The wave functions of relativistic bound systems

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Invariant wave functions that appear upon the consideration of a state vector defined on the surface of a light wave front are introduced for the description of bound systems. The main distinction of this parametrization from the nonrelativistic case can be interpreted as a deviation from spherical symmetry at small distances even in the S-wave. A light-cone diagram technique is discussed; the wave functions are related to the vertex functions of this diagram technique. The machinery developed here is necessary for the determination of the high-momentum components of nuclear wave functions, in the relativistic theory of nuclear reactions, and in composite models of elementary particles.

1. INTRODUCTION

The problem of the description of bound states has at present surpassed the framework of a purely academic problem and has acquired a special significance in applications to nuclear physics problems. The reason for this is that experiments have recently moved into a region where the knowledge of nuclear wave functions (WF) is necessary at relativistic values $q \sim m$ of the nucleon momenta, a region where the nonrelativistic WF lose their meaning. Thus, the formfactor of the deuteron is known up to momentum transfers 6 (GeV/c)^2. [1]

The problem consists in finding a formalism in which the relativistic WF would "abut" as closely as possible on the nonrelativistic ones: they should admit a probabilistic interpretation, should depend on three-vectors like their nonrelativistic counterparts and at the same time exhibit relativistic invariance, they should admit a representation in coordinate space and should go over in the nonrelativistic limit into the known nonrelativistic functions. It is clear that only within the framework of an invariant formalism which bares to the limit the physical meaning of the WF and the nature of their functional dependences can one hope to obtain a successful resolution of a quite complex problem: to find the WF of nuclei at nucleon momenta of the order of their masses. The problem of determination of relativistic WF was posed and considered in coordinate space in Shaprio's paper. [2]

The purpose of the present paper is to introduce WF which exhibit the enumerated properties. Preliminary results have been published in the review[3] and were summarized in[4].

Before going over to the formal side of the matter we explain what we have in mind when we talk about relativistic WF and describe the result we have obtained. Our point of view is that studying a relativistic bound system (in general, having many components and containing an indefinite number of particles) means a determination of the probabilities with which one can find a definite number of particles in the system, as well as their momentum distribution. The answer to this problem is given by an expansion of the state vector $\Phi(p)$ of the system in terms of states with fixed particle number:

$$\Phi(p) = N^{-1/2} \sum_n \int C(k_1, \ldots, k_n, p) a^+(k_1) \ldots a^+(k_n) |0\rangle$$
$$\times \delta^{(3)}(k_1 + \ldots + k_n - p) \frac{d^3 k_1}{\sqrt{2\varepsilon_1}} \ldots \frac{d^3 k_n}{\sqrt{2\varepsilon_n}}, \tag{1}$$

with all vectors k_i on the mass shell $k^2 = m^2$, $p^2 = M^2$. The coefficients of this expansion are the Fock components $C(k_1, \ldots, k_n, p)$ and determine the probabilities mentioned above. We note that other kinds of WF (e.g., the Bethe–Salpeter functions), although they contain information on the properties of the bound system, do not directly determine the probabilities of the momentum distribution of various components, since they are not coefficients in the expansion (1). Relativistic dynamics is such, however, that these probabilities are not invariant quantities (cf. infra) and differ from reference frame to reference frame. This manifests itself, for instance, in the fact that after the exclusion of k_2 the two-particle component $C(k_1, k_2, p)$ is not a function of only the relative momentum, but depends in addition on the momentum p of the system. The total two-particle probability

$$w_2(p) = \frac{1}{N} \int |C_2(k_1, p)|^2 \frac{d^3 k_1}{4\varepsilon(k_1)\varepsilon(p - k_1)}$$

also depends on the momentum of the system, and only the sum $\sum_n w_n$ is equal to one in all systems. Formally the noninvariance is a consequence of the fact that the state vector $\Phi(p)$ is given on the hyperplane $t = 0$ belonging to a definite frame of reference.

We formulate an invariant theory of bound systems considering the state vector defined on an arbitrary flat spacelike hypersurface (a hyperplane) $\lambda x = 0$, where λ is a four-vector (λ_0, λ) such that $\lambda^2 = 1$, $\lambda_0 > 0$. In the sequel it will be convenient to choose the surface of a light-wave front $\omega x = 0$, where $\omega^2 = 0$, $\omega_0 > 0$. In such a theory the WF will be invariant quantities which are parametrized in an invariant manner. There remains, of course, a dependence on a "redundant" variable, but even this can be given a simple geometric interpretation. As a whole the formalism takes on such a form that the theoretical description of experimental data with the help of Fock rows becomes quite realistic.

We shall consider the case of spinless particles and bound states with total angular momentum equal to zero.

We explain the parametrization of the invariant WF on the example of the two-particle component, which is parametrized in the following manner:

$$C = C(\mathbf{q}, \mathbf{n}), \qquad (2)$$

where n is a unit vector $|\mathbf{n}| = 1$, the meaning of which will be explained somewhat later, and q is the relativistic relative momentum, for which the explicit expression in terms of the particle momenta is also unimportant. Under rotations and Lorentz transformations the vectors q and n are subject only to rotations and the function $C(\mathbf{q}, \mathbf{n})$ is Lorentz-invariant if it is invariant relative to rotations of the vectors q and n.

The only complication in the parametrization of the relativistic WF consists in the appearance of the additional argument n compared to the nonrelativistic case. This appearance could be interpreted as the appearance of deviations from spherical symmetry at small distances even in the S-wave. In the nonrelativistic limit the n-dependence disappears and the function $C(\mathbf{q}, \mathbf{n})$ goes over into the nonrelativistic WF $\psi(\mathbf{q})$. We stress the fact that the problem of selecting the direction n does not arise. Rather, we are dealing with an additional variable varying between -1 and $+1$, which it is convenient to consider as the cosine of a certain angle. In those WF in which relativity barely begins to manifest itself, the dependence on this variable is weak $(\sim \mathbf{q} \cdot \mathbf{n}/m)$.

Thus, to make the wave function of the deuteron relativistic one must introduce, besides the functional dependence of the two-nucleon component for $q \sim m$ and the introduction of additional components (isobars, pionic components), an additional variable—the unit vector n. Those are the WF one has to use for a self-consistent description of the ensemble of experimental data in the relativistic region. With an incorrect parametrization one would need different WF for the description of different experiments.

We now clarify in more detail the origin of the variable n, variable which does not occur in the nonrelativistic case. For infinitesimal Poincaré transformations $x \to x' = x + \delta x$, $\delta x_i = \delta \varepsilon_i + \delta \omega_{ik} x_k$ the state vector $\Phi(p)$ transforms as follows:

$$\Phi(p) \to \Phi'(p') = \hat{U}\Phi(p) = (1 + \delta\hat{U})\Phi(p),$$

$$\delta\hat{U} = i\hat{P}_k \delta\varepsilon_k + \frac{i}{2}\hat{M}_{ik}\delta\omega_{ik},$$

where $\hat{P}_k, \hat{M}_{ik}$ are the generators of the Poincaré group. In view of the fact that $\hat{P}_k$ transforms according to the law

$$\hat{P}_k' = U^{-1}\hat{P}_k\hat{U} = \hat{P}_k - \delta\omega_{ki}\hat{P}_i,$$

we obtain

$$\Phi(p_k - \delta\omega_{ki}p_i) = (1 + \delta\hat{U})\Phi(p_k). \qquad (3)$$

In the nonrelativistic case, for a two-particle system the equation (3) leads to Galilei invariance of the WF:

$$C_2(\mathbf{k}_1 - m\delta\mathbf{v}, \ \mathbf{k}_2 - m\delta\mathbf{v}, \ \mathbf{p} - M\delta\mathbf{v}) = C_2(\mathbf{k}_1, \ \mathbf{k}_2, \ \mathbf{p}).$$

Eliminating $\mathbf{k}_2$ with the help of the equality $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{p}$, we reach the conclusion that the function must depend on a definite combination of p and $\mathbf{k}_1$ from which $\delta\mathbf{v}$ is missing: $\mathbf{q} = \mathbf{k}_1 - m\mathbf{p}/M$, i.e., the function depends on a single variable.

