Asymptotic behavior of non-linear wave systems integrated by the inverse scattering method

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We find, for large times, the asymptotic solutions of some non-linear wave systems integrated by the inverse scattering method (non-linear Schrödinger equation, Korteweg-de Vries equation, sine-Gordon equation), directly in terms of the initial data.

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INTRODUCTION

Recently it has been established that a whole number of non-linear wave equations of importance to physics can be integrated exactly by means of a new mathematical method—the inverse scattering method. Among those we have the Korteweg–de Vries equation,[1] the non-linear Schrödinger equation,[2,3] the sine–Gordon equation,[4,5] and others. The formalism of the inverse method enables us to find easily extensive classes of exact solutions which describe the propagation and interaction of solitary waves—solitons, and to determine their asymptotic behavior.[2,3,6–8] However, the problem of the asymptotic behavior of an arbitrary initial condition is in this way far from exhausted; it is still necessary to determine the asymptotic behavior of the "non-soliton" part of the solution which is connected with the continuous spectrum of the corresponding differential operator.[1)] The present paper is devoted to the solution of that problem.

One must note that the structure of the asymptotic solutions is determined also without invoking the inverse method. Indeed, by analogy with the linear problem it is natural to expect that the asymptotic solution is locally periodic with a slowly varying amplitude and frequency and can thus be easily found using one asymptotic method or another, such as, e.g., Whitham's method.[10] For instance, for the non-linear Schrödinger equation

$$i\psi_t+\psi_{xx}\pm2|\psi|^2\psi=0 \tag{1}$$

the solution of this kind has the form $(t\to\infty)$

$$\psi(x,t)=\frac{1}{\sqrt{t}}f\left(\frac{x}{t}\right)\exp\left\{i\frac{x^2}{4t}\pm2i\left|f\left(\frac{x}{t}\right)\right|^2\ln t\right\}+O\left(\frac{1}{t^{3/2}}\right) \tag{2}$$

with an arbitrary function f. The problem consists, firstly, in proving the fact that (2) is, indeed, the asymptotic solution of the Cauchy problem and, secondly, in determining the function $f(\xi)$ from a given initial condition $\psi(x, t)$ for $t=0$. Asymptotic methods are clearly useless for solving these problems.[2)]

In the present paper we show how one can find the "non-soliton" asymptotic behavior for various integrable systems using a standard procedure in which the central place is occupied by the solution of the problem of parametric resonance for a linear change in the frequency of the perturbation. We give two versions of this procedure. The first of them uses only the direct spectral problem for the "integrating" operator and in that sense is more "elementary" than the second one which is based upon the equations of the inverse problem. Using the direct scattering problem we evaluate the asymptotic behavior of the non-linear Schrödinger equation for different signs of the non-linear term (describing the propagation of wavepackets in self-focusing and defocusing media) (§1, 2) and also the problem of the asymptotic behavior of the Korteweg–de Vries equation (§3). We see then that the asymptotic behavior of the non-linear Schrödinger equation has the form (2) for all x whereas such a "quasi-linear"[3)] asymptotic behavior occurs for the KdV equation only for $x>t$; in the remaining region, however, the asymptotic behavior is determined by the self-similar solution which joins up with the quasi-linear one at $x\sim t$. We apply the procedure based upon the inverse problem equations to an evaluation of the asymptotic behavior of the sine–Gordon equation (§4).

§1. NON-LINEAR SCHRÖDINGER EQUATION (SELF-FOCUSING MEDIUM)

Equation (1) with the "plus" sign in front of the non-linear term describes the propagation of spectrally narrow wavepackets in a non-linear self-focusing medium. The inverse scattering method connects this equation with the spectral problem (see[2]):

$$i\frac{\partial u_1}{\partial x}+\psi^*u_2=\frac{\lambda}{2}u_1,\qquad -i\frac{\partial u_2}{\partial x}-\psi u_1=\frac{\lambda}{2}u_2, \tag{3}$$

or the equivalent set of equations

$$i\frac{\partial v_1}{\partial x}+Ae^{-i\tilde\Phi}v_2=0,\qquad i\frac{\partial v_2}{\partial x}+Ae^{i\tilde\Phi}v_1=0, \tag{4}$$

where we have introduced the notation

$$\psi=Ae^{i\Phi}\ (\operatorname{Im}A=0,\ A\geqslant0),\qquad \tilde\Phi=\Phi-\lambda x,$$
$$v_1=u_1e^{i\lambda x/2},\qquad v_2=u_2e^{-i\lambda x/2}.$$

The spectral problem (3) may have discrete eigenvalues lying in the complex λ-plane. They correspond to exact solutions of Eq. (1), exponentially decreasing with x, namely solitons. The dynamics of solitons was studied in detail in[2]. Here we shall assume that solitons (eigenvalues of the system (3)) are not present. A sufficient condition for the absence of a discrete spectrum is, e.g., the inequality

$$\int_{-\infty}^{\infty}|\psi(x)|dx<\ln(2+\sqrt{5}).$$

We shall in what follows denote by $\xi(x, \lambda)$ the exact solution of the set (4) with asymptotic behavior $\xi_1\to 0$, $\xi_2\to 1$ as $x\to\infty$. We write $\xi_1(-\infty, \lambda)=b(\lambda)$, $\xi_2(-\infty, \lambda)=a(\lambda)$. If $\psi(x)$ in the problem (3) depends on the time, the functions $a(\lambda)$ and $b(\lambda)$ will, in general, also be time-dependent. It is, clearly, impossible to give the explicit time-dependence of a and b for an arbitrary t-dependence of $\psi(x)$. However, there is an important fact: if $\psi(x, t)$ in the set (3) changes with time according to Eq. (1), $a(\lambda)$ will be time-independent and

