_0^s does not depend on the ratio p_4^2/p_3^2 (property (b) in Sec. 5), then it can only be determined by means of the ε-expansion. If the variable $\zeta=\max(p_3^2, p_4^2, p_s^2)$ differs from $\eta=p_s^2$ we obtain for $\Gamma_T^i(\xi,\eta,\zeta)$ (cf. (18))[14]

$$\Gamma_T{}^i(\xi,\eta,\zeta)=c^i\int_\eta^\xi\Gamma_T{}^i(\xi,z,z)\Gamma_T{}^i(\zeta,z,z)\,dz+\Gamma_T{}^i(\xi,\zeta,\zeta)$$
$$=\frac{\varepsilon}{2c^i}\frac{(1-\delta_T{}^i)}{(1-2\delta_T{}^i)}\big[(1-\delta_T{}^i)(\xi^{\varepsilon/2})^{-\delta_T{}^i}(\zeta^{\varepsilon/2})^{-\delta_T{}^i}(\eta^{\varepsilon/2})^{-(1-2\delta_T{}^i)}$$
$$-\delta_T{}^i(\xi^{\varepsilon/2})^{-\delta_T{}^i}(\zeta^{\varepsilon/2})^{\delta_T{}^i-1}\big].$$

From here we have, to accuracy δ_0^s,

$$\gamma_0{}^s(p_s{}^2/p_3{}^2,\ p_4{}^2/p_3{}^2)=(p_s{}^2/p_3{}^2)^{\delta_0{}^s}=(p_s{}^2/p_4{}^2)^{\delta_0{}^s}.\tag{35}$$

The relation obtained in this way may be used as the zeroth approximation of an iteration method, substituting it into the right-hand side of the four-dimensional parquet equations. The analysis carried out above shows that: 1) no divergent integrals appear, i.e., there is no "vanishing charge," 2) at high energies the zeroth approximation apparently determines the principal part of the exact solution. One can therefore expect that the iteration method will be numerically convergent.

The last assertion is not in fact proved, since in this paper the iterations have not been carried through. The problem of carrying these through seems possible and quite interesting. Effecting the first iteration will allow one to consider the problem of validity of the property (b) (Sec. 5), and if it is not valid, to determine the correct form of the function γ_0^s (in place of (35)).

We note that for small ε the nonparquet graphs of Fig. 3b, c contain an extra factor g_1^{k-2} (k is the order of the graph), compared to the parquet graph, and are therefore unimportant. They also do not contribute to the zeroth approximation for $\varepsilon=2$ since in this case g_1 remains numerically small: we have (at the second fixed point) $g_1/2=0.05$, $f_1/2=-0.03$. Owing to the rapid convergence of all the integrals except the integral over the last momentum, there appears a factor $1/(k-1)!$ in the contributions of these graphs. This factor seems to compensate, at least for not too large k, the growth of the number of graphs.

Similar considerations show that the structures of the form $p_\alpha p_\beta$ which have not been taken into account above also do not contribute to the zeroth approximation.

8. THE EQUATIONS FOR THE PROPAGATOR AND AN ESTIMATE OF THE INDEX Δ

We have found above the solutions to the parquet equations under the condition that $\beta(p^2)=1$. We now substitute into them β in the form (11). It is easy to notice that this simply leads to the substitution $\varepsilon\to\varepsilon+4\Delta$ ($x=(\xi)^{\varepsilon/2+2\Delta}$ etc.) in Eqs. (27), (29) and in the expressions for f_1 and g_1. In order to find Δ it is necessary to complement the system of parquet equations with an equation for the propagator.

