The phase stochastization mechanism and the structure of wave turbulence in dissipative media

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Turbulence mode models in dissipative media with hydrodynamic type of nonlinearity are considered. It is proposed that only one mode, viz., the decay mode, has a growth rate. By qualitative methods and a computer experiment it is shown that stochasticity is already possible in two- and three-mode models. In cases when the decay conditions are multiple-valued, the competition between various pairs produced upon decay of an unstable mode is investigated. Turbulence with a broad instability range is described within the framework of the kinetic equation for complex quasiparticles—stochastic triplets.

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INTRODUCTION

Investigations of turbulence in dissipative media are called for in many problems of physics, such as Langmuir turbulence in a collision-dominated plasma, acoustic turbulence in solids, investigations of thermal conduction, of the boundary layer, etc. In the majority of cases, turbulence in dissipative media is not similar to the most thoroughly investigated turbulence in media characterized by the presence of a wide inertia interval in which both the instability and the damping can be neglected. The turbulence that is established within the inertia interval usually has a universal spectrum k^s, [1] is fully described by the energy flux over the spectrum, and is essentially independent of the cause of its excitation, be it the intrinsic instability outside the inertia interval, the action of external fields, etc. In the case of a weak nonlinearity, such a turbulence is described by the kinetic equation for the waves, [1,2] which can be obtained by using the random-phase approximation (RPA). On the other hand, if the inertia interval is either small or nonexistent, then the phases of the individual modes can no longer be regarded as independent even in the case of a weak nonlinearity, and the random-phase approximation cannot be used. This situation, wherein the phases of the individual waves are interconnected in the case of multiwave interaction, is customarily called strong wave turbulence. [3]

The structure and properties of the strong turbulence already depend significantly on the method of its excitation and on the width of the instability interval. [4,5] It is just these questions which are discussed in this article as applied to turbulence in which the energy flows towards the lower end of the spectrum. Examples of such turbulence are: plasma turbulence connected with excitation of low-frequency waves such as ion or magnetic sound; sound turbulence in solids; turbulence of internal waves in the ocean when they are excited by surface waves, and others. The investigation is carried out within the framework of a traditional approach based on expansion in modes. We start with the equations for the time-varying complex amplitudes. We consider turbulence in dissipative media with a nonlinearity of the hydrodynamic type, i.e., a quadratic and conservative nonlinearity. It is assumed that one of the modes has a linear growth rate and the others are damped.

With the aid of qualitative methods and a computer experiment, it is shown that in the simplest (2- and 3-mode nonconservative) models, stochasticity can set in with properties that are different under finite changes of the initial conditions. This problem is of independent physical interest, since it arises in any situation in which a linearly amplified mode is stable with respect to decay into two damped modes, i.e., when it excites a pair of resonantly coupled modes in a dissipative medium. In the more general case, when the ambiguity of the decay conditions is taken into account but the instability region is spectrally narrow, a discrete set of decay pairs is produced. The phases of the pairs that are close in frequency are mutually synchronized, i.e., **k**-space regions are produced in which the phase is constant. Such regions will be called for brevity "**k**-domains." Different pairs of **k**-domains, which produce a triplet as a result of coupling with one unstable mode, compete with one another and the turbulence is made up of random pulsations with varying characteristic scales.

If the instability region in **k**-space greatly exceeds the domain width, then the coexistence of a large number of stochastic triplets becomes possible. To describe the turbulence in this case, a kinetic equation is derived for the unstable modes that generate different triplets. This equation is obtained by using the circumstance that the phases of the modes from different triplets are not correlated because of their stochastization within the triplet. By way of example we consider acoustic turbulence in a medium in which the spectrum of the damped low-frequency waves is analogous, in the linear approximation, to the spectrum of ion sound in a plasma or of helicons in a solid.

1. MODEL OF TURBULENCE WITH SPECTRALLY NARROW EXCITATION REGION

If an unstable mode exists in a nonlinear dissipative medium with dispersion, it is necessary in the general case to take into account the following nonlinear processes:

1) Generation of harmonics of the growing mode and

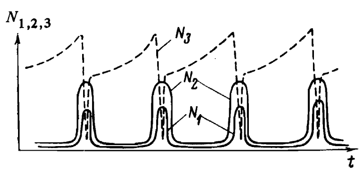

FIG. 1. Qualitative dependence of the intensity of the unstable HF mode and of the damped LF modes on the time for the steady stabilization regime.

the resultant synchronization of the phases of the harmonics of the nonlinear periodic waves whose waveform depends on the magnitude of the dispersion for the given type of waves.

