The Hawking process at the boundary and in the interior of a black hole

I. D. Novikov

Institute for Cosmic Research, USSR Academy of Sciences
(Submitted February 18, 1976)
Zh. Eksp. Teor. Fiz. **71**, 393–398 (August 1976)

The consequences of particle production and vacuum polarization inside and near the boundary r_g of a black hole are considered. Restrictions are obtained on the components of the energy-momentum tensor of "real" heavy particles near r_g, for a black hole which is in thermodynamic equilibrium with external radiation. The internal inconsistency of calculations yielding a nonzero energy flux from a black hole on the background of a Kruskal space-time metric without matter and neglecting the reaction of the process on the metric is pointed out.

PACS numbers: 97.70.Ss, 95.20.+g

The purpose of the present note is to consider some aspects of the Hawking process[1] describing particle production and vacuum polarization by a black hole (BH) in a freely falling reference frame, and to derive some physical consequences from this discussion.

We shall consider the BH without rotation in a freely falling Lemaître coordinate system with the free-fall velocity vanishing at infinity, and characterized by the metric

$$ds^2=d\tau^2-e^\lambda dR^2-r^2(d\theta^2+\sin^2\theta d\varphi^2). \tag{1}$$

Of course, taking into account quantum processes, this system differs (albeit by a small quantity) from the corresponding Lemaître system for $T_{ik}=0$.[2] We shall call the latter the unperturbed system. In the metric (1) there are no physical singularities on the horizon (cf. for this below).

We first show that from a formal point of view the Hawking process has the result that the boundaries of the R- and T-regions[3,4] (we denote $r_{RT}=r_{RT}(R,\tau)$) and the horizon (we use the notation $r_g=r_g(R,\tau)$) do not coincide.[1)] According to Penrose's theorem[6] the horizon is formed by null-geodesics. Owing to the Hawking process $dr_g/d\tau<0$. This means that at the horizon the world line $r, \theta, \varphi \equiv$ const. is situated outside the light cone along r_g and is thus spacelike. The same holds for a region $r>r_g$ on account of continuity, i.e., in some region outside the horizon the line $r, \theta, \varphi\equiv$ const. is spacelike. By the definition of the R- and T-regions[3,4] these points belong to the T-region. The boundary of the T-region is determined by the equation

$$e^{-\lambda}=\left(\frac{dr/d\tau}{dr/dR}\right)^2. \tag{2}$$

The statement above that this is a "splitting" of the Schwarzschild sphere into the horizon and the boundary of the R- and T-regions has a formal character, related to the fact that the distance between r_{RT} and r_g is extremely small and when quantum effects are taken into account, it has no direct physical meaning. Let us, indeed, estimate this distance operating with concepts of a nonquantum metric.

According to[7] the characteristic time of variation of r_g measured by the clock of a distant observer is $\tau\simeq 10^3 r_g^3/r_{\rm pl}^2$, where $r_{\rm pl}$ is the Planck length, $r_{\rm pl}\simeq 10^{-33}$ cm, $c=1$.

Consequently, in the Lemaître system the variation of the slope of the null-geodesics in the $(R\tau)$ plane compared to the slope in the classical case $T_{\mu\nu}=0$ near the horizon r_g will be

$$\delta_1\left(\frac{dR}{d\tau}\right)\approx\frac{r_g}{\Delta\tau}\approx\frac{10^{-3}r_{pl}^2}{r_g^2}, \tag{3}$$

where $\Delta\tau\simeq 10^3 r_g^3/r_{\rm pl}^2$ is the evaporation time of the BH.

On the other hand the slope of the null-geodesics of the unperturbed solution (with $T_{\mu\nu}=0$) is

$$dR/d\tau=\pm\sqrt{r_g/r}. \tag{4}$$

The change of this slope in a displacement by a distance δr along points with different r near r_g is

$$\delta_2\left(\frac{dR}{d\tau}\right)\approx\frac{\delta r}{r_g}. \tag{5}$$

Equating (3) and (5) we obtain the variation δr corresponding to the distance from r_g to the point where the slope of the null-geodesics corresponds to the slope in the unperturbed solution $r=$ const for the null-geodesic, i.e., to the boundary between the R- and T-regions, r_{RT}. Thus

$$r_{RT}-r_g=\delta r\approx 10^{-3}r_{pl}(r_{pl}/r_g). \tag{6}$$

In order of magnitude this difference corresponds also to the time difference $\Delta\tau$ for $R=$ const. The quantity δr in (6) is considerably smaller than the Planck length, and therefore has no direct physical meaning.