In the relativistic case the interaction Hamiltonian which enters into the generators $\hat{P}_i, \hat{M}_{ik}$ changes the numbers of particles and the condition (3) has the consequence that the two-particle function transforms not only in terms of the two-particle component, but also involves components with other particle numbers. There is no group transformation law for a separate function $C(\mathbf{k}_1, \mathbf{k}_2, \mathbf{p})$; this function is not invariant under Lorentz transformations and (after elimination of $\mathbf{k}_2$) it depends on two variables separately: $C = C(\mathbf{k}_1, \mathbf{p})$. One can realize some simplifications by going onto the light-cone[5] (into an infinite-momentum frame) where the dependence on the absolute value of p disappears, but there remains a dependence on the direction $\mathbf{p}/|\mathbf{p}| \,|_{\mathbf{p} \to \infty}$. The existence of this dependence manifests itself in the appearance of the additional argument n in the WF.

The appearance of the argument n corresponds to the following intuitive physical picture. The WF of a rapidly moving nonrelativistic (in its rest frame) bound system undergoes a Lorentz contraction, or in the momentum representation is stretched by a factor of $(1 - v^2/c^2)^{-1/2}$ which leads to an elliptic distribution of the probability density. Introducing a variable which compresses an ellipsoid into a ball (q in Eq. (2) is just such a variable) one can remove this trivial stretching. However, owing to the noninvariance of the individual components these do not follow exactly the $(1 - v^2/c^2)^{-1/2}$ elongation law in a relativistic multi-component system. After the "kinematic" stretching is removed by means of the variable q one obtains a residual "dynamical deformation," which is described as $v \to c$ by the variable n, the same for all the components. A knowledge of this dynamical deformation in the infinite momentum frame turns out to be sufficient for the description of scattering in an arbitrary system. As shown by Terent'ev,[6] the WF of a system consisting only of two relativistic particles depends only on one variable, coinciding with the variable q used in our paper. In our approach the computational formalism consists of a diagram technique in a three-dimensional field theory formulation proposed by Kadyshevskiǐ. The invariant wave functions turn out to be related to the vertex functions of this diagram technique. Our problem here consists only in writing out the WF in a manifestly invariant form and finding a convenient parametrization, since otherwise the determination of the WF would be extremely difficult and "unprofitable."

We stress that the expansion of the state vector in terms of states with a definite particle number is effective only in the case when their number is not too large. In the ultrarelativistic momentum region there is no particular basis for this situation (if one does not consider the deuteron as consisting of quarks). However, in the transitional region $q \sim m$ which is the one in which

we are mainly interested one may expect that only several components will be important and the description of this region by the proposed method will function to its full extent.

The plan of exposition is the following. In Sec. 2 we introduce the state vector on the hyperplane $\lambda x = 0$. In Sec. 3 we study the parametrization for the WF. In Sec. 4 we deal with the light-cone diagram technique. In Sec. 5 we obtain a spectral representation of the Green's functions. In Sec. 6 a relation is established between the WF and the vertex part. Section 7 contains concluding remarks.

2. THE INVARIANT FOCK COLUMN

In this section we consider a state vector defined on the hyperplane $\lambda x = 0$, the Fock rows of which are the invariant WF.

We consider the scalar field with the Lagrangian

$$\mathscr{L} = \frac{1}{2}\left[\left(\frac{\partial \varphi_H}{\partial x_i}\right)^2 - m^2 \varphi_H^2\right] + \mathscr{L}_{int}, \tag{4}$$

where φ_H is the Heisenberg-picture field operator. The energy-momentum tensor of the field has the form

$$T_{ik} = \frac{\partial \varphi_H}{\partial x_i}\frac{\partial \varphi_H}{\partial x_k} - g_{ik}\mathscr{L}, \tag{5}$$

and the four-momentum operator is

$$\hat{P}_i = \int_\sigma T_{ik} d\sigma_k. \tag{6}$$

The structure of the Fock components of the state vector is determined from the equation

$$\hat{P}_i \Phi(p) = p_i \Phi(p), \tag{7}$$

in which all quantities are expressed in the Schrödinger picture on the hyperplane $x = 0$. On this hyperplane we obtain

$$\hat{P}_i = \int T_{ik}\lambda_k \delta(\lambda x) d^4 x = \hat{P}_i^0 + \lambda_i H_{\lambda x}, \tag{8}$$

where $\hat{P}_i^0$ is the four-momentum operator that does not include the interaction, and

$$H_{\lambda x} = \int H_{int}(x)\delta(\lambda x)d^4 x. \tag{9}$$

Introducing the Fourier transform of the Hamiltonian H_{int}

$$\tilde{H}(p) = \int H_{int}(x)e^{-ipx}d^4 x \tag{10}$$

and the integral representation of the delta function

$$\delta(\lambda x) = \frac{1}{2\pi}\int e^{i(\lambda x)\tau}d\tau, \tag{11}$$

we rewrite the operator $H_{\lambda x}$ in the form

$$H_{\lambda x} = \int \tilde{H}(-\lambda\tau)\frac{d\tau}{2\pi}. \tag{12}$$

We express the operator $\hat{P}_i$ in terms of the Schrödinger picture operators on the hyperplane $\lambda x = 0$. For this purpose we first go over to the field $\varphi(x)$ in the interaction picture:

$$\varphi_H(x) = \exp(i\hat{P}x)\exp(-i\hat{P}^0 x)\varphi(x)\exp(i\hat{P}^0 x)\exp(-i\hat{P}x). \tag{13}$$

Substituting (13) into (5), making use of Eq. (8) and taking account of the fact that on the hyperplane $\lambda x = 0$ we have $\hat{P}x = \hat{P}^0 x$, we obtain for $\hat{P}_i^0$ the expression:

$$\hat{P}_i^0 = \int a^+(\mathbf{k})a(\mathbf{k})k_i d^3 k,$$

where $k_i = (\varepsilon(\mathbf{k}), \mathbf{k})$.

Resolving $\hat{P}_i$ and p_i into longitudinal (along λ) and transverse parts (orthogonal to λ)

$$\hat{P}_i = \hat{P}_i^\perp + \hat{P}_i^\|, \quad \hat{P}_i^\perp = \hat{P}_i^0 - \lambda_i(\hat{P}^0\lambda), \quad \hat{P}_i^\| = \lambda_i[(\hat{P}^0\lambda) + H_{\lambda x}],$$

we write down the equation for the transverse parts

$$(\hat{P}_i^0 - \lambda_i(\hat{P}^0\lambda))\Phi(p) = (p_i - \lambda_i(p\lambda))\Phi(p). \tag{14}$$

The equation for the longitudinal component leads to the mass spectrum problem.