$$b(\lambda, t)=b(\lambda, 0)e^{i\lambda^2 t}. \tag{5}$$

Equation (5) and also the fact that the mapping $\psi(x)\to a(\lambda)$, $b(\lambda)$ given by problem (3) is reciprocally unique also allows us to "integrate" Eq. (1). We must then proceed as follows for the solution of the Cauchy problem. Firstly, we must for a given initial condition $\psi(x, 0)$ for Eq. (1) use (3) to find $a(\lambda)$ and $b(\lambda)$; then we must use Eq. (5) and get the "scattering matrix" $a(\lambda)$, $b(\lambda, t)$, and after that find $\psi(x, t)$ which if substituted into (3) would lead to $a(\lambda)$, $b(\lambda, t)$. The last problem in this scheme is the subject of the inverse problem in scattering theory; its solution reduces to the solution of the linear integral equations of the inverse problem (see[2]). The explicit solution of these equations is known only in the purely soliton case. Here we show a method for solving the inverse problem when $b(\lambda)$ is a fast oscillating function of the spectral parameter λ which, according to (5) means $t\to\infty$.

We assume that the modulus and argument of $\psi(x)$ satisfy the following conditions:

$$\Phi_{xx}>0,\qquad \left|\frac{d}{dx}\ln A\right|^2\ll\Phi_{xx},\qquad \left|\frac{d}{dx}\ln\Phi_{xx}\right|^2\ll\Phi_{xx} \tag{6}$$

and that the integral $\int(A^2/\Phi_x)\,dx$ converges in the neighborhood of both infinities. Conditions (6) mean that the function $\psi(x)$ can on the basis of the set (3) in the neighborhood of any point be written as an exponential with constant amplitude and linearly changing frequency. For each λ we split off the resonance region—the vicinity of the point $x_0(\lambda)$ defined by the condition

$$\tilde{\Phi}_x(x_0)=\Phi_x(x_0)-\lambda=0.$$

As $\Phi_{xx}>0$ this equation has no more than a single solution for any λ. In the resonance region the set (4) simplifies to

$$i\frac{\partial v_1}{\partial y}+A(x_0)\exp\left\{-i\left(\Phi_0+\frac{1}{2}f_0y^2\right)\right\}v_2=0,$$
$$i\frac{\partial v_2}{\partial y}+A(x_0)\exp\left\{i\left(\Phi_0+\frac{1}{2}f_0y^2\right)\right\}v_1=0, \tag{7}$$
$$\tilde{\Phi}_0=\tilde{\Phi}(x_0),\quad f_0=\Phi_{xx}(x_0),\quad y=x-x_0.$$

Far from the point x_0 the set (4) can also be considerably simplified. In fact, in the region where

$$\tilde{\Phi}_x^2\gg\Phi_{xx}, \tag{8}$$

we get for v_1 and v_2

$$i\tilde{\Phi}_x\frac{\partial v_1}{\partial x}+A^2v_1=0,\qquad -i\tilde{\Phi}_x\frac{\partial v_2}{\partial x}+A^2v_2=0. \tag{9}$$

We shall call the region where condition (8) is satisfied the asymptotic region. In that region we can easily find v_1 and v_2 from (9). The general form of the solution in the resonance region can also easily be given. To do this we introduce the variable $z=y\Phi_{xx}^{1/2}(x_0)$ and eliminate the quantity v_1 from the set (7). We get for v_2

$$\frac{\partial^2 v_2}{\partial z^2}-iz\frac{\partial v_2}{\partial z}+\alpha^2v_2=0, \tag{10}$$

where

$$\alpha^2(\lambda)=A^2(x_0)/\Phi_{xx}(x_0),\qquad x_0=x_0(\lambda). \tag{11}$$

The general solution of Eq. (10) can be expressed in terms of parabolic cylinder functions (see, e.g., [13]):

$$v_2=e^{iz^2/4}\left(c_1D_{-1+i\alpha^2}\left(\frac{z}{\sqrt{i}}\right)+c_2D_{-1+i\alpha^2}\left(-\frac{z}{\sqrt{i}}\right)\right). \tag{12}$$

The function $v_1(y)$ can then be found from the second of Eqs. (7). The solution in the resonance region is thus determined by two arbitrary constants c_1 and c_2.

When conditions (6) are satisfied the resonance and asymptotic regions overlap. In the overlap region the solutions of Eqs. (9) and (10) must join up, after which we obtain the solution of problem (4) for the whole x-axis. In particular, we get from (9) for the solution $\xi(x, \lambda)$ in the asymptotic region for $x>x_0$, using the boundary conditions at $+\infty$ in x:

$$\xi_1(x,\lambda)=0,\qquad \xi_2(x,\lambda)=\exp\left(i\int_x^{\infty}(A^2/\tilde{\Phi}_x)\,dx\right). \tag{13}$$

In the overlap region (13) gives

$$\xi_1=0.$$
$$\xi_2=[\Phi_{xx}(x_0)(x-x_0)]^{-i\alpha^2}\exp\left[-i\int_{x_0}^{\infty}\ln\tilde{\Phi}_x\frac{d}{dx}\frac{A^2}{\Phi_{xx}}dx\right]. \tag{14}$$

The asymptotic behavior of the parabolic cylinder functions which interest us is the following ($z\to+\infty$)

$$D_{-1+i\alpha^2}\left(\frac{z}{i^{1/2}}\right)=e^{iz^2/4}i^{1/2}e^{\pi\alpha^2/4}z^{+i\alpha^2-1}+O(z^{-2}),$$
$$D_{-1+i\alpha^2}\left(-\frac{z}{i^{1/2}}\right)=\frac{\sqrt{2\pi}}{\Gamma(1-i\alpha^2)}e^{-\pi\alpha^2/4}z^{-i\alpha^2}e^{-iz^2/4}+e^{-3/4\pi i}e^{-3/4\pi\alpha^2}z^{-1+i\alpha^2}e^{iz^2/4}+\ldots \tag{15}$$