The average value of the components of the energy-momentum tensor for the BH in vacuum is everywhere nonzero owing to quantum effects. Far from the BH the values of $T_{\mu\nu}$ are determined by the flux of real particles produced by the hole and going off to infinity.[7] It is easy to show that at the horizon (the boundary of the BH) and in its vicinity, the average value of the energy density (the component T_0^0) must be negative.

Indeed, operating in an unquantized metric we write the slope of the radial null-geodesics at the points $r_g < r < r_{RT}$. It corresponds to the negative derivative

$$dr/d\lambda<0, \tag{7}$$

where λ is the effective affine parameter along the geodesic, which increases into the future. On the other hand, since these geodesics are situated outside the horizon $r > r_g$, they go off into lightlike infinity J^+, where $r \to \infty$. Hence at some point along the geodesic the derivative $dr/d\lambda$ must turn from being negative to positive. For a radial congruence r is $a^{1/2}$, where a is the cross section of the congruence of geodesics. According to the focusing theorem (cf., e.g., [8]), if $da^{1/2}/d\lambda \equiv dr/d\lambda$ is negative, it can become positive[2)] only if $T^0_0 < 0$, proving the assertion. Starobinskiĭ has obtained the condition $T^0_0 < 0$ near the gravitational radius from a direct calculation of the components T^k_i (private communication). Here we obtain the values of T^k_i near r_g in Schwarzschild coordinates starting from the physically obvious condition that the values of T^k_i be nonsingular in a freely falling coordinate system (the Lemaître system). We shall denote the components of the energy-momentum tensor in the Lemaître system by $\tilde{T}_{ik}$, and on r_g we denote these components as follows:

$$\tilde{T}_{00}=A,\ \tilde{T}_{10}=B,\ \tilde{T}_{11}=C. \tag{8}$$

We shall not be interested in the other components of $\tilde{T}_{ik}$, since they are not subject to transformations when one goes over to coordinate systems that move radially, and moreover they are finite.

We go over from the Lemaître coordinates to Schwarzschild coordinates.[3)] As a result we obtain near r_g (we have set $c = 1$).

$$\begin{aligned} T_{00}&=A+2B+C,\\ T_{01}&=\frac{1}{(1-r_g/r)}(A+2B+C),\\ T_{11}&=\frac{1}{(1-r_g/r)^2}(A+2B+C). \end{aligned} \tag{9}$$

Since near the gravitational radius the energy flux must be directed outwards (since r_g decreases with time), $T^1_0 = -T_{01}(1-r_g/r) > 0$. Consequently, $(A+2B+C) < 0$. We denote $-(A+2B+C) = D$ and rewrite (9) in terms of mixed components:

$$T^0_0=-\frac{T^1_0}{(1-r_g/r)}=-T^1_1=-\frac{D}{(1-r_g/r)}. \tag{10}$$

The quantity D is determined by the flux far from the BH and has the order of magnitude[4)] $D \simeq \hbar c/r_g^4$.

The quantities (10) correspond to negative energy density flowing into the BH. Any finite positive value of the energy density at r_g, flowing outward would lead to an infinite collision energy of the free falling matter with this flux. The values of the components $\tilde{T}_{\mu\nu}$ on r_g are not only finite (in distinction from the Schwarzschild coordinate system), but are even small in the sense that the additional curving of space-time produced by them is negligibly small compared to the curvature at r_g in the unperturbed solution. Indeed, in the Einstein equations the terms with $\tilde{T}^\nu_\mu$ are of the order $r_g^{-2}(r_{pl}/r_g)^2$, whereas the curvature of space-time is of the order r_g^{-2}.