It can be seen from Eq. (14) that any Fock component must contain a delta function as a factor in order to take into account the equality of the projections transverse to λ: $(k_1 + \ldots + k_n)^\perp = p^\perp$. It is convenient to represent this delta function by the integral

$$\int_{-\infty}^{+\infty} \delta^{(4)}(k_1 + \ldots + k_n - p - \lambda\tau)d\tau.$$

Thus, the general form of the solution of Eq. (7) is the following

$$\Phi(p) = N^{-1/2}\sum_n \int C(k_1, \ldots, k_n, p, \lambda\tau)a^+(\mathbf{k}_1)\ldots a^+(\mathbf{k}_n)|0\rangle$$

$$\times \delta^{(4)}(k_1 + \ldots + k_n - p - \lambda\tau)\frac{d^3 k_1}{\sqrt{2\varepsilon_1}}\ldots\frac{d^3 k_n}{\sqrt{2\varepsilon_n}}d\tau. \tag{15}$$

All vectors are on the respective mass shells. The four-momentum $\lambda\tau$ appearing in the argument of the Fock component will be called the spurion momentum. For $\lambda = 0$ and $\lambda_0 = 1$ the invariant expansion (15) goes over into the expansion (1). Thus, the functions $C(k_1, \ldots, k_n, p, \lambda\tau)$ represent an invariant form of the components of the expansion (1), defined in a frame which moves with four-velocity $\lambda\tau$.

For the sequel it will be convenient to represent the state vector and the momentum four-vector in a different form, introducing, according to Kadyshevskiĭ,[8,9] the spurion operator $a^+(\tau)$ and the operator $\hat{\tau}$ having the following properties:

$$a^+(\tau)|0\rangle = |\tau\rangle, \quad \langle\tau''|a(\tau)|\tau'\rangle = \delta(\tau'' + \tau - \tau'),$$
$$a^+(\tau) = a(-\tau), \quad a(\tau_1)a(\tau_2) = a(\tau_1 + \tau_2), \tag{16}$$
$$\langle\tau''|\hat{\tau}|\tau'\rangle = \tau'\delta(\tau' - \tau''), \quad [a(\tau'), \hat{\tau}]_- = \tau'a(\tau').$$

The quantity τ without circumflex is a number.

We introduce the operators $\hat{\mathscr{P}}_i$ and $\hat{\mathscr{H}}$:

$$\hat{\mathscr{P}}_i = \hat{P}_i^0 - \lambda_i \hat{\tau}, \quad \hat{\mathscr{H}} = 2\pi\hat{\tau} + \hat{H}, \tag{17}$$

where

$$\hat{H} = \int a(\tau)\hat{H}(-\lambda\tau)d\tau. \tag{18}$$

From the commutation relations (16) and translation invariance it follows that

$$[\hat{\mathscr{P}}_i, \hat{\mathscr{H}}] = 0.$$

We define the common eigenstates of the operators $\hat{\mathscr{P}}_i$ and $\hat{\mathscr{H}}$:

$$\hat{\mathscr{P}}_i|p, \gamma\rangle = p_i|p, \gamma\rangle, \quad \hat{\mathscr{H}}|p, \gamma\rangle = D_\gamma(p, \lambda)|p, \gamma\rangle. \tag{19}$$

The eigenvalues D_γ of the operator $\hat{\mathscr{H}}$ depend on the parameters p and λ. The index γ labels the states with different eigenvalues D_γ.

The expansion of the state vector $|p, \gamma\rangle$ defined by Eq. (19) in terms of states with fixed particle number (including the spurion) has the form

$$|p, \gamma\rangle = N^{-\frac{1}{2}} \sum_n \int C(k_1, \ldots, k_n, p, \lambda\tau) a^+(k_1) \ldots a^+(k_n)$$

$$\times a^+(\tau)|0\rangle \delta^{(4)}(k_1 + \ldots + k_n - p - \lambda\tau)\frac{d^3k_1}{\sqrt{2\varepsilon_1}} \ldots \frac{d^3k_n}{\sqrt{2\varepsilon_n}} d\tau. \tag{20}$$

We stress the fact that in Eqs. (19) and (20), in distinction from (15), the quantity p^2 is in general an arbitrary parameter which does not equal M^2. It is easy to see that for fixed $p^2 = M^2$ such that $D_\gamma(p, \lambda) = 0$, the equations (19) projected on the state $\int \langle\tau|d\tau$ goes over into the transverse and longitudinal parts of Eq. (7), and the vector (20) goes over into (15). The equation

$$D_\gamma(p, \lambda) = 0 \tag{21}$$

determines in the proposed formalism the mass spectrum of the bound states of the field.

Since the spectrum of the operators does not depend on the representation, the zeros of the function $D_\gamma(p, \lambda)$ do not depend on λ. The orthogonality and completeness conditions for the state vectors have the form

$$\langle p', \gamma'|p, \gamma\rangle = \delta^{(4)}(p-p')\delta_{\gamma\gamma'}, \tag{22}$$

$$\sum_\gamma \int |p, \gamma\rangle\langle\gamma, p|d^4p = 1. \tag{23}$$

The normalization condition (22) yields

$$N_\gamma = \sum_n \int |C_n^\gamma(k_1, \ldots, k_n, p, \lambda\tau)|^2 \delta^{(4)}(k_1 + \ldots + k_n - p - \lambda\tau)\frac{d^3k_1}{2\varepsilon_1} \ldots \frac{d^3k_n}{2\varepsilon_n} d\tau. \tag{24}$$
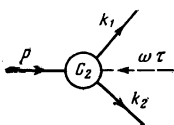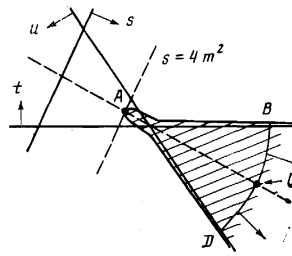

FIG. 1.

FIG. 2.

We omit the factorials that are due to the identity of the particles. In each concrete case they can be easily re-established. In the sequel we shall also no longer keep track of the index γ.

As will follow from the results of Sec. 3, the simplest parametrization of the functions appears if in place of λ one uses a four-vector ω such that $\omega^2 = 0$, $\omega_0 > 0$. Under such a substitution all the results of the present section remain in force. We show that the corresponding invariant functions represent the components of the Fock column (1) in the infinite-momentum frame which moves in the direction ω. This is clear already from the fact that the substitution $\lambda \to \omega$ can be achieved by means of the limiting process $\lambda \to \infty$. In this limit the four-vector λ gets closer to the light cone and the functions $C(k_1, \ldots, k_n, p, \lambda\tau)|_{\lambda \to \infty}$ are invariantly expressed components of the expansion (1), defined in the system with infinite four-velocity moving in the direction $\omega = \lambda/|\lambda||_{\lambda \to \infty}$. One can also give a more rigorous proof of this fact.

3. THE PARAMETRIZATION OF THE WAVE FUNCTION

We consider the parametrization of the two-particle WF $C(k_1, k_2, p, \omega\tau)$ $k_1 + k_2 = p + \omega\tau$. We represent $C(k_1, k_2, p, \omega\tau)$ graphically in the form of the four-point diagram (Fig. 1).

In the same manner as any Feynman four-point function the function $C(k_1, k_2, p, \omega\tau)$ can be parametrized by means of any pair of Mandelstam variables:

$$s = (k_1 + k_2)^2 = M^2 + 2\tau(\omega p), \quad t = (p - k_1)^2 = m^2 - 2\tau(\omega k_2),$$

$$u = (p - k_2)^2 = m^2 - 2\tau(\omega k_1), \quad s + t + u = 2m^2 + M^2. \tag{25}$$

As usual, the kinematically admissible region is determined by the Kibble equations and is represented in Fig. 2.