2) Decay of the growing wave $(\omega_0, \mathbf{k}_0)$ into pairs of low-frequency waves $(\omega_1, \mathbf{k}_1)$ and $(\omega_0-\omega_1, \mathbf{k}_0-\mathbf{k}_1)$ with synchronized phases, where the waves in each pair can be of different types, for example Langmuir and ion-sound or surface and internal.

3) The interaction between the different nonlinear waves and the decay pairs.

If the medium has a sufficiently strong dispersion for all the wave types that participate in the interaction, then no harmonics can be produced and the principal elementary process in the general picture of the turbulence is the decay of the growing mode into pairs that are parametrically coupled with it.

1. Stochastic triplet. Qualitative analysis

The decay interaction of an unstable mode with a pair of modes that is damped in the linear approximation, in a medium with quadratic nonlinearity, is described by the equations (see, e.g., [6,7]).

$$\dot{A}_{1,2}=-\sigma A_3A_{2,1}\sin\Phi-\nu_{1,2}A_{1,2},\quad \dot{A}_3=\sigma A_1A_2\sin\Phi+\tilde{\gamma}A_3,$$
$$\dot{\Phi}=\sigma(A_1A_2A_3)^{-1}(A_1^2A_3^2+A_2^2A_3^2-A_1^2A_2^2)\cos\Phi-\Delta\omega. \tag{1}$$
$$\Phi=\varphi_3-\varphi_2-\varphi_1.$$

Here A_i and φ_i are the amplitude and phases of the interacting waves, σ is the coefficient of nonlinear interaction, $\nu_{1,2}$ characterizes the damping of the low-frequency (LF) waves, $\tilde{\gamma}$ is the growth rate of the high-frequency (HF) waves, and $\Delta\omega$ is the deviation from synchronism. The system (1) has no stable equilibrium state. This means that in this case, in contrast to interaction of modes with random phases, if stabilization via decay is possible, a periodic or quasiperiodic process will be produced. Let us explain its properties.[1)]

The character of the solutions of the system (1) is determined by the intensity of the interacting waves and the amplitude-phase relations between them. At a large initial energy of the damped modes $A_{1,2}(0)\sim A_3$, when $\sigma A_3^2 \gg \tilde{\gamma}$, ν and $|\Phi(0)| \ll \pi/2$, the solutions can be easily shown to be close to the solution of the conservative triplet, and the instability and the damping can be regarded as small. All the characteristics of this solution can be obtained by using the method of averaging over elliptic functions (see[8]). On the other hand, in cases when as a result of the evolution of the solution or the initial conditions we have, for example, $A_{1,2}(0) \ll A_3(0)$ at $\Phi\sim\pi/2$, then the solution of (1) will differ qualitatively from the conservative analog even at small $\nu_{1,2}$ and $\tilde{\gamma}$. Indeed, at $A_3\sin\Phi > -\nu_{1,2}/\sigma$ (we assume that $A_1\sim A_2$) the damped modes, being small, prevent each other from growing until A_3, owing to its exponential growth that is independent of A_1 or A_2, becomes so large that the parametric growth rates exceed the damping. The duration of the "slow" stage, i.e., the time that the amplitudes of the low-frequency modes stay near zero, can be arbitrarily longer than the time of the "fast" motion—the nonlinear exchange of energy between $A_{1,2}$ and A_3. Thus, in this case the stabilization regime will be of the relaxation type: $A_{1,2}(t)$ will take the form of a sequence of spikes, and $A_3(t)$ will constitute pulses with exponential "roofs" (see Fig. 1). This is precisely the relaxation regime which is of interest from the point of view of the onset of stochasticity in a system of the coupled stable and unstable modes.

An investigation of the process of interest to us can be simplified by recognizing that at $\nu_1=\nu_2=\nu$

$$\frac{d}{dt}(A_1^2-A_2^2)=-\nu(A_1^2-A_2^2) \tag{2}$$

and the intensity of the LF modes, starting with a certain time, should be regarded as equal $A_1^2=A_2^2=A^2$. We then obtain instead of (1) the system

$$\dot{X}=Z+\delta Y-2Y^2+\gamma X,$$
$$\dot{Y}=-\delta X+2XY+\gamma Y,\quad \dot{Z}=-2Z(X+1), \tag{3}$$

where

$$\tau=\nu t,\quad \delta=\Delta\omega/\nu,\quad \gamma=\tilde{\gamma}/\nu,$$
$$X=\sigma A_3\sin\Phi/\nu,\quad Y=\sigma A_3\cos\Phi/\nu,\quad Z=\sigma^2A^2/\nu^2$$

