Kinetic equations for sound and Alfvén waves in a plasma with random inhomogeneities

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Nonlinear interaction between Alfven and sound waves in a plasma layer with random concentration inhomogeneities is considered. Equations are derived for the interacting-wave intensities averaged over the inhomogeneity ensemble. Solutions of the equations are presented in an approximation with a given Alfven wave field. Results of a numerical calculation of the plasma-layer parameters are reported.

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The propagation of nonlinear signals in media with random inhomogeneities has been recently discussed in the literature (see[1-10]). The problems considered dealt with the interaction of averaged (over the ensemble of inhomogeneities) fields of quasimonochromatic wave packets[1-3] as well as with the propagation of nonlinear waves (solitons, shock waves) in systems with weak[4-6] and strong dispersion.[7] It is shown in these papers that random inhomogeneities lead to the average-field damping that can be of either high frequency (such as viscosity) or low frequency (friction) in media having resonant properties. Bogatyrev and one of us,[9] using a transmission line with random parameters as an example, have confirmed the main results predicted theoretically in[2].

It should be noted that the cited papers left open the question of the behavior of the fluctuating component of the field into which a fraction of the average-field energy is transformed as a result of scattering. This is the most complicated problem and pertains to the still unanswered question of the dynamics of the development of self-consistent turbulence. To a certain degree, an answer to this question might be obtained from a solution of the problem of the behavior of the total field in a randomly inhomogeneous nonlinear medium, since it constitutes the sum of the average field and its fluctuating component. In this paper, using the interaction of Alfvén and acoustic waves as an example, we obtain reduced equations for the complex amplitudes of the total fields of these waves in a plasma with one-dimensional stationary inhomogeneities of the density. We derive kinetic equations for the average field intensities in the randomized-phase approximation. We analyze some general properties of the solutions. We report also the results of a numerical calculation of the distribution of the average wave intensities in a plasma layer for different system parameters. The results are, in essence, of general theoretical interest, since the calculation procedure proposed in this paper can be used to describe a rather extensive class of nonlinear interactions in media with random parameters.

1. The initial system of one-dimensional ($\mathbf{H}_0 \parallel Ox$, $\mathbf{H}_0$ is the external magnetic field) magnetohydrodynamic equations describing the propagation of Alfvén and acoustic waves in a plasma with random density inhomogeneities along the Ox axis is of the form[1)]

$$\begin{gathered}
\frac{\partial v_x}{\partial t}+\frac{C_s^2}{\rho_0}\frac{\partial \rho}{\partial x}=\mu\left(\frac{C_s^2}{\rho_0^2}\rho\frac{\partial\rho}{\partial x}-\frac{1}{4\pi\rho_0}H_y\frac{\partial H_y}{\partial x}\right)+\nu\frac{C_s^2}{\rho_0^2}\frac{\partial}{\partial x}(\rho\delta\rho(x)),\\
\frac{\partial\rho}{\partial t}+\rho_0\frac{\partial v_x}{\partial x}=-\mu\frac{\partial}{\partial x}(\rho v_x)-\nu\frac{\partial}{\partial x}(\delta\rho v_x),\\
\frac{\partial v_y}{\partial t}-\frac{C_A^2}{H_0}\frac{\partial H_y}{\partial x}=-\mu\left(v_x\frac{\partial v_y}{\partial x}+\frac{C_A^2}{H_0\rho_0}\rho\frac{\partial H_y}{\partial x}\right)-\nu\frac{C_A^2}{H_0\rho_0}\delta\rho\frac{\partial H_y}{\partial x},\\
\frac{\partial H_y}{\partial t}-H_0\frac{\partial v_y}{\partial x}=-\mu\frac{\partial}{\partial x}(v_xH_y),
\end{gathered}\tag{1}$$

where v_x, v_y, and H_y are the projections of the velocity and magnetic-field perturbations on the axes x and y; $\delta\rho(x)$ is the specified fluctuation deviation of the plasma density from its unperturbed value ρ_0 (it is assumed that the plasma inhomogeneity is maintained by a corresponding external random force field), $\rho(x,t)$ is the wave perturbation; the parameters $\mu \ll 1$ and $\nu \ll 1$ have been introduced to designate the weak nonlinearity and inhomogeneity of the plasma (the right-hand sides of the system (1)); C_s is the speed of sound and $C_A = H_0/(4\pi\rho_0)^{1/2}$ is the Alfvén velocity.