The problem of determining the two-particle component of the state vector of a relativistic bound system now reduces to a determination of the function of two variables $C = C(s, t)$. However, for the practical solution of this problem it is more convenient to introduce another parametrization, which is closer to the nonrelativistic one, and allows in addition to introduce a relativistic coordinate space.[10] For this purpose we shall make use of the shift operation on the hyperboloid. For two vectors $\mathbf{k}$ and $\mathbf{p}$ situated on the mass-hyperboloid of mass m this shift is defined in the following manner (cf.[10]):
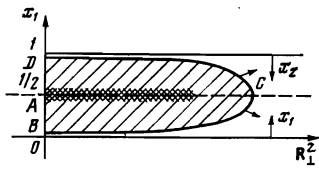
FIG. 3.

$$q_0 = L(\mathbf{p})\mathbf{k} = \mathbf{k}(-)\mathbf{p} = \mathbf{k} - \frac{\mathbf{p}}{m}\left[\varepsilon(\mathbf{k}) - \frac{\mathbf{k}\mathbf{p}}{m+\varepsilon(\mathbf{p})}\right],$$ (26)

$$\varepsilon(\mathbf{k}) = \sqrt{\mathbf{k}^2+m^2}, \quad \varepsilon(\mathbf{p}) = \sqrt{\mathbf{p}^2+m^2}.$$

The operation (26) represents a Lorentz transformation (boost) $L(\mathbf{p})$ on the vector $\mathbf{k}$ into a frame moving with velocity $\mathbf{v} = \mathbf{p}/\varepsilon(\mathbf{p})$. In the nonrelativistic limit we obviously obtain the difference $\mathbf{k} - \mathbf{p}$. We note the equalities

$$(\mathbf{k}(-)\mathbf{p})^2 = m^{-2}(kp)^2 - m^2,$$
$$(\mathbf{k}_1(-)\mathbf{p})(\mathbf{k}_2(-)\mathbf{p}) = m^{-2}(k_1 p)(k_2 p) - (k_1 k_2).$$ (27)

Introducing the notation

$$Q = p + \omega\tau, \quad Q^2 = s,$$ (28)

we define the following variables:

$$\mathbf{q}_1 = \mathbf{k}_1(-)\frac{m}{\sqrt{Q^2}}Q = \mathbf{k}_1 - \frac{Q}{\sqrt{Q^2}}\left[\varepsilon(\mathbf{k}_1) - \frac{(\mathbf{k}_1 Q)}{\sqrt{Q^2}+Q_0}\right],$$ (29)

$$\mathbf{q}_2 = \mathbf{k}_2(-)\frac{m}{\sqrt{Q^2}}Q = -\mathbf{q}_1,$$ (30)

$$\omega' = \omega - \frac{Q}{\sqrt{Q^2}}\left[\omega_0 - \frac{(\omega Q)}{\sqrt{Q^2}+Q_0}\right],$$ (31)

$$\mathbf{n} = \omega'/|\omega'| = \sqrt{Q^2}\omega'/(\omega p).$$ (32)

The vectors introduced here have the meaning of momenta in the center-of-inertia system (c.m.s.) of the particles which form the bound system, a fact responsible for their convenience. In view of (27) under Lorentz transformations of the 4-vectors k_1, k_2, p, and ω the vectors $\mathbf{q}_1$, $\mathbf{q}_2$, $\mathbf{n}$ are subjected only to rotations, and the angles between them do not vary.

Without dwelling on the group-theoretical questions we note that the vector $\mathbf{q}_0$ (Eq. (26)) undergoes the following transformation under a Lorentz transformation Λ:

$$\mathbf{q}_0' = R(\Lambda)\mathbf{q}_0, \quad R(\Lambda) = L(\Lambda p)\Lambda L^{-1}(\mathbf{p}),$$ (33)

with $R(\Lambda_1 \cdot \Lambda_2) = R(\Lambda_1)R(\Lambda_2)$.

The wave function $C(k_1, k_2, p, \omega\tau)$ may be considered as depending on the two vectors $\mathbf{q} \equiv \mathbf{q}_1$ and $\mathbf{n}$: $C = C(\mathbf{q}, \mathbf{n})$. Indeed, $|\mathbf{n}| = 1$, and on account of Eqs. (27) we have

$$\mathbf{q}^2 = s/4 - m^2,$$ (34)

$$(\mathbf{n}\mathbf{q}) = (u-t)\sqrt{s}/2(s-M^2).$$ (35)

Therefore the parametrization (28)–(32) is equivalent to the parametrization in the variables s and t.

In the nonrelativistic limit $(2m - M \ll m, k_1^2, k_2^2, p^2 \ll M^2)$

the conservation law $p + \omega\tau = k_1 + k_2$ goes over into its nonrelativistic counterpart $\mathbf{p} = \mathbf{k}_1 + \mathbf{k}_2$. Indeed, it follows from (25) that $\tau = (s - M^2)2(p\omega)$ and the spatial part of the equation acquires the form

$$\mathbf{p} + \omega \frac{s-M^2}{2(p\omega)} = \mathbf{k}_1 + \mathbf{k}_2.$$

In the nonrelativistic limit $(p\omega) \sim M\omega_0$, $s - M^2 \sim k^2$ the second term in the left-hand side of the equation also has the order $(\omega/\omega_0)k^2/2M$. Since $|\omega/\omega_0| = 1$, it is small of order $\sim k/m$ compared to the other terms of the equation, and can be neglected. The WF of the nonrelativistic system becomes a function of one variable on account of Galilei-invariance, i.e., the dependence on the variable $\mathbf{q} \cdot \mathbf{n}$ disappears from $C(\mathbf{q}, \mathbf{n})$.

As was noted in Sec. 2, the function $C(k_1, k_2, p, \omega\tau)$ is related to the invariant Fock component in the infinite-momentum frame. We show how their parametrizations are related. We recall that $C(\mathbf{k}_1, \mathbf{k}_2, \mathbf{p})|_{p\to\infty}$ is parametrized in the following manner (cf. e.g., [11]). The vectors $\mathbf{k}_1$, and $\mathbf{k}_2$ are resolved into their longitudinal parts $\mathbf{k}_1^{\|}, \mathbf{k}_2^{\|}$, and the transverse parts $\mathbf{k}_1^{\perp}, \mathbf{k}_2^{\perp}$. The following variables are introduced: $x_1 = k_1^{\|}/p$, $x_2 = k_2^{\|}/p$, $0 \le x_1, x_2 \le 1$, $x_1 + x_2 = 1$, $\mathbf{R}_1 = \mathbf{k}_1 - x_1\mathbf{p}$, $\mathbf{R}_2 = \mathbf{k}_2 - x_2\mathbf{p}$, $\mathbf{R}_1 = -\mathbf{R}_2$, $\mathbf{R}_1 \cdot \mathbf{p} = \mathbf{R}_2 \cdot \mathbf{p} = 0$. One may consider that the WF depends on the two variables x_1 and $\mathbf{R}_\perp^2 \equiv \mathbf{R}_1^2 = \mathbf{R}_2^2$: $C = C(x_1, \mathbf{R}_\perp^2)$. The region of variation of the variables x_1, x_2, and $\mathbf{R}_\perp^2$ is the unit half-strip represented in Fig. 3.

Starting with these definitions one can find the relation between the variables s, t, u and x_1, x_2, $\mathbf{R}_\perp^2$. This relation has the form

$$x_1 = \frac{m^2-u}{s-M^2} = \frac{(\omega k_1)}{(\omega p)},$$ (36)

$$x_2 = \frac{m^2-t}{s-M^2} = \frac{(\omega k_2)}{(\omega p)},$$ (37)

$$\mathbf{R}_\perp^2 = s\frac{(m^2-t)(m^2-u)}{(s-M^2)^2} - m^2.$$ (38)

Equations (36)–(38) transform the Kibble region of Fig. 2 into the strip of Fig. 3. The mapping is realized in the following manner. The finite region $ABCD$ of Fig. 2 goes into the region $ABCD$ of Fig. 3, and as the line BCD in Fig. 2 tends to infinity, the shaded region of Fig. 3 fills the whole strip.