The Dyson–Schwinger equation turns out to be inconvenient in our case, since in order to solve it, it is necessary to know the vertex to a higher accuracy,[9] than was done before. We therefore utilize the unitary expansion of $S^{-1}(p)$ with respect to "jets," as formulated by Polyakov.[22]

In view of numerical smallness of the asymptotic values of the coupling constants f_1 and g_1 we restrict our attention to the first "three-jet" term of this expansion, which in this case yields the largest contribution. We then obtain for $S(p)$ the equation

$$\operatorname{Im}S^{-1}(p)=\int\Gamma_{\alpha\beta}(p,k,q-k,p-q)\operatorname{Tr}[O_\alpha\operatorname{Im}S(k)O_{\alpha'}\operatorname{Im}S(k-q)]$$
$$\times O_\beta\operatorname{Im}S(p-q)O_{\beta'}\Gamma_{\alpha'\beta'}(k,q-k,p-q,p)\frac{d^4k\,d^4q}{\pi^4}.\tag{36}$$

In the case of the power-law form (11) the propagator is

$$\theta(p_0)\,\mathrm{Im}\,\beta=\sin(\pi\Delta)\,(p^2/\lambda^2)^{\Delta}\theta(p),$$

where $\theta(p)=\theta(p^2)\theta(p_0)$ and $p^2=p_0^2-\mathbf{p}^2$.

We estimate the integral in the right-hand side under the assumption that the vertex functions $\Gamma_{\alpha\beta}$ vary slowly inside the integration region and are equal to

$$\Gamma_{f\alpha\beta}=g_{\alpha\beta}\frac{f_1\lambda^{4\Delta}}{(-p^2)^{1+2\Delta}},\quad \Gamma_{g\alpha\beta}=g_{\alpha\beta}\frac{g_1\lambda^{4\Delta}}{(-p^2)^{1+2\Delta}}. \qquad (37)$$

We do not know whether this form of the vertex is a good approximation. However, the calculations carried out in this section show how one can determine the index Δ.

The equation (36) for the vertices of the form (37) has the following form

$$\mathrm{Im}\,S^{-1}(p)=\left[\frac{f_1^2}{2}\left(N+\frac{1}{2}\right)+\frac{3g_1^2}{2}\left(N-\frac{1}{2}\right)+\frac{3}{2}f_1g_1\right]\lambda^{2\Delta}\sin^3\pi\Delta$$
$$\times\int\frac{\mathrm{Tr}[O_\alpha\hat{k}O_\beta(\hat{k}-\hat{q})]O_\alpha(\hat{p}-\hat{q})O_\beta}{(-p^2)^{2+2\Delta}[k^2(k-q)^2(p-q)^2]^{1-\Delta}}\theta(k)\theta(q-k)\theta(p-q)\frac{d^4k\,d^4q}{\pi^4}.$$

The integral with respect to d^4k equals the discontinuity across the cut of the quantity

$$I_{\mu\nu}=\int\frac{\mathrm{Tr}[O_\mu\hat{k}O_\nu(\hat{q}-\hat{k})]}{[(-k^2-i0)(-(q-k)^2-i0)]^{1-\Delta}}\frac{d^4k}{\pi^2 i}.$$

This is easy to see if one notes that the discontinuity of $I_{\mu\nu}$ appears, as usual, when the integration path is situated between the singularities of the denominator. The divergent part of $I_{\mu\nu}$ does not have a discontinuity, and the convergent one equals

$$\mathrm{disc}\,I_{\mu\nu}=4i\pi^2\sin(2\pi\Delta)\left[\left(\frac{1+\Delta}{1+2\Delta}\right)g_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\right]\frac{\Gamma^2(2+\Delta)\Gamma(-2\Delta)(q^2)^{1+2\Delta}}{\Gamma(4+2\Delta)\Gamma^2(1-\Delta)\pi^2}.$$

A similar calculation of $\int d^4q$ yields

$$\mathrm{Im}\,S^{-1}(p)=\hat{p}\left(\frac{p^2}{\lambda^2}\right)^{-\Delta}\frac{8\pi}{\Gamma(3+3\Delta)\Gamma(5+3\Delta)}\left[\frac{\Gamma(2+\Delta)}{\Gamma(1-\Delta)}\right]^3$$
$$\times\left[\frac{f_1^2}{2}\left(N+\frac{1}{2}\right)+\frac{3g_1^2}{2}\left(N-\frac{1}{2}\right)+\frac{3}{2}f_1g_1\right].$$

This equation has the approximate solution

$$\Delta(N)={}^1/_6[{}^1/_2 f_1(N+{}^1/_2)+{}^3/_2 g_1^2(N-{}^1/_2)+{}^3/_2 f_1g_1]$$

($\Delta(4)=0.06$). For the reasons mentioned above the value of the index obtained here may be valid only in order of magnitude.