We consider the interaction of opposing Alfvén and acoustic waves in a plasma layer. The frequencies and the wave vectors should satisfy the synchronism conditions

$$\omega_1-\omega_2=\Omega,\quad k_1=k_2+q,\tag{2}$$

where $\omega_{1,2}$, Ω, and $k_{1,2}, q$ pertain respectively to the Alfvén and sound waves. Under real conditions the following inequality is frequently satisfied:

$$C_A \gg C_s.\tag{3}$$

Relations (2) are then satisfied if $\Omega \ll \omega_{1,2}$. It is easy to see that in this case we have $\omega_1 \approx \omega_2 = \omega$, $k_1 \approx k_2 = k$ and $q \approx 2k$.

Inasmuch as scattering by the plasma inhomogeneities gives rise to waves propagating oppositely to the incident waves, the solution of the system (1) at $\mu \ll 1$ and $\nu \ll 1$ must be sought in the form[2)]

$$\begin{gathered}
\rho(x,t)\approx\int_{-\infty}^{\infty}b(x,t,\Omega)\exp\{i[\Omega t-q(\Omega)x]\}d\Omega+\int_{-\infty}^{\infty}\bar{b}(x,t,\Omega)\\
\times\exp\{i[\Omega t+q(\Omega)x]\}d\Omega,\\
H_y(x,t)\approx\int_{-\infty}^{\infty}a(x,t,\omega)\exp\{i[\omega t-k(\omega)x]\}d\omega+\int_{-\infty}^{\infty}\bar{a}(x,t,\omega)\\
\times\exp\{i[\omega t+k(\omega)x]\}d\omega,
\end{gathered}\tag{4}$$

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where $a(x, t, \omega)$ and $b(x, t, \Omega)$ are slowly varying ($\partial a/\partial x$, $\partial b/\partial x$, $\partial a/\partial t$, $\partial b/\partial t \sim \mu$, $\nu \ll 1$) complex amplitudes of the waves propagating from left to right in the plasma layer (towards positive values of x), and the superior bar denotes waves traveling in the opposite direction.

In a homogeneous medium, three-wave interaction of two opposing Alfvén waves ($a_1, \bar{a}_2$) and a sound wave ($b(x, t, \Omega)$) is described by a system of three equations for the complex amplitudes a_1, $\bar{a}_2$, and b. In an inhomogeneous plasma, backscattered waves are produced, and this should obviously lead to the appearance of an additional system of three equations for $\bar{a}_1$, a_2, and $\bar{b}$.

Substituting (4) in (1) and averaging over spatial (L_0) and temporal (τ_0) scales greatly exceeding $2\pi/\Omega$ and $2\pi/q$, respectively, we obtain the equations for the spectra of the interacting waves. Since we shall consider henceforth the stationary problem for a plasma layer, we assume below that all the quantities are independent of time ($\partial/\partial t = 0$). Since the system of six equations for the complex amplitudes is rather cumbersome, we write down by way of example the equation for the spectrum a_1 of the high-frequency Alfvén wave:

$$\frac{da_1}{dx} = i\sigma_1 \int b\bar{a}_2 \delta(\omega_1 - \omega_2 - \Omega) e^{i\Delta kx} d\omega_2 \, d\Omega - \frac{iq}{4L_0\rho_0}\left\{a_1 \int_{x-L_0/2}^{x+L_0/2} \delta\rho(\xi) d\xi - \bar{a}_1 \int_{x-L_0/2}^{x+L_0/2} \delta\rho(\xi) e^{iq\xi} d\xi\right\}, \quad (5)$$

where $\sigma_1 = \omega_1/2\rho_0 C_A$; $\Delta k = (\omega_1 + \omega_2)/C_A - \Omega/C_s$.