A freely falling coordinate system is a natural system for the consideration of effects near the event horizon. It is free from singularities at the horizon and allows in a natural way to avoid the paradoxes which occur in the other systems which are not applicable at r_g, e.g., the Schwarzschild system. The simplest example are the values of T^ν_μ in the Schwarzschild coordinates, (10), which diverge at r_g and the values (8) in the Lemaître coordinates, which are finite at r_g. The reason for the divergence in (10) is the fact that the Schwarzschild coordinates do not exist at r_g, i.e., there is an unphysical requirement that the accelerations increase without bound as $r \to r_g$ in this coordinate system.

The discussion in the Lemaître coordinates allows one to clarify the situation with a BH in thermodynamical equilibrium with external radiation of the same temperature as the BH. In the Schwarzschild metric this problem has been considered by Zel'dovich[9] (cf. also[10]). In[9] the conclusion was reached that there exists a considerable number of real heavy energetic particles which are created near r_g even at low temperature at infinity, and which do not leave the region.

The discussion in Lemaître coordinates (1) allows one to estimate the values of the energy-momentum tensor of "real" heavy particles, $T^\nu_{\mu(*)}$ near r_g. The tensor $T^\nu_{\mu(*)}$ is determined by the local thermodynamic equilibrium; therefore in the Schwarzschild coordinate system there are no energy fluxes and $T^1_{0(*)} \equiv 0$. The transition to Lemaître coordinates is determined by the equations ($c = 1$)

$$\begin{aligned} \tilde{T}^0_{0(*)}&=\frac{1}{(1-r_g/r)}\left(T^0_{0(*)}-\frac{r_g}{r}T^1_{1(*)}\right),\\ \tilde{T}_{0(*)}&=\frac{1}{(1-r_g/r)}(T_{0(*)}-T_{1(*)}),\\ \tilde{T}^1_{1(*)}&=\frac{1}{(1-r_g/r)}\left(-\frac{r_g}{r}T^0_{0(*)}+T^1_{1(*)}\right). \end{aligned} \tag{11}$$

The components $\tilde{T}^\nu_{\mu(*)}$ must be finite and should not exceed (8). Hence, the components $T^\nu_{\mu(*)}$ in Schwarzschild coordinates must decrease at least proportionally to $(r-r_g)/r_g$ as r_g is approached. We recall that the components T^ν_μ in (10) increase like $r_g/(r-r_g)$ as one approaches r_g. Thus, at least at the horizon, the energy density of "real" heavy particles in the proper coordinate system (where $T^1_{0(*)} \equiv 0$) must be equal to zero. If it differs from zero in the neighborhood of r_g, then inside the horizon it is probably in a certain sense equivalent to negative energy for an observer at infinity.

We recall that heavy particles can be excited only at distances Δl from the horizon which are smaller than $\hbar/mc$, i.e., at distances not exceeding the Compton wavelength of the particle, and that a change of particle densities in this zone has a limited meaning.

Finally, we turn to phenomena which occur inside the event horizon owing to the particle production process. We introduce inside r_g in the T-region in the unperturbed solution the T-coordinate system with the metric[11]

$$ds^2=\frac{dT^2}{(r_g/T-1)}-\left(\frac{r_g}{T}-1\right)dx^2-T^2(d\theta^2+\sin^2\theta\,d\varphi^2). \qquad (12)$$

Here T is the temporal coordinate and x is the radial spatial coordinate. This system is spatially homogeneous (in the unperturbed solution!). Of course, if one takes into account the Hawking process, r_g is not a constant and the homogeneity is violated. But in a region bounded in x (of size of the order of several r_g) the system is homogeneous to a good approximation.[5)] Near the real singularity $T=0$ there occurs particle production; the spatial curvature is here unimportant and the process occurs as in the flat homogeneous model.[13] In the course of contraction $T\to 0$ and the energy density related to the particle production process is[13]

$$\varepsilon\approx\hbar/c^3T^4. \qquad (13)$$

For $T\simeq r_g/c$ this quantity can be rewritten as $\varepsilon=\hbar c/r_g^4$, coinciding in order of magnitude with the absolute value of T_0^0 in (10), related to the Hawking process.