We also connect the variables $\mathbf{q}$ and $\mathbf{n}$ with the variables in the infinite momentum frame:

$$\mathbf{R}_\perp^2 = \mathbf{q}^2 - (\mathbf{q}\mathbf{n})^2, \quad x_1 = \frac{1}{2}\left(1 - \frac{\mathbf{n}\mathbf{q}}{\varepsilon(\mathbf{q})}\right), \quad x_2 = \frac{1}{2}\left(1 + \frac{\mathbf{n}\mathbf{q}}{\varepsilon(\mathbf{q})}\right).$$ (39)

Let us analyze the transition to the nonrelativistic limit in the WF parametrized in terms of the parameters x and $\mathbf{R}_\perp^2$. How does the nonrelativistic character of the WF $\psi(\mathbf{q})$ expressed in terms of the variables x and $\mathbf{R}_\perp^2$ manifest itself when its qualitative behavior is for certain? Expressing s in Eq. (34) by means of the formulas (36)–(38) in terms of x_1, x_2 and $\mathbf{R}_\perp^2$ we obtain

$$\mathbf{q}^2 = \frac{\mathbf{R}_\perp^2 + m^2(1-4x_1 x_2)}{4x_1 x_2} = \frac{\mathbf{R}_\perp^2 + 4m^2(1/2-x_1)^2}{4x_1(1-x_1)}.$$ (40)

Thus, in the nonrelativistic limit the ψ-function de-

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pends on the combination of the variables x_1 and $\mathbf{R}_\perp^2$ defined by Eq. (40). The dependence on $\mathbf{R}_\perp^2$ for fixed x_1 is close to the dependence on $\mathbf{q}^2$. The dependence on x_1 is very peculiar. Since $\mathbf{q}^2 \ll m^2$, the effective region of variation of x_1 is very narrow and concentrated near the values $x_1 = \frac{1}{2}$ (the doubly cross-hatched region in Fig. 3). Its width is of the order $|\frac{1}{2} - x_1| \sim (|\,\varepsilon\,|/2m)^{1/2}$. In this region the WF varies just as strongly as in the whole region of $\mathbf{R}_\perp^2$. It is therefore clear that the parametrization in terms of the variables x_1 and $\mathbf{R}_\perp^2$ would be very inconvenient even if its dependence on these variables would not reduce to a single one. This circumstance serves as an indication that the parametrization in the variables x_1, x_2, $\mathbf{R}_\perp^2$ of the relativistic WF may also not be optimal, at least as far as its determination from experimental data is concerned.

The parametrization of the WF in terms of the variables $\mathbf{q}$ and $\mathbf{n}$ seems more optimal for the reason that the variable $\mathbf{q}$ (as follows from its definition (29)) removes the dilation by the factor $(1 - v^2/c^2)^{-1/2}$ common for all the functions, and the individual properties of the WF manifest themselves more clearly in this case.

The contribution to the normalization integral (24) from the two-particle component has the form

$$N = \sum_n I_n,$$

$$I_2 = \int |C(k_1, k_2, p, \omega\tau)|^2 \delta^{(4)}(k_1 + k_2 - p - \omega\tau)$$

$$\times \theta(\omega k_1)\theta(\omega k_2)\frac{d^3k_1}{2\varepsilon_1}\frac{d^3k_2}{2\varepsilon_2}d\tau = \frac{1}{(\omega p)}\int |C(\mathbf{q}, \mathbf{n})|^2 \frac{d^3q}{2\varepsilon(\mathbf{q})}. \quad (41)$$

In the case of n particles one has to introduce the variables

$$\mathbf{q}_i = \mathbf{k}_i (-)\frac{m}{\sqrt{Q^2}}\mathbf{Q}, \quad i = 1, \dots, n, \ . \quad (42)$$

and the WF depends on $n - 1$ momenta and on the unit vector $\mathbf{n}$:

$$C_n = C_n(\mathbf{q}_1, \dots, \mathbf{q}_{n-1}, \mathbf{n}). \quad (43)$$

The normalization integral has the form

$$I_n = \int |C_n(\mathbf{q}_1, \dots, \mathbf{q}_n, \mathbf{n})|^2 \delta^{(3)}(\mathbf{q}_1 + \dots + \mathbf{q}_n)$$

$$\times \frac{(\varepsilon_1 + \dots + \varepsilon_n)}{(\omega p)}\frac{d^3q_1}{2\varepsilon_1} \dots \frac{d^3q_n}{2\varepsilon_n}. \quad (44)$$

The delta function in (44) allows one to eliminate one of the vectors and to obtain the parametrization (43). One can, however, use the more convenient $n - 1$ combinations of the vectors $\mathbf{q}_1, \dots, \mathbf{q}_n$ analogous to the Jacobi variables in the nonrelativistic case. It poses no difficulty to generalize the results obtained here to the case of particles of different masses.

On the plane $\lambda x = 0$ the WF can be parametrized in analogy with expression (43). Here the unit vector $\mathbf{n}$ is replaced by the vector $\mathbf{a} = \lambda\tau(-)\tau\mathbf{Q}(Q^2)^{-1/2}$ and Q is determined by the equation (28) with the substitution $\omega \to \lambda$:

$$C = C(\mathbf{q}_1, \dots, \mathbf{q}_{n-1}, \mathbf{a}). \quad (45)$$

The result is an additional variable, the absolute value of the vector $\mathbf{a}$.

The WF considered by us is necessary for the solution of the scattering problem when the momenta of the system are different before and after the scattering. However, there exist problems in which it suffices to know the state vector in some system, e.g., in the c.m.s. (e.g., in the spectral problem) and therefore one can limit oneself in them to a simpler function not depending on the supplementary argument. It can be obtained from the function (45) setting $\mathbf{a} = 0$, which is easily seen to correspond to the $(\mathbf{p} = 0)$ component of the Fock column in the c.m.s.

We note that the WF of the quasipotential approach[12] is a special case of the WF considered here; the latter goes over in the quasipotential WF under the condition that in (45) $\mathbf{a} = 0$. For this purpose it suffices to show that the quasipotential WF is an invariantly written component of the Fock column in the rest frame. According to[12] we have

$$\chi(t, \mathbf{r}_1, \mathbf{r}_2) = \langle 0 | \varphi_H(\mathbf{r}_1, t)\varphi_H(\mathbf{r}_2, t) | n \rangle. \quad (46)$$

Going over to the Schrödinger picture

$$\varphi_H(\mathbf{r}, t) = e^{iHt}\varphi(\mathbf{r}, 0)e^{-iHt},$$

we obtain

$$\chi(t, \mathbf{r}_1, \mathbf{r}_2) = e^{-iE_n t}\langle 0 | \varphi(\mathbf{r}_1, 0)\varphi(\mathbf{r}_2, 0) | n \rangle. \quad (47)$$

Representing $|n\rangle$ in the form of the expansion (1), expanding $\varphi(\mathbf{r}, 0)$ in terms of creation-annihilation operators and carrying out a Fourier transformation we see that the matrix element (47) is proportional to $C(\mathbf{k}_1, \mathbf{k}_2, \mathbf{p} = 0)$. The result is valid also for the case of an arbitrary particle number. We note that in[13] Garsevanishvili et al. go over to the light-cone coordinates in the framework of the quasipotential approach.