Substituting these formulae into v_1 and v_2 and joining the expressions obtained with (14) we find

$$c_2=\frac{\Gamma(1-i\alpha^2)}{\sqrt{2\pi}}e^{\pi\alpha^2/4}[\Phi_{xx}(x_0)]^{-i\alpha^2/2}$$
$$\times\exp\left\{-i\int_{x_0}^{\infty}\ln\tilde{\Phi}_x\frac{d}{dx}\frac{A^2}{\Phi_{xx}}dx\right\},\qquad c_1=c_2e^{-\pi\alpha^2}.$$

We have thus found $\xi(x, \lambda)$ in the resonance region. Joining then similarly the solution of Eqs. (9) in the overlap region $x < x_0$ with the already-found solution, we determine the solution in the asymptotic region to the left of the resonance point. Letting after that x tend to $-\infty$ we get the scattering matrix:

$$a(\lambda)=\exp\left[-\pi\alpha^2+i\int_{-\infty}^{\infty}\frac{A^2(x)}{\Phi_x}dx\right], \tag{16}$$

$$b(\lambda)=\frac{\sqrt{-2\pi i}}{\alpha\Gamma(i\alpha^2)}e^{-\pi\alpha^2/2}\exp\{-i[\Phi_0+L_1-L_2+\alpha^2\ln\Phi_{xx}(x_0)]\}, \tag{17}$$

where

$$L_1=\int_{x_0}^{\infty}\ln\Phi_x\frac{d}{dx}\frac{A^2}{\Phi_{xx}}dx,\quad L_2=\int_{-\infty}^{\infty}\ln|\Phi_x|\frac{d}{dx}\frac{A^2}{\Phi_{xx}}dx. \tag{18}$$

In equations (16) to (18) we must put $x_0 = x_0(\lambda)$ found from the equation $\Phi_x(x_0) = \lambda$. We have thus explicitly solved the direct scattering problem for the class of functions which satisfy conditions (6).

One verifies easily that the scattering matrix we have obtained possesses the necessary properties. Firstly, using the formula

$$|\Gamma(i\alpha^2)|^2=\pi/\alpha^2\operatorname{sh}\pi\alpha^2,$$

we can satisfy ourselves that $|a(\lambda)|^2 + |b(\lambda)|^2 = 1$. We note, secondly, that the exponential index in Eq. (16) can be written in the form

$$\frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{(-\pi\alpha^2(\xi))}{\xi-\lambda-i0}d\xi,$$

which means that $\ln a(\lambda)$ can be analytically continued from the real axis into the upper λ half-plane; this in turn means that $a(\lambda)$ is analytical in the region $\operatorname{Im}\lambda > 0$ and has no zeroes there. These statements about the scattering matrix are valid in the general case (here, however, we established them for "potentials" satisfying conditions (6) by directly evaluating $a(\lambda)$, $b(\lambda)$).

Having Eqs. (16) to (18) available we can with the same accuracy solve also the inverse scattering problem—recover $A(x)$, $\Phi(x)$ for given $a(\lambda)$, $b(\lambda)$. To do this we note that

$$\alpha^2(\lambda)=\frac{A^2}{\Phi_{xx}}=\frac{1}{\pi}\ln\frac{1}{|a(\lambda)|}$$

is a given function of the spectral parameter λ. We write $\arg b(\lambda) = \Theta(\lambda)$ and note that the integrals L_1 and L_2 of (18) can be expressed solely in terms of $\alpha(\lambda)$:

$$L_1=\int_{\lambda}^{\infty}\ln(\xi-\lambda)\frac{d}{d\xi}\alpha^2(\xi)d\xi,\quad L_2=\int_{-\infty}^{\lambda}\ln(\lambda-\xi)\frac{d}{d\xi}\alpha^2(\xi)d\xi.$$

We put

$$\tilde\Theta(\lambda)=\Theta(\lambda)+L_1(\lambda)-L_2(\lambda)-{}^3/_4\pi+\arg\Gamma(i\alpha^2).$$

From (17) we get the relation

$$\tilde\Theta(\lambda)=\lambda x-\Phi(x)-\alpha^2\ln\Phi_{xx},\quad \Phi_x=\lambda. \tag{19}$$

We consider first of all the case when we can neglect the term $\alpha^2\ln\Phi_{xx}$ in (19). $\Phi(x)$ is then found by a Legendre transformation

$$x=\partial\tilde\Theta/\partial\lambda,\quad \Phi(x)=\lambda x-\tilde\Theta(\lambda).$$

Evaluating $\alpha^2\ln\Phi_{xx}$ as a correction we note that

$$\Phi_{xx}=\frac{\partial}{\partial x}\frac{\partial\Phi}{\partial x}=\frac{\partial\lambda}{\partial x}=\left(\frac{\partial x}{\partial\lambda}\right)^{-1}=\tilde\Theta_{\lambda\lambda}^{-1},$$

so that we can write (19) in the form

$$\Phi(x)=\lambda x-\tilde\Theta(\lambda)+\alpha^2\ln\tilde\Theta_{\lambda\lambda},\quad x=\partial\tilde\Theta/\partial\lambda. \tag{20}$$