We note the following important circumstance. The Hawking process has so far been calculated without taking into account its reaction on the metric, i.e., the calculations were done in the unperturbed system. As we have seen above, the energy flux is nonzero, $T_0^1\neq 0$, both in Lemaître and in Schwarzschild coordinates. This component is different from zero in the T-system (12), a fact which is easily obtained by going over from the Lemaître coordinates (1) to the metric (12). In (12) $T_0^1=D(r_g/T-1)^{-2}$. But an unbounded unperturbed T-system is homogeneous! Both directions of the radial coordinate in it are equivalent. This means that the calculation of quantum processes may give a nonvanishing flux T_0^1 in it either if there exists an asymmetry with respect to the T-region of other space-time regions which are not encompassed by the T-system (we shall return to this elsewhere), or if account is taken of the inhomogeneity of the self-consistent solution, i.e., if the reaction of the process on the metric is taken into account, or, finally, if account is taken of the inhomogeneities in the boundary conditions, i.e., of the fact that in the collapse of the star the T-system (12) is bounded in x and must be "joined smoothly" to the matter. However, it is sometimes asserted[14,15] that Hawking's result can be obtained considering an unbounded everywhere empty Kruskal metric. In this case the T-system is completely symmetric relative to the expanding T-region and the two R-regions of the Kruskal diagram (cf., e.g.,[4]). Both directions of the x-axis of the T-system are completely equivalent and it seems paradoxical that one obtains a T_0^1 which is different from zero in the T-system.

Finally, we note that the Hawking process radically changes the topology of phenomena similar to the collapse of a charged ball[16] and others, where if quantum phenomena are not taken into account there would be more than one causally connected region with Euclidean infinity. These questions will be dealt with in another publication.

1)This is the situation for regions where $T_{ik}\neq 0$ (cf., e.g.,[5]).
2)At least in some reference systems.
3)The transformation is carried out, of course, by means of the formulas of the unperturbed solution.
4)Starobinskiĭ has noted (private communication) that taking account of the conservation law $T^k_{i;k}$ allows one to establish a relation among the coefficients in the following terms of the expansion (10) in powers of $(1-r_g/r)$.
5)For formal reasons the system (12) does not enter into the Bianchi classification of homogeneous models, however, from a physical point of view, it is homogeneous (cf.[12]).

1S. W. Hawking, Nature, 248, 30 (1974).
2L. D. Landau and E. M. Lifshits, Teoriya polya (Classical theory of fields), Nauka, 1973 [English Translation, Pergamon, 1974].
3I. D. Novikov, Soobshcheniya GAISh (Reports of the Sternberg Observatory), No. 132, 3 (1964).
4Ya. B. Zel'dovich and I. D. Novikov, Teoriya tyagoteniya i evolyutsiya zvezd (Gravitation Theory and Stellar Evolution), Nauka, 1971 [Engl. Transl. Univ. of Chicago Press, 1971].
5I. D. Novikov, Soobsch. GAISh (cf.[3]), No. 132, 43 (1964).
6R. Penrose, in "Batelle Rencontres," C. de Witt and J. A. Wheeler, Eds., Benjamin, N. Y., 1968.
7D. N. Page, Particle Emission Rate from a Black Hole, Preprint OAP-419, Caltech, 1975.
8C. W. Misner, K. S. Thorne and J. A. Wheeler, Gravitation, Freeman, San Francisco, 1973.
9Ya. B. Zel'dovich, Preprint IBM (Inst. Appl. Math.) No. 121 (1975).
10G. W. Gibbons and M. J. Perry, Black Holes in Thermal Equilibrium, Preprint, Cambridge University, 1976.
11I. D. Novikov, Astron. Zh. 38, 564 (1961) [Sov. Astron. 5, 423 (1961)].
12L. P. Grishchuk, Candidate (Ph.D.) Dissertation, GAISh, 1967.
13Ya. B. Zel'dovich and A. A. Starobinskiĭ, Zh. Eksp. Teor. Fiz. 61, 2161 (1971) [Sov. Phys. JETP 34, 1159 (1972)].
14T. Damour and R. Ruffini, Preprint, Princeton University, 1976.
15J. B. Hartle and S. W. Hawking, Preprint, OAP-433, Caltech, 1975.
16I. D. Novikov, Pis'ma Zh. Eksp. Teor. Fiz. 3, 223 (1966) [JETP Lett. 3, 142 (1966)].

Translated by M. E. Mayer