9. CONCLUSION

In the Hamiltonian (2) considered above the fermion fields are grouped into doublets

$$\begin{pmatrix}\nu_e\\ e\end{pmatrix}\quad\begin{pmatrix}\nu_\mu\\ \mu\end{pmatrix}$$

Such a combination of fields is characteristic for the unified theories of weak and electromagnetic interactions.[2] In order to include the hadrons, one introduces by analogy with the leptons the quark doublets

$$\begin{pmatrix}p\\ n_\theta\end{pmatrix}\quad\begin{pmatrix}p'\\ \lambda_\theta\end{pmatrix},$$

where

$$n_\theta=n\cos\theta+\lambda\sin\theta,\quad \lambda_\theta=-n\sin\theta+\lambda\cos\theta,$$

θ is the Cabibbo angle. The charmed quark[23] is introduced in order that the Hamiltonian should not contain neutral strange currents, which have not been observed experimentally. Without taking the strong interactions into account the point quarks p, p', n, λ may be considered as particles of the same nature as the leptons. Their inclusion into the Hamiltonian leads simply to an increase of the number N of doublets for the realistic weak interactions apparently $N=4$ or $N=8$ (if the quarks are colored[24]).

The Hamiltonian (2) can be rewritten in the form

$$H=\frac{F_0}{4}\left(\sum_{i=1}^{N}\bar{\nu}_i\nu_i+\sum_{i=1}^{N}\bar{e}_ie_i\right)^2+\frac{G_0}{4}\left(\sum_{i=1}^{N}\bar{\nu}_i\nu_i-\sum_{i=1}^{N}e_ie_i\right)^2$$
$$+G_0\left(\sum_{i=1}^{N}\bar{\nu}_ie_i\right)\left(\sum_{i=1}^{N}\bar{e}_i\nu_i\right).$$

The interaction of the charged currents is characterized by the part $G_0(\bar{\nu}_\mu\mu)(\bar{e}\nu_e)$, that of the neutral currents is characterized by the parts $\frac{1}{2}(F_0+G_0)(\bar{\mu}\mu)(\bar{e}e)$ and $\frac{1}{2}(F_0-G_0)(\bar{\nu}_\mu\nu_\mu)(\bar{e}e)$. The second part describes the interaction of the lower components of the doublets with the lower components and the interaction of the upper components with the upper components (with a coupling constant $G_n=\frac{1}{2}(F_c+G_c)$), and the third describes the interaction of upper components with lower components (with the coupling constant $G_n'=\frac{1}{2}(F_c-G_c)$).

Above we have obtained the relation $f=ag$, where

$$a=\frac{N+1-(N^2+20N+28)^{1/2}}{2N+3},$$

between the effective coupling constants f and g at high energies $G_cp^2\gg 1$; the above solutions are valid in this region. For $N=4$ the slope is $a=-0.56$, for $N=8$, $a=-0.36$. For smaller energies the value of a may change substantially. However, if $a<0$ and is of the same order of magnitude as for high energies, one should expect that $|G_n'|>|G_n|$. In any case the relation $G_n+G_n'=G_c$ must hold. If the quantity a is close to zero, the cross section of the process produced by the neutral currents must be four times smaller than the cross section of the process mediated by the charged currents.

One obtains a unique prediction for the sign of the Fermi constant G_c at low energies. Since the straight line $g=0$ is a singular solution (cf. Fig. 6), G_c cannot change sign as the energy decreases and for large energies one must have $G_c>0$. Experimentally the sign of G_c can be determined, e.g., from experiments on determining the asymmetry which appears in the scattering of polarized charged leptons and antileptons with energies >100 GeV on nucleons.[25]