By virtue of the properties of the Legendre transformation the condition $\Phi_{xx} > 0$ leads to $\tilde\Theta_{\lambda\lambda} > 0$ and vice versa; the conditions (6) can be written in the form

$$\left|\frac{d}{d\lambda}\ln\alpha^2(\lambda)\right|^2\ll\tilde\Theta_{\lambda\lambda},\quad \frac{d}{d\lambda}(|\tilde\Theta_{\lambda\lambda}|)^{1/2}\ll\frac{d}{d\lambda}\tilde\Theta_\lambda. \tag{21}$$

Integrating (21) we find that excluding the vicinity of zero the term $\alpha^2(\lambda)\ln\tilde\Theta_{\lambda\lambda}$ is small compared to $\tilde\Theta$ which justifies the preceding considerations. We now remember that by virtue of Eq. (5)

$$\tilde\Theta(\lambda,t)=\tilde\Theta_0(\lambda)+\lambda^2 t.$$

It is clear that as $t\to\infty$ the conditions (21) turn out to be satisfied independent of the form of $a(\lambda)$, $\tilde\Theta_0(\lambda)$. This fact also enables one to find the asymptotic form of the initial problem for Eq. (1).

We have[4)] for $t\to\infty$

$$x=2\lambda t+x_1(\lambda),\quad x_1(\lambda)=\partial\tilde\Theta_0/\partial\lambda,\quad \lambda\approx x/2t,$$

i.e.,

$$\Phi_{xx}=1/2t,\quad \alpha^2(\lambda)=\alpha^2(x/2t).$$

We have thus finally:

$$A^2(x,t)=\Phi_{xx}\alpha^2\left(\frac{x}{2t}\right)=\frac{1}{2\pi t}\ln\frac{1}{|a(x/2t)|}, \tag{22}$$

$$\Phi(x,t)=\frac{x^2}{4t}+\alpha^2\left(\frac{x}{2t}\right)\ln 2t-\tilde\Theta_0\left(\frac{x}{2t}\right). \tag{23}$$

Equations (22) and (23) completely determine the main term in the asymptotic form of $\psi(x, t)$. We note that Eq. (22) was obtained earlier by one of the authors[14] by directly solving the inverse problem equations. We note that the asymptotic formulae (22), (23) have the same structure as (2).

In conclusion, a few words on the nature of the approximations made. The condition (6) $A_{xx} \ll A\Phi_x^2$ is for Eq. (1) equivalent to the "non-linear geometric optics" approximation or to the equations of the one-dimensional gas dynamics with a negative pressure. The condition that there be no solitons means in that terminology the choice of initial conditions for which the negative pressure effects do not appear as a matter of principle. The quantity A^2 corresponds to the gas density, Φ to the hydrodynamic potential, and Φ_x to the velocity.

The condition $\Phi_{xx}>0$ means that gas separation takes place with a monotonic velocity profile, and the asymptotic regime corresponds to the self-similar separation regime. The method developed by us enables us to study the evolution of initial data, which satisfy conditions (6), up to the emergence into the asymptotic regime, according to the following scheme:

$$\Phi(x,0),\quad A(x,0)\xrightarrow{\text{I}}\alpha^2(\lambda),\quad \tilde{\Theta}_0(\lambda)\xrightarrow{\text{II}}\alpha^2(\lambda),\quad \tilde{\Theta}(\lambda,t)\xrightarrow{\text{III}}\Phi(x,t),\quad A(x,t).$$

In the first stage of the scheme $\tilde{\Theta}_0(\lambda)$ is found from Eq. (19), in the third state $\Phi(x, t)$ is recovered according to (20), while $\alpha^2(\lambda)$ is expressed directly in terms of A and Φ. Since in each stage of this scheme we have only operations which are local in x, we can get rid of the requirement that $A^2(x)$ decreases as $|x|\to\infty$ and assume that $A(x)$ increases in an arbitrary manner without contradicting conditions (6).

§2. NON-LINEAR SCHRÖDINGER EQUATION (DEFOCUSING MEDIUM)

Equation (1) with the lower sign describes the propagation of wavepackets in a defocusing medium. This equation can be integrated by means of the spectral problem[3]

$$i\frac{\partial u_1}{\partial x}+\psi^* u_2=\frac{\lambda}{2}u_1,\qquad -i\frac{\partial u_2}{\partial x}+\psi u_1=\frac{\lambda}{2}u_2.$$

Under the condition that $|\psi|$ decreases as $x\to\pm\infty$ the scattering matrix $a(\lambda)$, $b(\lambda)$ is determined as before; Eq. (5) also remains valid. The method for solving the direct and inverse problems when conditions (6) are satisfied differs in no essential way from the one given in §1. We give therefore at once the final formulae retaining the notation of the preceding section:

$$a(\lambda)=\exp\{\pi\alpha^2-i(L_1+L_2)\},$$

$$b(\lambda)=\frac{\sqrt{2\pi i}}{\alpha\Gamma(i\alpha^2)}\exp\left\{\frac{\pi\alpha^2}{2}+i\Phi_0+i(L_2-L_1)-i\alpha^2\ln\Phi_{xx}(x_0)\right\}\tag{24}$$

(so that $|a(\lambda)|^2-|b(\lambda)|^2=1$ and $a(\lambda)$ is analytical in the upper λ-half-plane).

The inverse scattering problem is solved by means of the formulae

$$\alpha^2(\lambda)=\frac{1}{\pi}\ln|a(\lambda)|,\qquad \Phi(x)=\lambda x-\tilde{\Theta}(\lambda)+\alpha^2\ln\Phi_{xx},\qquad x=\frac{\partial\tilde{\Theta}}{\partial\lambda}.$$

For the asymptotic behavior we have the formulae

$$\Phi\to\frac{x^2}{4t}-\alpha^2\left(\frac{x}{2t}\right)\ln 2t-\tilde{\Theta}_0\left(\frac{x}{2t}\right),$$

$$A^2\to\frac{1}{2t}\alpha^2\left(\frac{x}{2t}\right)=\frac{1}{2\pi t}\ln\left|a\left(\frac{x}{2t}\right)\right|.$$

We note that in contrast to the preceding section the formulae given here describe the asymptotic behavior of an arbitrary initial condition as there are no solitons in this problem.