In the general case it is impossible to obtain from the system (5) a closed system of equations for the wave intensities. We consider therefore a particular case, but one of practical interest, when the randomized-phase approximations can be used. [12] To this end it is obviously necessary that the spatial scale L_{coh} of the randomization of the phases of the complex amplitudes of the interacting fields be small in comparison with the characteristic scale L_{nl} of the nonlinear interaction. This condition will henceforth be assumed satisfied. The system (5), by using a procedure described in sufficient detail by Tsytovich[12] for averaging over a scale $L_{\text{coh}} \ll L_{\text{av}} \ll L_{\text{nl}}$, yields then equations for the wave intensities. To be sure, in our case, in contrast to Tsytovich's book, [12] it is necessary to carry out an additional averaging over the ensemble of the density inhomogeneities $\delta\rho(x)$, although this averaging does not differ in principle from that indicated above. As a result we obtain for the average dimensionless wave intensities the following system of equations[3)]:

$$\frac{dN}{dx_\delta} = -\Gamma_1(N - \bar{N}) - \frac{\pi}{8}\beta\Phi(N, \bar{P}, M), \quad \frac{d\bar{P}}{dx_\delta} = -\Gamma_1(P - \bar{P}) - \frac{\pi}{8}\beta\Phi(N, \bar{P}, M),$$

$$\frac{dM}{dx_\delta} = -\Gamma_2(M - \bar{M}) + \frac{\pi}{4}\beta^2\Phi(N, \bar{P}, M), \quad \frac{d\bar{N}}{dx_\delta} = -\Gamma_1(N - \bar{N}) + \frac{\pi}{8}\beta\Phi(\bar{N}, P, \bar{M}), \quad (6)$$

$$\frac{dP}{dx_\delta} = -\Gamma_1(P - \bar{P}) + \frac{\pi}{8}\beta\Phi(\bar{N}, P, \bar{M}), \quad \frac{d\bar{M}}{dx_\delta} = -\Gamma_2(M - \bar{M}) - \frac{\pi}{4}\beta^2\Phi(\bar{N}, P, \bar{M}),$$

where $x_\delta = qx$ is the dimensionless coordinate, $\Gamma_1 = q/16\pi\sigma_\rho^2 f_\rho(q)$; $\Gamma_2 = q/\pi\sigma_\rho^2 f_\rho(2q)$ are scattering coefficients, $\sigma_\rho^2 = \langle(\delta\rho/\rho_0)^2\rangle$; f_ρ is the Fourier spectrum of the normalized correlation function of the density fluctuations $\delta\rho(x)$;

$$N = \langle|a_1|^2\rangle \omega_1 H_0^{-2}; \; P = \langle|a_2|^2\rangle \omega_2 H_0^{-2}; \; M = \langle|b|^2\rangle \Omega\rho_0^{-2}$$

are dimensionless quantities proportional to the intensities of the Alfvén and sound waves; $\beta = C_A/C_s \gg 1$; $\Phi = 2\beta NP + (N - \bar{P})M$; the symbol $\langle \ldots \rangle$ denotes statistical averaging; the quantities $\langle|a_1|^2\rangle$, $\langle|a_2|^2\rangle$ etc. are defined by the relation

$$\langle a_1(x, \omega) a_1^*(x, \omega')\rangle = \langle|a_1^2|\rangle \delta(\omega - \omega'). \quad (7)$$

It is easily seen that from (6) we obtain equations for three-wave interaction of Alfvén and sound waves, if we put $\Gamma_1 = \Gamma_2 = 0$ (there is no scattering) and $\bar{N} = P = M = 0$. In the presence of scattering we have a system of six coupled nonlinear equations for the wave intensities.