APPENDIX 1

For reference we write down the complete set of parquet equations in the notations of Sec. 2

$$\Phi_f^s=a(d)\int\{[\Gamma_f-\Phi_f^s]\Gamma_f^s+3[\Gamma_g-\Phi_g^s]\Gamma_g^s\}\frac{\beta(k^2)\beta((p_s-k^2))}{(p_s-k)^2}\frac{d^dk}{\pi^{d/2}i},$$

$$\Phi_g^s=a(d)\int\{[\Gamma_g-\Phi_g^s]\Gamma_f^s+[\Gamma_f-\Phi_f^s]\Gamma_g^s-2[\Gamma_g-\Phi_g^s]\Gamma_g^s\}\frac{\beta(k^2)\beta((p_s-k)^2)}{(p_s-k)^2}\frac{d^dk}{\pi^{d/2}i},$$

$$\Phi_f^u=-b(d)\int\{[\Gamma_f-\Phi_f^u]\Gamma_f^u+3[\Gamma_g-\Phi_g^u]\Gamma_g^u\}\frac{\beta(k^2)\beta((p_u-k)^2)}{(p_u-k)^2}\frac{d^dk}{\pi^{d/2}i},$$

$$\Phi_g^u=-b(d)\int\{[\Gamma_f-\Phi_f^u]\Gamma_g^u+[\Gamma_g-\Phi_g^u]\Gamma_f^u+2[\Gamma_g-\Phi_g^u]\Gamma_g^u\}\frac{\beta(k^2)\beta((p_u-k)^2)}{(p_u-k)^2}\frac{d^dk}{\pi^{d/2}i},\tag{A.1}$$

$$\Phi_f^t=-\frac{c(d)}{2}\int\Big\{2N[\Gamma_f-\Phi_f^t]\Gamma_f^t+\frac{\varepsilon}{2}[\Gamma_f-\Phi_f^t]\Gamma_f^u+\frac{\varepsilon}{2}[\Gamma_f-\Phi_f^u]\Gamma_f^t+\frac{3\varepsilon}{2}[\Gamma_f-\Phi_f^t]\Gamma_g^u+\frac{3\varepsilon}{2}[\Gamma_g-\Phi_g^u]\Gamma_f^t\Big\}\frac{\beta(k^2)\beta((p_t-k)^2)}{(p_t-k)^2}\frac{d^dk}{\pi^{d/2}i},$$

$$\Phi_g^t=-\frac{c(d)}{2}\int\Big\{2N[\Gamma_g-\Phi_g^t]\Gamma_g^t+\frac{\varepsilon}{2}[\Gamma_g-\Phi_g^t]\Gamma_f^u+\frac{\varepsilon}{2}[\Gamma_f-\Phi_f^u]\Gamma_g^t-\frac{\varepsilon}{2}[\Gamma_g-\Phi_g^t]\Gamma_g^u-\frac{\varepsilon}{2}[\Gamma_g-\Phi_g^u]\Gamma_g^t\Big\}\frac{\beta(k^2)\beta((p_t-k)^2)}{(p_t-k)^2}\frac{d^dk}{\pi^{d/2}i}.$$

The coefficients

$$\left.\begin{matrix}a(d)\\ b(d)\end{matrix}\right\}=\frac{3}{2}-\frac{1}{d}\pm f(d)\frac{(d-1)(d-2)(d-3)}{2d},\quad c(d)=\frac{d-2}{2}$$

appear from the transformation of the spin structure. The function $f(d)$ is defined in Sec. 4A. For $d=4$ we have $a(4)=2$, $b(4)=c(4)=\frac{1}{2}$ and the system (A.1) coincides with the system of four-dimensional equations (9). For $d=2$ we have $a(2)=b(2)=1$, $c(2)=0$ and the system (A.1) coincides with the two-dimensional system (19). In Secs. 5 and 7 one can find the solution of the system (A.1) for $a=2$, $b=c=\frac{1}{2}$ but with scalar integrals defined for nonintegral $d=2+\varepsilon$.

In the system (A.1) Γ_i^s, Γ_i^t, Γ_i^u denote the vertex $\Gamma_i(p_1p_2p_3p_4)$ in the regions where the momenta p_1 and p_2, p_1 and p_3, p_1 and p_4, are large, respectively. Hence the amplitude $\hat{T}$ has for large p_1 and p_3 the following form

$$\hat{T}=-{}^1/_2[\Gamma_f^t(O_\alpha)_{31}(O_\alpha)_{42}-\Gamma_f^u(O_\alpha)_{32}(O_\alpha)_{41}]-{}^1/_2[\Gamma_g^t(O_\alpha\tau)_{31}(O_\alpha\tau)_{42}-\Gamma_g^u(O_\alpha\tau)_{32}(O_\alpha\tau)_{41}].$$