§3. KORTEWEG-DE VRIES EQUATION

The Korteweg–de Vries (KdV) equation

$$u_t-6uu_x-u_{xxx}=0$$

can be integrated by means of the spectral problem

$$\frac{d^2\psi}{dx^2}+u\psi=-\frac{k^2}{4}\psi.\tag{25}$$

The scattering data $a(k)$ and $b(k)$ are determined by means of the solution

$$\psi\to e^{ikx/2},\qquad x\to+\infty,$$

as elements of its asymptotic form as $x\to-\infty$:

$$\psi\to a(k)e^{ikx/2}+b(k,t)e^{-ikx/2},$$

while

$$b(k,t)=b(k,0)e^{ik^3t}.\tag{26}$$

We put

$$u=2A\cos\Phi,$$

where A and Φ satisfy conditions (6) and also the condition

$$A^2\ll\Phi_x^2.$$

Under those assumptions the spectral problem (25) is equivalent to the problem of parametric resonance for a small perturbation of the frequency while the perturbation is quasi-periodic and has a linearly changing frequency. We look for the solution of the problem (25) in the form

$$\psi=\psi_1e^{ikx/2}+\psi_2e^{-ikx/2},$$

where we shall assume that ψ_1 and ψ_2 satisfy the set of equations

$$\frac{\partial^2\psi_1}{\partial x^2}+ik\frac{\partial\psi_1}{\partial x}+A(e^{i\Phi}+e^{-i\Phi})e^{-ikx}\psi_2=0,$$

$$\frac{\partial^2\psi_2}{\partial x^2}-ik\frac{\partial\psi_2}{\partial x}+A(e^{i\Phi}+e^{-i\Phi})e^{ikx}\psi_1=0.$$

The assumptions we have made mean that this set can be simplified to

$$i\frac{\partial\psi_1}{\partial x}+\tilde{A}e^{i\tilde{\Phi}}\psi_2=0,\qquad \tilde{A}=\frac{A}{k},$$

$$-i\frac{\partial\psi_2}{\partial x}+\tilde{A}e^{-i\tilde{\Phi}}\psi_1=0,\qquad \tilde{\Phi}=\Phi-kx,$$

i.e., be reduced to the problem which we have already studied above. For the direct scattering problem we have Eq. (24), but now

$$\alpha^2(k)=\frac{A^2}{\Phi_x^2\Phi_{xx}}=\frac{1}{\pi}\ln|a(k)|;$$

as before $\Phi_x=k$ and

$$\tilde{\Theta}(k)=\arg b(k)+L_2(k)-L_1(k)-\pi/4-\arg\Gamma(i\alpha^2).$$

The time evolution is according to (26) given by the formula

$$\tilde{\Theta}(k, t) = \tilde{\Theta}_0(k) + k^3 t.$$

For $\Phi(x, t)$ we have

$$\Phi(x, t) = kx - \tilde{\Theta}(k, t) - \alpha^2(k) \ln \tilde{\Theta}_{kk}.$$

As $t \to \infty$ we get

$$x = 3k^2 t, \quad k = (x/3t)^{1/2},$$

$$\Phi(x,t) = \left(\frac{4}{27}\frac{x^3}{t}\right)^{1/2} - \alpha^2\left(\left(\frac{x}{3t}\right)^{1/2}\right)\ln\left(6t\left(\frac{x}{3t}\right)^{1/2}\right) - \tilde{\Theta}_0\left(\left(\frac{x}{3t}\right)^{1/2}\right),$$

$$A^2(x,t) = \left(\frac{x}{108t^3}\right)^{1/2}\frac{1}{\pi}\ln\left|a\left(\left(\frac{x}{3t}\right)^{1/2}\right)\right|. \qquad (27)$$

Equations (27) are applicable if $xt^{-1/3} \gg 1$. In the range $t^{1/3} \ll x \ll t$ we have

$$\Phi \approx \left(\frac{4}{27}\frac{x^3}{t}\right)^{1/2} + \ldots, \quad A \to \alpha(0)\left(\frac{x}{108t^3}\right)^{1/4},$$

i.e., the asymptotic form is self-similar

$$u_{s.s} \approx 2\frac{\alpha(0)}{t^{2/3}}\left(\frac{\xi}{108}\right)^{1/4}\cos\left(\frac{4}{27}\xi^3\right)^{1/2}, \quad \xi = \frac{x}{t^{1/3}}, \qquad (28)$$

which agrees with the well known fact that the KdV equation has a self-similar solution[15]

$$u = \frac{1}{t^{2/3}} f\left(\frac{x}{t^{1/3}}\right)$$

with the asymptotic behavior (28). It was shown in[16] that in the region $x \lesssim t^{1/3}$ in which our consideration is inapplicable any initial condition also emerges into the self-similar solution. When $\xi \gg 1$ the self-similar solution joins up with the asymptotic form found by us, thus giving a complete solution of the problem when there are no solitons.