2. An investigation of the system (6) in general form is difficult, since the analysis must be carried out in six-dimensional phase space. Nonetheless, we can find two independent integrals

$$N + P - \bar{N} - \bar{P} = C_1, \quad N - \bar{N} + \frac{1}{2\beta}(M - \bar{M}) = C_2, \quad (8)$$

that express the conservation laws for the energy fluxes of the Alfvén and sound waves. To be sure, their presence unfortunately does not facilitate the investigation of (6). We therefore consider here the simpler problem of the distribution of the fields generated by the waves in a given field of two opposing Alfvén waves ($N = N_0 = \text{const}$, $P = P_0 = \text{const}$). We assume that the intensities of the sound M and $\bar{M}$ and of the scattered Alfvén waves $\bar{N}$ and $\bar{P}$ are small enough everywhere inside the plasma layer, so that we can neglect the terms $\sim \bar{N}P$ and $(\bar{N} - P)\bar{M}$ in equations (6). As a result we obtain a system of linear equations for M, $\bar{M}$, $\bar{N}$, and P.

The solutions of this system for the specified boundary conditions $M(0) = P(0) = 0$ and $\bar{M}(L_\delta) = \bar{N}(L_\delta) = 0$ are

$$M = C_1 \exp(-\lambda_1 x_\delta) + C_2 \exp(-\lambda_2 x_\delta) - G/\alpha,$$

$$\bar{M} = C_1 \frac{\Gamma_2 - \alpha - \lambda_1}{\Gamma_2} \exp(-\lambda_1 x_\delta) + C_2 \frac{\Gamma_2 - \alpha - \lambda_2}{\Gamma_2} \exp(-\lambda_2 x_\delta) - \frac{G}{\alpha}, \quad (9)$$

$$\bar{N} = N_0\{1 - \exp[-\Gamma_1(L_\delta - x_\delta)]\}, \quad P = P_0[1 - \exp(-\Gamma_1 x_\delta)];$$

$$\lambda_{1,2} = -\frac{1}{2}\alpha[1 \pm (1 - 4\Gamma_2/\alpha)^{1/2}], \quad \alpha = \frac{1}{4}\pi\beta^2(N_0 - P_0),$$

$$C_1 = \frac{G}{\alpha} \frac{\Gamma_2 - (\Gamma_2 - \alpha - \lambda_2)\exp(-\lambda_2 L_\delta)}{(\Gamma_2 - \alpha - \lambda_1)\exp(-\lambda_1 L_\delta) - (\Gamma_2 - \alpha - \lambda_2)\exp(-\lambda_2 L_\delta)}, \quad (10)$$

$$C_2 = \frac{G}{\alpha} \frac{\Gamma_2 - (\Gamma_2 - \alpha - \lambda_1)\exp(-\lambda_1 L_\delta)}{(\Gamma_2 - \alpha - \lambda_1)\exp(-\lambda_1 L_\delta) - (\Gamma_2 - \alpha - \lambda_2)\exp(-\lambda_2 L_\delta)}, \quad G = \frac{\pi}{2}\beta^3 N_0 P_0.$$

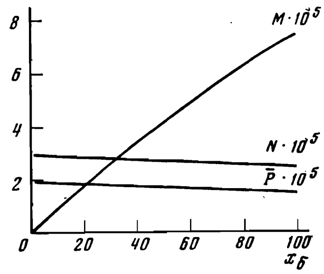

FIG. 1

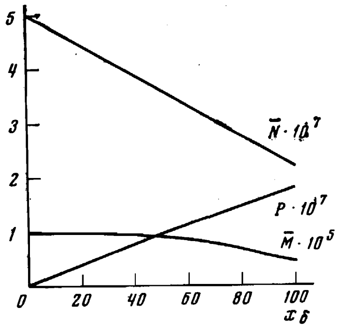

FIG. 2.

It is seen from (9) and (10) that the acoustic and scattered Alfvén waves increase in amplitude along the propagation direction, and the sound amplification has an aperiodic character at $\alpha > 4\Gamma_2$ (strong pumping). In the opposite case ($\alpha < 4\Gamma_2$) the sound intensity increases in oscillatory fashion.