According to Kadyshevskiĭ et al.[10] one can introduce a relativistic coordinate space with the help of an expansion in terms of Shapiro functions[14]:

$$\xi(\mathbf{q}, \boldsymbol{\rho}) = \left(\frac{\varepsilon(\mathbf{q}) - \mathbf{\rho q}/\rho}{m}\right)^{-1-i\rho m}. \quad (48)$$

The functions $\xi(\mathbf{q}, \boldsymbol{\rho})$ span an irreducible unitary representation of the Lorentz group and exhibit the orthogonality and completeness properties:

$$\frac{1}{(2\pi)^3}\int \xi^*(\mathbf{q}, \boldsymbol{\rho})\xi(\mathbf{q}', \boldsymbol{\rho})d^3\rho = \delta^{(3)}(\mathbf{q}(-)\mathbf{q}') = \frac{\varepsilon(\mathbf{q})}{m}\delta^{(3)}(\mathbf{q} - \mathbf{q}'). \quad (49)$$

$$\frac{1}{(2\pi)^3}\int \xi^*(\mathbf{q}, \boldsymbol{\rho})\xi(\mathbf{q}, \boldsymbol{\rho}')\frac{md^3q}{\varepsilon(\mathbf{q})} = \delta^{(3)}(\boldsymbol{\rho} - \boldsymbol{\rho}').$$

In the nonrelativistic limit $\xi(\mathbf{p}, \boldsymbol{\rho})$ goes over into an exponential function:

$$\xi(\mathbf{q}, \boldsymbol{\rho})|_{m\to\infty} \to e^{i\mathbf{q}\boldsymbol{\rho}}. \quad (50)$$

Expanding the function $C(\mathbf{q}, \mathbf{n})$ in Shapiro functions we obtain a representation of the WF in relativistic coordinate space:

$$C(\rho, \mathbf{n}) = \frac{1}{(2\pi)^{3/2}} \int C(\mathbf{q}, \mathbf{n})\, \xi(\mathbf{q}, \rho)\, \frac{m\, d^3 q}{\varepsilon(q)}. \qquad (51)$$

The normalization integral takes the form

$$I_2 = \frac{1}{2m(\omega p)} \int |C(\rho, \mathbf{n})|^2\, d^3\rho. \qquad (52)$$

The n-particle function depends on $n-1$ coordinates related by the Shapiro transformation to the appropriate momentum-space variables. As pointed out by Shapiro[2] the coordinate space introduced in this manner is the Fourier conjugate of the rapidity space.

4. THE LIGHT-CONE DIAGRAM TECHNIQUE

We now go over to a reformulation of the diagram technique developed by Kadyshevskiĭ[7] for the case where one uses in place of λ the four-vector ω, $\omega^2 = 0$. This diagram technique is the basis of the computational apparatus and for such a choice of ω there occur simplifications both in the structure of the diagrams (the vacuum diagrams disappear), and in the procedure of their computation. The vertex parts of the Kadyshevskiĭ diagram technique turn out to be connected to the WF introduced before.

We recall how the four-vector λ appears in the diagram technique.[7] The T-product in the S-matrix can be written in terms of the θ-functions:

$$S^{(n)} = (-i)^n \int H(x_1)\theta(\lambda(x_1 - x_2))H(x_2)\dots\theta(\lambda(x_{n-1} - x_n))H(x_n)\, d^4x_1 \dots d^4x_n. \qquad (53)$$

The signs of $t_1 - t_2$ and $\lambda(x_1 - x_2)$ in (53) coincide for $(x_1 - x_2)^2 \geqslant 0$, and for $(x_1 - x_2)^2 < 0$ the operators $H(x_1)$ and $H(x_2)$ commute and their order becomes unimportant. A transformation of the expression (53) to normal form leads to the Kadyshevskiĭ diagram technique,[7,8] where the dependence of the amplitude on the vector λ becomes fictitious. In the case $\omega^2 = 0$ the situation changes somewhat: for $(x_1 - x_2)^2 = 0$ the signs of $t_1 - t_2$ and $\omega(x_1 - x_2)$ may be different. However, the region where this can happen represents a line on the light cone. Splitting the integral over d^4x into the integrals over d^3x and dt we see that in the $t = \text{const}$ plane there is only one point where the integrand differs from the initial expression, which cannot change the value of the integral. Thus the theory with $\omega^2 = 0$ yields the correct matrix elements of the S-matrix.

We go over to the rules of the diagram technique for the case with $\omega^2 = 0$. The role of the propagator in this technique is played by the expression $\theta(k_0)\delta(k^2 - m^2)$ which is conveniently represented in the form $\theta(\omega k)\delta(k^2 - m^2)$. As an example we consider the theory with the Hamiltonian $H = -g : \varphi^3 :$. We formulate by analogy with Kadyshevskiĭ[7,8] the rules of the diagram technique for
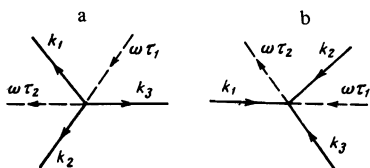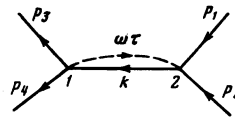

FIG. 4.

the transition matrix element from the m-particle state into the n-particle state, $F_{nm}\delta^{(4)}(\dots)$, defined as follows:

$$S_{nm} = 1 + iT_{nm}, \qquad (54)$$

$$T_{nm} = \delta^{(4)}\left(\sum_{i=1}^{n} p_i' - \sum_{j}^{m} p_j\right)\frac{F_{nm}}{(2\varepsilon_1 \dots 2\varepsilon_m \cdot 2\varepsilon_1' \dots 2\varepsilon_n')^{1/2}}, \qquad (55)$$

and related to the cross section for the process $1 + 2 \to 3 + 4 + \dots + n$:

$$d\sigma = \frac{(2\pi)^2}{j}|F|^2 \frac{d^3k_3}{2\varepsilon_3} \dots \frac{d^3k_n}{2\varepsilon_n}\delta^{(4)}(k_1 + k_2 - k_3 - \dots - k_n), \qquad (56)$$

where j is the incident-particle flux density.

1. In the Feynman diagram of n-th order we label the vertices in an arbitrary way by numbers. We connect the vertices by directed lines in the order of increasing labels. On the full lines we orient the arrows in the direction from the larger number to the smaller one. Diagrams which contain vertices with three incoming or outgoing full lines (the vacuum vertices, Fig. 4a and b) may be left out of consideration, as will be shown below. To the i-th dotted line (we call it a spurion line) we attribute the momentum $\omega\tau_i$.

2. To each full (particle) line we set in correspondence the propagator $\theta(\omega k)\delta(k^2 - m^2)$, to each spurion line we associate the factor $1/2\pi(\tau - i0)$.

3. With each vertex we associate $(2\pi)^{-1/2}g\delta^{(4)}(\dots)$, where the delta function takes in account four-momentum conservation, including the spurion momentum.

4. We integrate over all internal line momenta, with respect to d^4k and with respect to $d\tau$ over spurion lines.

5. We repeat the procedure outlined in rules 2–4 for all $n!$ possible numberings of the vertices.

The vacuum vertices (Fig. 4a and b) vanish owing to the impossibility to satisfy for them the conservation laws. Indeed, since the squared momentum $\omega\tau$ of the spurion vanishes: $(\omega\tau)^2 = 0$, the existence of the vertex of Fig. 4a would mean that a massless particle (e.g., the photon) could decay in vacuum into two particles, some of which have mass. But since the particle momenta are on the mass shell this is impossible.

A similar property of vanishing of vacuum expectation values is exhibited by old-fashioned perturbation theory in the infinite-momentum frame.[15] The theory with $\omega^2 = 0$ expounded here may be considered as a special, relativistic-invariant formulation of old perturbation theory in an infinite-momentum frame.

We illustrate the rules 1–5 on the instance of the diagram in Fig. 5. The amplitude F corresponding to this diagram has the form
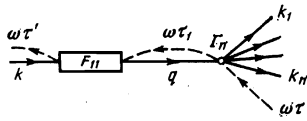

FIG. 5.