§4. SINE-GORDON EQUATION

The equation

$$u_{tt} - u_{xx} + \sin u = 0$$

can be integrated by means of the spectral problem[5]:

$$-\psi_{2x} + \frac{iw}{4}\psi_2 + \frac{1}{16\lambda}e^{iu}\psi_1 = \lambda\psi_1,$$
$$\psi_{1x} + \frac{iw}{4}\psi_1 + \frac{1}{16\lambda}e^{-iu}\psi_2 = \lambda\psi_2, \qquad (29)$$

where

$$w = u_t + u_x.$$

Since we apply to this problem a method different from the one expounded above it is necessary for us to discuss in detail the properties of the solution of the set (29).

We consider two solutions of the set $\varphi(x, \lambda)$ and $\psi(x, \lambda)$ which are defined by the asymptotic forms

$$\psi \to \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}\exp\left\{i\left(\lambda - \frac{1}{16\lambda}\right)x\right\}, \quad x \to +\infty,$$
$$\varphi \to \frac{-i}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix}\exp\left\{-i\left(\lambda - \frac{1}{16\lambda}\right)x\right\}, \quad x \to -\infty.$$

The solutions

$$\psi(x,\lambda), \quad \tilde{\psi}(x,\lambda) \triangleq \begin{pmatrix}\psi_2^*\\ -\psi_1^*\end{pmatrix}$$

form a fundamental system so that the solutions φ, ψ, and $\tilde{\psi}$ are linearly dependent:

$$\varphi(x, \lambda) = a(\lambda)\tilde{\psi}(x, \lambda) + b(\lambda, t)\psi(x, \lambda). \qquad (30)$$

This relation is the definition of the scattering data (as above we consider the situation without solitons). In that case

$$b(\lambda, t) = b(\lambda)\exp\left\{2i\left(\lambda + \frac{1}{16\lambda}\right)t\right\}.$$

The function

$$\varphi(x,\lambda)\exp\left\{i\left(\lambda - \frac{1}{16\lambda}\right)x\right\}, \quad \psi(x,\lambda)\exp\left\{-i\left(\lambda - \frac{1}{16\lambda}\right)x\right\}$$

is analytical in the upper λ-halfplane and regular in the point $\lambda = 0$. $a(\lambda)$ has the same properties. A direct consequence of this fact is the integral relation

$$\tilde{\psi}(x,\lambda)\exp\left\{i\left(\lambda - \frac{1}{16\lambda}\right)x\right\} = -\frac{i}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix}$$
$$+ \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{b(\lambda', t)}{a(\lambda')}\psi(x, \lambda')\frac{\exp\{i(\lambda' - 1/16\lambda')x\}}{\lambda' - \lambda + i0}d\lambda', \qquad (31)$$

which is the set of equations of the inverse scattering problem, if we consider it as an equation for the functions $\psi_{1,2}(x, \lambda)$.

We consider the solution of this equation as $t \to \infty$. We shall then follow[14]. There is in principle no difference between the details of the calculations and those given in[14] so that we shall omit them.

As $u(x, t) \to 0$ when $t \to \infty$ the functions ψ_1 and ψ_2 will differ little from an exponential in the vicinity of any point x. We put

$$\psi = U_1\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}\exp\left\{i\left(\lambda - \frac{1}{16\lambda}\right)x\right\}$$
$$-U_2\frac{i}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix}\exp\left\{-i\left(\lambda - \frac{1}{16\lambda}\right)x - 2i\left(\lambda + \frac{1}{16\lambda}\right)t\right\}, \qquad (32)$$

where U_1 and U_2 are functions of x and t, and are slow compared to the exponential indexes. From the conservation of the Wronskian of the set (29) it follows that

$$|U_1|^2 + |U_2|^2 = 1. \qquad (33)$$

Let, further, $x = Vt$. Substituting (32) into (31) and using the well known relation from the theory of generalized functions

$$\lim_{t\to\infty}\frac{e^{iF(\lambda')t}}{\lambda' - \lambda - i0} = \begin{cases}-2\pi i\delta(\lambda - \lambda')e^{iF(\lambda)t}, & F'(\lambda) < 0\\ 0, & F'(\lambda) > 0\end{cases},$$

we find that

$$U_2^*(\xi, \lambda) = \frac{b(\lambda)}{a(\lambda)}U_1(\xi, \lambda)\theta(\xi^2 - \lambda^2),$$

where

$$\xi^2=\frac{1}{16}\frac{1-V}{1+V},\quad -1<V<1,\quad \theta(x)=\begin{cases}1, & x>0,\\ 0, & x<0.\end{cases} \tag{34}$$

and we see, moreover, that $U_1(\xi,\lambda)$ is analytical in the upper λ half-plane and has no zeroes there, if $a(\lambda)$ has no zeroes (the zeroes of $a(\lambda)$ correspond to solitons). This fact enables us to determine U_1 and U_2:

$$\begin{aligned} U_1&=\exp\left\{\frac{1}{\pi i}\int\limits_{-\infty}^{\infty}\frac{\theta(\xi^2-\lambda'^2)\ln|a(\lambda')|}{\lambda'-\lambda-i0}d\lambda'\right\},\\ U_2^{*}&=b(\lambda)\theta(\xi^2-\lambda^2)\exp\left\{-\frac{1}{\pi i}\int\limits_{-\infty}^{\infty}\frac{\theta(\lambda'^2-\xi^2)\ln|a(\lambda')|}{\lambda'-\lambda-i0}d\lambda'\right\}. \end{aligned} \tag{35}$$

The solution (35) is valid for all ξ except a narrow ($\propto t^{-1/2}$) region near $\xi=\lambda$.