For small ε,

$$\Gamma_f^t(\xi^t,\eta^t)=F+\Phi_f^s(\xi^t)+\Phi_f^t(\xi^t,\eta^t)+\Phi_f^u(\xi^t),$$
$$\Gamma_f^u(\xi^u,\eta^u)=F+\Phi_f^s(\xi^u)+\Phi_f^t(\xi^u)+\Phi_f^u(\xi^u,\eta^u),$$
$$\Gamma_g^t(\xi^t,\eta^t)=G+\Phi_g^s(\xi^t)+\Phi_g^t(\xi^t,\eta^t)+\Phi_g^u(\xi^t),$$
$$\Gamma_g^u(\xi^u,\eta^u)=G+\Phi_g^s(\xi^u)+\Phi_g^t(\xi^u)+\Phi_g^u(\xi^u,\eta^u),$$

where $\xi^t=\max\{\bar{p}_1^2,\ \bar{p}_3^2,\ \bar{p}_t^2\}$, $\eta^t=\bar{p}_t^2$ and the variables ξ, η; ξ^u, η^u are defined in Sec. 4C.

APPENDIX 2

We list for Γ^t and Γ^u expressions similar to Eqs. (27). Here $x=(\xi^t)^{\varepsilon/2+2\Delta}$, $y=(\eta^t)^{\varepsilon/2+2\Delta}$. We have

$$\Gamma_f^u(x,y)=\frac{\varepsilon}{4}(1-\delta_0^u)x^{-\delta_0^u}y^{-1+\delta_0^u}+\frac{3\varepsilon}{4}(1-\delta_1^u)x^{-\delta_1^u}y^{-1+\delta_1^u},$$

$$\Gamma_g^u(x,y)=\frac{\varepsilon}{4}(1-\delta_0^u)x^{-\delta_0^u}y^{-1+\delta_0^u}-\frac{\varepsilon}{4}(1-\delta_1^u)x^{-\delta_1^u}y^{-1+\delta_1^u}\tag{A.2}$$

$$\Gamma_f^t(x,y)=\frac{\varepsilon}{2N}[(1-\delta_0^t)x^{-\delta_0^t}y^{-1+\delta_0^t}+(1-\delta_0^u)x^{-\delta_0^u}y^{-1+\delta_0^u}],$$

$$\Gamma_g^t(x,y)=\frac{\varepsilon}{2N}[(1-\delta_1^t)x^{-\delta_1^t}y^{-1+\delta_1^t}-(1-\delta_1^u)x^{-\delta_1^u}y^{-1+\delta_1^u}],$$

where

$$\delta_0^u=\frac{2N+5-5a}{2(N+4-2a)},\quad \delta_0^t=\frac{(2N+5)(1-a)}{2(N+4-2a)},$$
$$\delta_1^u=\frac{2N+9-5a}{2(N+4-2a)},\quad \delta_1^t=\frac{9-5a}{2(N+4-2a)}.\tag{A.3}$$

And finally, for $\Phi(x)$ we obtain expressions similar to (29):

$$\Phi_f^u(x)=\frac{\varepsilon}{4x}[(1-\delta_0^u)^2+3(1-\delta_1^u)^2],$$
$$\Phi_g^u(x)=\frac{\varepsilon}{4x}[(1-\delta_0^u)^2-(1-\delta_1^u)^2],$$
$$\Phi_0^t(x)=\frac{\varepsilon}{Nx}[(1-\delta_0^t)^2-(1-\delta_0^u)^2],\tag{A.4}$$
$$\Phi_1^t(x)=\frac{\varepsilon}{Nx}[(1-\delta_1^t)^2-(1-\delta_1^u)^2].$$