At arbitrary values of the pump intensities, we have integrated the system (6) numerically with allowance for the following boundary conditions: $N(0) = N_0 = 3\times10^{-5}$, $\bar{P}(L_6 = 100) = 1.6\times10^{-5}$, $M(0) = 0$, $\bar{N}(L_6 = 100) = 2.2\times10^{-7}$, $P(0) = 0$, $\bar{M}(L_6 = 100) = 5\times10^{-6}$, $\beta = 10$, $\Gamma_1 = 10^{-2}$, and $\Gamma_2 = 0.16$ (the equality $\Gamma_2 = 16\Gamma_1$ is possible, as can be easily seen, only in the case of small-scale inhomogeneities $ql \ll 1$). The results of the calculation are shown in Figs. 1 and 2. The reliability of this calculation is evidenced by the satisfaction of the conservation law (8) at an arbitrary cross section x_6 of the plasma layer ($0 \leqslant x_6 \leqslant 100$). We note that the numerical data obtained here are rather general in character, inasmuch as similarity conditions are satisfied for the system (6)—this system is invariant to the substitutions:

$$N, \bar{N}, P, \bar{P}, M, \bar{M} \to \gamma N, \gamma\bar{N}, \gamma P, \gamma\bar{P}, \gamma M, \gamma\bar{M};$$
$$\Gamma_{1,2} \to \gamma\Gamma_{1,2}; \quad x_6 \to \gamma^{-1}x_6,$$

where γ is the similarity coefficient.

Thus, the results of the analytic investigation (formulas (9) and (10)) and of the numerical calculations show that in a number of cases the presence of random plasma inhomogeneities is an essential factor in nonlinear three-wave interaction. This manifests itself in additional generation of corresponding opposing waves, which in the case of a sufficiently thick plasma layer also take part in the nonlinear interaction. This effect, in particular, may turn out to be useful for the diagnostics of a weakly turbulent plasma. For example, in a given field of two opposing Alfvén waves, the solutions for the remaining waves as functions of the ratio of the nonlinearity parameter α and the scattering coefficient Γ_2 have either an aperiodic or an oscillatory character. Thus, at $\alpha < 4\Gamma_2$ it is possible to estimate Γ_2 from the period of the field oscillations. By determining the relative fluctuations of the electron density by another independent method, we can determine the scale l of the plasma inhomogeneities. Moreover, inasmuch as $\Gamma_2 \sim f_\rho(2q)$, by performing similar measurements at different frequencies of the incident waves we can investigate the low-frequency spectrum $f_\rho(2q)$ of the plasma turbulence as a function of the wave number q.

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[1)] For simplicity we consider here the case of one-dimensional inhomogeneities, when both the incident and scattered waves can propagate only in the x direction. The calculation of the problem for the case of three-dimensional inhomogeneities is much more complicated, but for the equations obtained for the average intensities integrated over the transverse coordinates y and z should agree qualitatively with those obtained below. We note also that the stationary-inhomogeneity approximation employed here is valid when the characteristic time of alternation of the realizations of the random field is large in comparison with the other times of the problem $2\pi/\Omega$, L/C_s, etc. (L is the plasma-layer thickness). This situation can be realized, for example, for a plasma whose turbulence in the region of low frequency is determined by the drift instability,[11] where $C_T/C_s \sim r_H/L \ll 1$ (r_H is the ion gyromagnetic radius, L is the scale of the concentration gradient, and C_T is the pulsation velocity).

[2)] Since we are investigating a boundary-value wave-interaction problem, we assume ω and Ω in the solution (4) to be the running frequencies, and the vectors k and q to be functions of ω and Ω; the representation of the solution in the form (4) is possible for sufficiently quasi-monochromatic waves ($\Delta\omega \ll \omega$, $\Delta\Omega \ll \Omega$).

[3)] Strictly speaking, the quantities N, $\bar{N}$, P, $\bar{P}$, M and $\bar{M}$ introduced here for convenience are not dimensionless wave intensities. They can be obtained by multiplying these quantities by by $\Delta\omega/\omega$ and $\Delta\Omega/\Omega$ for Alfvén and sound waves, respectively. In addition, as shown by estimates, Eqs. (5) and (6) are satisfied if the inequality $\sigma_\rho^{-1} \gg qf_\rho(2q) \gg (qL_0)^{-1}$ holds. Physically they mean that the most effective contribution to the backscattering is made by the resonant lattice with period $\sim (2q)^{-1}$.

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