Without making any concrete assumptions about the form of $u(x, t)$ we have thus determined the solution of the set (29) in the region which we called the asymptotic one above. To determine the solution in the resonance region it is necessary to make such assumptions. We shall look for $u(x, t)$ in the form

$$u=\frac{1}{\sqrt{t}}(C(V)t^{i\varkappa(V)}\exp\{-i(1-V^2)^{1/2}t\}+\text{c.c.}),\qquad V=\frac{x}{t}. \tag{36}$$

The motivation for such a choice of $u(x, t)$ has been given in sufficient detail in the Introduction. (Moreover, it is clear that outside the light cone $u(x, t)$ must rapidly turn to zero.) Substituting now (32), (36) and w, calculated with the necessary accuracy, into (29) we verify that the set (29) can, when $|\xi-\lambda|\ll 1$, be simplified to the form

$$\begin{aligned} &i\frac{1+16\lambda^2}{2\sqrt{t}}\frac{\partial U_1}{\partial\xi}+(2-V^2)C^{*}(V)t^{-i\varkappa}\exp\left\{-\frac{4i(\xi-\lambda)^2t}{\lambda(1+16\lambda^2)}\right\}U_2=0,\\ &i\frac{1+16\lambda^2}{2\sqrt{t}}\frac{\partial U_2}{\partial\xi}+(2-V^2)C(V)t^{i\varkappa}\exp\left\{\frac{4i(\xi-\lambda)^2t}{\lambda(1+16\lambda^2)}\right\}U_1=0. \end{aligned}$$

Introducing the notation

$$\begin{aligned} B(\lambda)&=\frac{1}{4}(2-V^2)(1-V^2)^{1/4}C(V)t^{i\varkappa(V)},\\ z&=\left(\frac{8t}{\lambda(1+16\lambda^2)}\right)^{1/2}(\xi-\lambda),\qquad \lambda^2=\frac{1}{16}\frac{1-V}{1+V}. \end{aligned} \tag{37}$$

we get

$$i\frac{\partial U_1}{\partial z}+B^{*}e^{-iz^2/2}U_2=0,\qquad i\frac{\partial U_2}{\partial z}+Be^{iz^2/2}U_1=0. \tag{38}$$

The set (38) is exactly the same as the set (7), the solution of which we know already. We can find the function $B(\lambda)$ in which we are interested without writing down equations such as (16), (17) (which, of course, can now easily be found). To do this we eliminate U_2 from the set (38). We find for U_1

$$\frac{d^2U_1}{dz^2}+iz\frac{dU_1}{dz}+|B|^2U_1=0. \tag{39}$$

The function U_2 satisfies Eq. (10) in which we must write $|B|^2$ instead of α^2. The expression for U_2 is given by Eq. (12); the solutions of (39) can also be expressed in terms of parabolic cylinder functions (it is sufficient to note that (39) is obtained from (10) by taking the complex conjugate of the latter):

$$U_2=e^{-iz^2/4}(d_1D_{-1-i|B|^2}(z\sqrt{i})+d_2D_{-1-i|B|^2}(-z\sqrt{i})). \tag{40}$$

The constants $c_{1,2}$ and $d_{1,2}$ occurring in (12) and (40) can be determined from joining these solutions with the solutions (35). After this we can easily find $B(\lambda)$ by substituting U_1 and U_2 in the second of Eqs. (38). The result has the form

$$\begin{aligned} |B(\lambda)|^2&=\frac{1}{\pi}\ln\frac{1}{|a(\lambda)|},\\ B(\lambda)&=\frac{b^{*}(\lambda)}{\sqrt{2\pi}}\frac{\Gamma(1-i|B|^2)}{|a(\lambda)|^{1/2}}\left[\frac{8t}{\lambda(1+16\lambda^2)}\right]^{i|B|^2}\\ &\quad\times\exp\left\{i\left(\Phi_2-\Phi_1+\frac{3\pi}{4}\right)\right\}, \end{aligned}$$

where

$$\Phi_1=\frac{1}{\pi}\ln|2\lambda|\ln|a(\lambda)|+\frac{1}{\pi}\int\limits_{-\lambda}^{\lambda}\ln|\lambda'-\lambda|\frac{d}{d\lambda'}\ln|a(\lambda')|d\lambda',$$

$$\Phi_2=-\frac{1}{\pi}\int\limits_{-\infty}^{-\lambda}\frac{\ln|a(\lambda')|}{\lambda'-\lambda}d\lambda'+\frac{1}{\pi}\int\limits_{\lambda}^{\infty}\ln|\lambda'-\lambda|\frac{d}{d\lambda'}\ln|a(\lambda')|d\lambda'.$$

1) An attempt to consider this problem was made in[9]. However, after one removes the calculational errors from that paper there remains a clear total lack of perspective for the approach used.

2) Exceptions are the cases when an averaging technique is applicable, starting at the time $t=0$, or, on the other hand, when the asymptotic solution has a self-similar character. In the latter case the form of the solution is determined solely by very rough characteristics of the initial condition (see[11,12]).

3) A characteristic feature of the oscillating asymptotic behavior of the one-dimensional non-linear equations is the presence in the phase of a term proportional to $\ln t$; this is the only difference between the non-soliton asymptotic behavior of the non-linear equations and that of the corresponding linearized problems. One can easily understand the reason for the occurrence of the logarithm: If the amplitude decreases as $t^{-1/2}$ (as in any linear problem) the non-linear shift in the frequency (quadratic in the amplitude) is proportional to t^{-1} and this gives $\ln t$ in the phase. As far as we know, L. D. Faddeev was the first to draw attention to this fact.

4) It is incorrect to retain in λ terms $\propto t^{-1}$ since, strictly speaking, the concept of a "resonance point" introduced above has a meaning only in the variables $\xi=\Phi_x$ so that $x=x(\xi)$ (see Sec. 4).

1C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura, Phys. Rev. Lett. 19, 1095 (1967).