APPENDIX 3

We show how the presence of saddle points in the phase plane allows one to fix the form of the interaction. We break the $SU(2)$ symmetry of the Hamiltonian (2) and consider the interaction

$$H={}^1/_4F_0J_0J_0+{}^1/_4G_{z0}J_zJ_z+G_0J_\perp J_\perp.$$

For the effective coupling constants we obtain the Gell-Mann–Low system of equations (cf. Sec. 5):

$$f'={}^1/_2\varepsilon f-f^2(N-{}^1/_2)+{}^3/_2g_z^2-fg_z-2fg,$$
$$g_z'={}^1/_2\varepsilon g_z-g_z^2(N+1)-5g^2+2fg_z+2gg_z,\tag{A.5}$$
$$g'={}^1/_2\varepsilon g-Ng^2-4gg_z+2fg.$$

The straight line $g=g_z=f/a$ is a singular solution on which the fixed point

$$g_1=g_{z1}=f_1/a=\varepsilon/2(N+4-2a)$$

is situated. In agreement with Sec. 6 this point is a saddle point in the fg plane, and is a saddle point in the gg_z plane for $N<8$ (the corresponding singular solution of the linearized system (A.5) is $g_1/(8-N)$). Thus, for $N<8$ the stability of the scale-invariant solution can be guaranteed only if the initial (physical) values of the coupling constants are situated on the singular solution. Physically, this means that only the $SU(2)$-invariant form of the interaction can exist.

In phase space there are also other singular solutions, on which other fixed points are situated. It would be very attractive to have a situation where a finite solution exists only at one $SU(2)$-symmetric point, similar to the fg plane (cf. Sec. 6).

[1]For $N=1$ the Fierz identity $((\bar{e}\nu)(\bar{\nu}e)\equiv(\bar{\nu}\nu)(\bar{e}e))$ implies that the isoscalar interaction coincides with the isovector interaction: $(\bar{\Psi}\Psi)(\bar{\Psi}\Psi)\equiv(\bar{\Psi}\tau\Psi)(\bar{\Psi}\tau\Psi)$.

[2)]The Fierz identity (cf. footnote [1)]) does not hold in the presence of isospin structure.

[3)]At $d=2$ the matrices O_α can be chosen such that they coincide with the Thirring γ-matrices $O_0=\gamma_0=\sigma_x$, $O_1=\gamma_1=i\sigma_y$. For $d=2$ the Fierz identity (cf. footnote [1)]) is not valid.

[4)]Terms of order εf^2 have been omitted, since, as can be seen from the solution $g\sim\varepsilon$, and $f\sim\varepsilon^2$.

[5)]Tr 1 cannot be exactly established with our method of analytic continuation.

[6)]Even to first approximation in ε this solution differs from the solution 22) of the system (A.1) with $a(2)=b(2)=1$, $c(2)=0$, obtained in Sec. 4. It is natural to expect that for $\varepsilon=2$ (22) will not be an approximate solution of the system (9), since it does not reflect the symmetry of the four-dimensional problem (the Fierz identities, cf. footnote[1)]).

[7)]Here we do not discuss the property (b). It can be investigated by means of iteration of the equations (A.1) with the zeroth approximation in the form of the solution obtained below (Sec. 7).

[8)]The whole ε-dependence is included in the variables $x=\xi^{\varepsilon/2+2\Delta}$ and $y=\eta^{\varepsilon/2+2\Delta}$. For the role of Δ cf. Sec. 8.

[9)]This assertion remains valid also if the region of small k^2 is taken into account in (31), $(0\leqslant k^2\leqslant y)$. The quantities f_1, g_1 change insignificantly.[21]

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Verification of a possible asymmetry in the polarization of thermal neutrons after reflection from a mirror

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The neutron polarization asymetry observed previously by K. Berndofer [Z. Phys. **243**, 188 (1971)] has not been confirmed by experiments with a polarizing neutron guide. In view of the spin-orbit effects currently discussed in the literature, measurements have been carried out of the polarization of neutrons, singly reflected from magnetic and nonmagnetic mirrors. It was found that the polarization asymmetry was absent to an accuracy of 10^{-4}–10^{-3}.

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When the polarization of thermal neutrons transmitted through a polarizing neutron guide was investigated,[1] a polarization asymmetry was found, depending on the direction of the magnetic field of the polarizer and the direction of the curvature of the uniformly bent neutron guide. The difference in the polarization of the neutron beam was found to be up to $\Delta P\approx 30\%$ with maximum polarization $P\approx 80\%$. This experimental result was un-