FIG. 6.

$$F = \frac{g^2}{2\pi} \int \theta(\omega k)\delta(k^2-m^2)\delta^{(4)}(p_1+p_2+\omega\tau-k)\frac{d\tau}{2\pi(\tau-i0)}d^4k. \quad (57)$$

Integrating over d^4k we obtain

$$F = -\frac{g^2}{(2\pi)^2}\theta(\omega(p_1+p_2))\int_{-\infty}^{+\infty}\delta((p_1+p_2)^2+2\omega(p_1+p_2)\tau-m^2)\frac{d\tau}{\tau-i0}$$

$$= \frac{g^2}{(2\pi)^2}\frac{1}{m^2-s-i0}, \quad (58)$$

where $s = (p_1+p_2)^2$ and it was taken into account that $\theta(\omega(p_1+p_2))=1$.

As can be seen from Eqs. (57) and (58) the integration over $d\tau$ with the help of the delta function leads to a linear equation for τ. This property remains valid also in the general case. This case also has the advantage of having $\omega^2 = 0$ which simplifies the calculations since for $\lambda^2 = 1$ a quadratic equation is obtained for τ.

5. THE SPECTRAL REPRESENTATION OF THE GREEN'S FUNCTIONS

The parametrization of the WF introduced above is close to the nonrelativistic one. This analogy extends also further and manifests itself in the spectral representations of the Green's functions and in the relation between the vertex function and the component of the Fock column.

We write the Green's functions G_0 and G according to Kadyshevskii[8]:

$$G_0 = 1/2\pi\hat{\tau}, \quad (59)$$

$$G = G_0 - G_0\hat{H}G. \quad (60)$$

The operators $\hat{\tau}$ and $\hat{H}$ are defined in Eqs. (16) and (18).

The Green's function G is an operator inverse to the Hamiltonian $\hat{\mathcal{H}} = 2\pi\hat{\tau} + \hat{H}$:

$$\hat{\mathcal{H}} = G^{-1}, \quad \hat{\mathcal{H}}G = G\hat{\mathcal{H}} = 1. \quad (61)$$

The matrix elements of the operator T which is defined in terms of G by[8]:

$$G = G_0 + G_0 T G_0 \quad (62)$$

and satisfies the equation

$$T = -\hat{H} - \hat{H}G_0 T, \quad (63)$$

are related to the S-matrix elements by Eq. (54).

We write the spectral representation of the Green's function G:

$$G = \sum_\gamma \int \frac{|p,\gamma\rangle\langle\gamma,p|}{D_\gamma(p,\omega)}d^3p, \quad (64)$$

where $|p,\gamma\rangle$ are the eigenstates of the Hamiltonian $\hat{\mathcal{H}}$;

D_γ are the eigenvalues, and the sum is over the whole spectrum. We recall that the values $p^2 = M^2$ for which $D_\gamma(p,\omega)$ vanishes determine the mass spectrum of the bound system. It is obvious that the function G represented in the form (64) satisfies the conditions (61).

Making use of the expansion (15) for the state vectors we obtain an expression for the transition matrix element from an m-particle state with a spurion into an n-particle state with a spurion:

$$\langle n|G(p)|m\rangle = \sum_\gamma \frac{C_n^{*\gamma}(k_1',\ldots,k_n',p,\omega\tau')C_m^\gamma(k_1,\ldots,k_m,p,\omega\tau)}{N_\gamma(p,\omega)D_\gamma(p,\omega)(2\varepsilon_1'\ldots2\varepsilon_n'\cdot2\varepsilon_1\ldots2\varepsilon_m)^{\frac{1}{2}}}$$

$$\times\delta^{(4)}(k_1+\ldots+k_m-\omega\tau-k_1'-\ldots-k_n'+\omega\tau)|_{p=k_1+\ldots+k_m-\omega\tau}. \quad (65)$$

If γ contains a discrete part (corresponding to discrete masses M), any Green's function matrix element has poles for values of p^2 equal to the masses of the bound states and factorizes at the pole:

$$\langle n|G(p^2\to M^2)|m\rangle = \frac{C_n^{*\gamma}C_m^\gamma}{N_\gamma(p,\omega)\partial D(p,\omega)/\partial p^2}\bigg|_{p^2=M^2}$$

$$\times\frac{\delta^{(4)}(\ldots)}{(2\varepsilon_1'\ldots2\varepsilon_n'\cdot2\varepsilon_1\ldots2\varepsilon_m)^{\frac{1}{2}}}\frac{1}{p^2-M^2} \quad (66)$$

We note that all that was said in this section is true both in the case $\omega^2 = 0$ and the case $\lambda^2 = 1$.

6. THE RELATION OF THE WAVE FUNCTION WITH THE VERTEX PART

We define the irreducible n-particle vertex part $\Gamma_n(k_1,\ldots,k_n,q-\omega\tau_1,\omega\tau)$ containing one incoming and n outgoing particle lines and one incoming and one outgoing spurion line (the right-hand side of Fig. 6). We shall consider Γ_n irreducible if it cannot be cut into two parts, one of which contains only the external incoming particle line and outgoing spurion line and is connected to the remainder by one particle line and one spurion line. Figure 6 represents a reducible vertex part.

We note that in a theory with $\omega^2 = 0$ the spurion line may enter the diagram only at the point where one of the particle lines leaves it and may come out of the diagram only at the entrance point of a particle. Otherwise the diagram contains a vacuum vertex and vanishes. For this reason the variables q and $\omega\tau_1$ (Fig. 6) appear in the vertex part only in the combination $q - \omega\tau_1$.

We express the matrix element F_{n1} in terms of the amplitude F_{11} and the irreducible vertex part Γ_n. We consider that to bound states there correspond bare particles in the Hamiltonian. Calculating the diagram of Fig. 6, we obtain

$$F_{n1} = F_{11}\Gamma_n\frac{1}{2\pi\tau'\cdot2(\omega k)}. \quad (67)$$

Relating by means of the equality (62) the amplitudes F_{11}, F_{n1} and the Green's function matrix elements $\langle 1|G|1\rangle$, $\langle n|G|1\rangle$, going in the variable $(k-\omega\tau')^2 = p^2$ to the pole $p^2 = M^2$ and equating the residues we obtain the relation between the component of the Fock column C_n and the irreducible vertex Γ_n:

$$C_n^*(k_1,\ldots,k_n,p,\omega\tau) = \frac{1}{2\pi}\frac{\Gamma_n(k_1,\ldots,k_n,p,\omega\tau)}{s-M^2}, \quad (68)$$
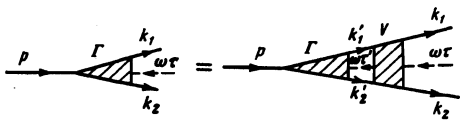
FIG. 7.

where $s = (k_1 + \ldots + k_n)^2$ and the momenta are related by: $p + \omega\tau = k_1 + \ldots + k_n$.

For the description of the nonrelativistic two-particle system we introduce the function $\psi(\mathbf{q}) = (2\pi)^{3/2} C_2/2mN^{1/2}$, normalized by

$$\int |\psi(\mathbf{q})|^2 \frac{d^3q}{(2\pi)^3} = 1,$$

the nonrelativistic normalized amplitude $M^{nr} = \frac{1}{4}(2\pi)^{1/2} \times m^{-3/2}\Gamma_2$, the constant $z^{1/2} = \frac{1}{2}m^{-1/2}N^{-1/2}$ which renormalizes the charge $(g_R = z^{1/2}g)$, and take into account that $s \to 4(k^2 + \varkappa^2)$, where $\varkappa^2 = m|\varepsilon_d|$. As a result we obtain the usual nonrelativistic formula:

$$\psi(\mathbf{q}) = \frac{mz^{\prime h}}{q^2 + \varkappa^2} M^{nr} = \frac{m}{q^2 + \varkappa^2} M_R^{nr}. \tag{69}$$

The amplitude M_R^{nr} contains the renormalized charge g_R.