2V. E. Zakharov and A. B. Shabat, Zh. Eksp. Teor. Fiz. 61, 118 (1971) [Sov. Phys. JETP 34, 62 (1972)].

3V. A. Zakharov and A. B. Shabat, Zh. Eksp. Teor. Fiz. 64, 1627 (1973) [Sov. Phys. JETP 37, 823 (1973)].

4M. J. Ablowitz, D. J. Kaup, A. C. Newell, and H. Segur, Phys. Rev. Lett. 30, 1262 (1973).

5V. E. Zakharov, L. A. Takhtadzhyan, and L. D. Faddeev, Dokl. Akad. Nauk SSSR 219, 1334 (1974) [Sov. Phys. Dokl. 19, 824 (1975)].

6V. E. Zakharov, Zh. Eksp. Teor. Fiz. 60, 993 (1971) [Sov. Phys. JETP 33, 538 (1971)].

[7]S. V. Manakov, Zh. Eksp. Teor. Fiz. 65, 505 (1973) [Sov. Phys. JETP 38, 248 (1974)].
[8]L. A. Takhtadzhyan, Zh. Eksp. Teor. Fiz. 66, 476 (1974) [Sov. Phys. JETP 39, 228 (1974)].
[9]M. J. Ablowitz, and A. C. Newell, J. Math. Phys. 14, 1277 (1973).
[10]Transl. in: Nelinieĭnaya teoriya rasprostraneniya voln (Non-linear Theory of Wave Propagation) Mir, 1970.
[11]A. V. Gurevich and L. P. Pitaevskiĭ, Zh. Eksp. Teor. Fiz. 65, 590 (1973) [Sov. Phys. JETP 38, 291 (1974)].
[12]E. Ya. Khruslov, Pis'ma Zh. Eksp. Teor. Fiz. 21, 469 (1975) [JETP Lett. 21, 217 (1975)].
[13]A. Erdelyi *et al.*, Higher Transcendental Functions, McGraw-Hill, New York, 1953, Vol. 2.
[14]S. V. Manakov, Zh. Eksp. Teor. Fiz. 65, 1392 (1973) [Sov. Phys. JETP 38, 693 (1974)].
[15]V. I. Karpman, Nelinieĭnye volny v dispergiruyushchikh sredakh (Non-linear Waves in Dispersive Media) Nauka, 1973 [translation published by Pergamon Press, Oxford, 1975].
[16]A. B. Shabat, Dokl. Akad. Nauk SSSR 211, 1310 (1973).

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Electric conductivity of a non-ideal plasma

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Results are presented of measurements of the electric conductivity of a dense plasma with strong interparticle Coulomb interaction. Experiments with air, neon, argon and xenon were carried out with an explosive nonideal-plasma generator. A four-point probe recording technique was used. The Coulomb component of the electrtic conductivity is compared with that predicted by theories of a non-ideal plasma.

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1. INTRODUCTION

Electric conductivity is one of the most essential plasma characteristics that determine its dissipative heating and the interaction with the electromagnetic field in the operation of magnetohydrodynamic and magnetocumulative generators, thermonuclear, laser, and other pulsed devices that require a considerable energy concentration.[1] In view of the high charge density, the average electrostatic-interaction energy turns out to be of the order of the kinetic energy of particle motion, so that deviations of the plasma from ideal determine the equilibrium and kinetic properties of such a medium.[2]

At the present time, a consistent theoretical calculation of the transport characteristics of disordered electron systems can be carried out only in the case of weak interaction at $\Gamma = e^2/kTr_D \ll 1$ ($r_D = \sqrt{kT/8\pi n_e e^2}$) on the basis of the kinetic equations or by the method of time-dependent correlation functions.[3,4] As the deviation from ideal increases, however, it becomes quite difficult to justify the initial kinetic equations and the methods for their solutions. In particular, in view of the strong collective interaction in a dense plasma it is impossible to separate unambiguously the characteristic times of the elementary processes, and the time of evolution of the system under the influence of an external field is no longer, generally speaking a Markov process.[4] Allowance for the bound states in a partly ionized plasma[3] constitutes a special problem, owing to the absence of the corresponding kinetic equations and transport cross sections that would permit the use of approximate semi-empirical methods. The results of the theoretical calculations of the low-frequency ($\omega_2 \lesssim \omega_p^2 = 4\pi n_e e^2/m_e$) conductivity of the plasma therefore begin to differ noticeably starting with $\Gamma \sim 0.1$.[5] Extrapolation of these theories to the region $\Gamma \gtrsim 1$ of increased deviation from ideal, as a rule, leads to unphysical divergences that are connected with the use of a finite number of Sonine polynomials when solving the corresponding kinetic equations by the Chapman-Enskog method.

The difficulties in the experimental study of the electrophysical properties of a non-ideal plasma are connected with the need for highly concentrating the energy, and with the absence of well-developed methods for measurements in optically dense media. The region of parameters up to $\Gamma \sim 0.7$ can be reached relatively easily in stationary (see[5]) and in pulsed[6–9] experiments, the results of which, however, frequently contradict each other because of considerable experimental errors and interpretation inaccuracies.[10] The transition to increased deviations from ideal entails great difficulties in the generation and the diagnostics of the plasma. The number of pertinent experiments is quite limited[11–15] and most are of qualitative character, in view of the lack of directly recorded and reliable information on the physical parameters of the plasma.[11–13]

We present here the results of the measurement of the electric conductivity of a dense low-temperature plasma in a wide range of non-ideality parameters $\Gamma \sim 0.3$–4.5. The absence of complicated molecular and ion-molecular formations, the fact that the cross sections of the elementary processes have been investigated in detail, and the high molecular weight have dictated the choice of inert gases as the investigation