We stress the fact that in the derivation of the results of this section the assumption that the spectrum of $\hat{\mathcal{H}}$ has a discrete part was essential. Otherwise the whole sum over γ turns into an integral, changing the character of the singularities of the Green's function.

Let us derive an equation for the WF. We first write the equation for the two-particle vertex (Fig. 7), considering it homogeneous:

$$\Gamma(k_1, k_2, p, \omega\tau) = \int \Gamma(k_1', k_2', p, \omega\tau')\theta(\omega k_1')\theta(\omega k_2')\delta(k_1'^2 - m^2)$$

$$\times \delta(k_2'^2 - m^2)\delta^{(4)}(p + \omega\tau' - k_1' - k_2')\frac{1}{2\pi(\tau' - i0)}$$

$$\times V(k_1', k_2', \omega\tau', k_1, k_2, \omega\tau)d^4k_1'd^4k_2'd\tau'. \tag{70}$$

The kernel $V(k_1', k_2', \omega\tau', k_1, k_2, \omega\tau)$ represents the irreducible block in Fig. 7 and may be parametrized in the following manner:

$$V = V(\mathbf{q}', \mathbf{q}, \mathbf{n}, s_1), \tag{71}$$

where $s_1 = (k_1 + k_2 - \omega\tau)^2$ and the momenta $\mathbf{q}$ and $\mathbf{q}'$ are constructed according to the rule (42). Introducing the wave function $C_2 = \Gamma/2\pi(s - M^2)$ and noting that $\tau = (s - M^2)2(\omega p)$, we obtain

$$(4\varepsilon^2(\mathbf{q}) - M^2)C(\mathbf{q}, \mathbf{n}) = \frac{1}{\pi}\int C(\mathbf{q}', \mathbf{n}')V(\mathbf{q}', \mathbf{q}, \mathbf{n}, M^2)\frac{d^3q'}{2\varepsilon(\mathbf{q}')}. \tag{72}$$

The vector $\mathbf{n}$ plays the role of a parameter on which the kernel V depends. The equation (72) has an illustrative character, since it does not reflect the fact that in going over to the variables $\mathbf{q}$ and $\mathbf{n}$ one must use different vectors in the wave functions in the left-hand side and in the right-hand side:

$$Q = p + \omega\tau \text{ and } Q' = p + \omega\tau'.$$

However, in the present paper we shall not discuss this in more detail.

Even for quite general assumptions on the kernel $V(\mathbf{q}', \mathbf{q}, \mathbf{n}, M^2)$ the investigation of Eq. (72) may allow one to understand the basic properties of $C(\mathbf{q}, \mathbf{n})$.

7. CONCLUSION

The most unusual features of the formalism under consideration are the dependence of the amplitudes of processes on the arbitrary 4-vector ω and the dependence of the WF on the vector $\mathbf{n}$. Both these circumstances, although they have common origins, are in general different aspects of the problem. Thus the dependence on $\mathbf{n}$ in the WF means only a peculiar parametrization containing two parameters more than the nonrelativistic case (the two angles which determine the direction of n relative to the system of vectors $q_1, \ldots, q_{n-1}$). For the two-particle function we have only one extra parameter. If the WF would not depend on $\mathbf{n}$, this would not remove the dependence on the scattering amplitudes on ω.

Let us assess the extent to which the ω-dependence makes the theory ambiguous. As already pointed out, the ω-dependence of the matrix elements calculated in any order of perturbation theory is fictitious, according to Kadyshevskiǐ.[7] It stops being fictitious if the problem is solved approximately, however, it has the order of magnitude of the terms which were omitted, so that the exact amplitude does not depend on ω.

Thus, the discussion is self-consistent: the uncertainties related to the arbitrariness in the choice of ω do not exceed the accuracy of the chosen approximation. But this accuracy may also depend on ω. Therefore there arises the problem of the optimal choice of ω in the amplitudes to be calculated. We recall that the vector ω was chosen to lie on the light cone in order to simplify the theory. The next step is how to place ω in an optimal manner on the light cone. We note that the same uncertainty occurs in calculations using the old perturbation theory: the dependence on the coordinate system, or, for calculations in the infinite-momentum frame, the dependence on the direction of motion of the system. Here, however, this problem is posed clearly and can be done in terms of invariant variables, the scalar products of the 4-vector ω with the 4-momenta of the particles participating in the reaction. In the general case there are two such variables, thus, in this approach the amplitudes depend on additional scalar parameters and it is necessary to find a method for their optimal choice. Such a choice is perhaps the requirement that the singularities of the amplitudes, if they depend on these parameters, should be so situated that the diagram under consideration should be maximally distinguished.

In the present paper we have considered a state vector defined on a hyperplane. We note that it is quite tempting to go beyond the class of flat hypersurfaces and to define the state vector on hypersurfaces which are the most natural in a space with pseudoeuclidean metric, namely on a hyperboloid.

In order to utilize the proposed formalism in concrete calculations one must extend it to the case of particles with spin. The first attempts at describing the deuteron in pd-scattering in the backward direction with the help of a WF in ρ-space (neglecting the n-dependence and taking the spin into account nonrelativistically) were done in Ref. 16.

I express my profound gratitude to I. S. Shapiro for posing the problem, useful discussion, and criticism. The author is grateful to A. E. Kudryavtsev, Yu. A. Simonov, M. V. Terent'ev as well as to I. P. Volobuev, V. G. Kadyshevskiĭ, S. Shch. Mavrodiev and R. M. Mir-Kasimov, for useful discussions.

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Translated by M. E. Mayer

Study of the detecting properties of a counter filled with solid argon

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Joint Institute for Nuclear Research
(Submitted February 20, 1976)
Zh. Eksp. Teor. Fiz. **71**, 417–423 (August 1976)

The article reports on a study of a cylindrical filamentary counter (filament diameter 10 μ) filled with condensed argon. It is established that the ionization regime of operation is characteristic of both solid and liquid argon. In a counter filled with solid argon, a fraction of the pulses are observed to be amplified to a height exceeding the height of the ionization pulses by about 100 times. The contribution of these pulses to the total pulse-height distribution depends on the experimental conditions and the exposure time and does not exceed 30%.

PACS numbers: 29.40.Br

1. INTRODUCTION

In recent years many laboratories in various countries have investigated electronic methods of detecting particles in condensed noble gases.[1-10] Development of detectors using such a medium with high spatial and time resolution is necessary for solution of a number of physical problems, for example, for fast detection of neutral radiations and, particularly, neutrinos.[1,4] Most of the studies have used liquid argon or xenon as a working medium. Several studies have been made of detectors employing solid argon and xenon.[2,5,11,12]

Pisarev *et al.*[5] previously reported observation of electron multiplication (a gain up to 150 in the proportional region) near the filament in a counter filled with crystalline argon, and also with xenon. In the present article we report the results of more detailed study of the operation of the counter described previously.[5]

Liquid and crystalline argon were used as the working medium.

2. EXPERIMENTAL ARRANGEMENT

The experimental apparatus included a counter, a gas purification system, a temperature regulating system, and the detecting electronics. Descriptions of the basic arrangement of the apparatus and the design of the counter have been given previously.[5] We recall that the counter had a cylindrical cathode of diameter 6 mm and the anode consisted of a gold-plated tungsten wire 10 microns in diameter. A block diagram of the electronics used in the present studies is shown in Fig. 1. The signal from the counter anode was fed to a charge-sensitive preamplifier PA with a sensitivity of 7×10^{12} volts per coulomb,[13] and then to a linear amplifier A (maximum gain 2000) in which the signal was simultaneously amplified and shaped (the integration and differentation