(the dot denotes the derivative with respect to τ). For a qualitative analysis of the solutions of (3) we investigate the behavior of the trajectories in phase space of this system, assuming γ to be small ($\gamma \ll 1$).[2)] We subdivide the phase space in this case into regions of fast and slow motions. The latter are located near the line $Y=\delta/2$ and $Z=0$, and are defined by the inequalities

$$|Z| \lessapprox \gamma|X|,\quad |2Y^2-\delta Y| \lessapprox \gamma|X|.$$

The system (3) has only two equilibrium states, both unstable, of the "saddle-focus" type. One is located at the origin and the other has coordinates

$$X=-1,\quad Y=\delta/(2-\gamma),$$
$$Z=\gamma[1+\delta^2/(2-\gamma)^2],$$

and is located at the boundary of the fast and slow motions.

A. Exact synchronism ($\delta=0$). The phase space of the system (3) contains at $\delta=0$ two integral surfaces, which the trajectories approach arbitrarily closely but cannot intersect. These surfaces are the planes $Z=0$ and $Y=0$. At the intersection of the integral planes are located the cones of the slow motions—see Fig. 2. The phase portraits of these planes are shown in Fig. 3. On the plane $Z=0$ the region of slow motions is bounded by the parabola $\gamma|X|=2Y^2$, while at $X>0$ it is the isocline of the horizontal tangents. Outside this region,

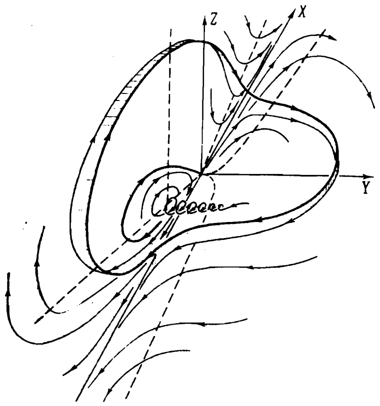

FIG. 2. Phase portrait of the system (3) in the case of exact synchronism. The equilibrium state is a saddle-node at the origin and a saddle-focus on the integral plane $y=0$. The steady limit cycle corresponding to a stable single-period motion is illustrated qualitatively.

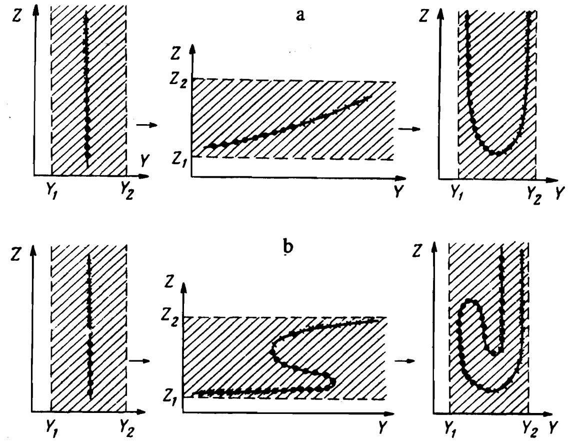

FIG. 4. Schematic form of the mappings of narrow strips on the $X=0$ plane into each other: a) onset of "simple horseshoe"; b) onset of "double horseshoe."

the influence of the small growth rate is negligible and the trajectories hardly differ from the circles[3] $X^2+Y^2=\text{const}$, on which the energy is conserved. The motions on this plane is asymptotically stable with respect to perturbations of Z in the region $X>-1$.

Motions on the integral plane $Y=0$ are asymptotically stable with respect to the perturbations of Y in the region $X<-\gamma/2$. On this plane, the region of slow motions is bounded by the lines $Z=\pm\gamma X$, while at $X<0$ this is the isocline of the vertical tangents (see Fig. 3a). On the integral surfaces $Y=0$ and $Z=0$, the behaviors of the trajectories are very similar—they tend asymptotically to the X axis as $t\to\infty$.

We now describe in detail the character of the motions corresponding to trajectories that are not too far from the planes $Z=0$ and $Y=0$. We emphasize by way of introduction that in the system (3) there is a strong nonlinear crowding together of the trajectories. In our case, $\delta=0$, they tend asymptotically to the plane $Y=0$ at $X<-\gamma/2$ and to the plane $Z=0$ at $X>-1$. Tending in this half-plane to the plane $Z=0$, the representative point moves over the circle $X^2+Y^2=\text{const}$ until it falls into the region of slow motions near the surface $Y=0$, (see Figs. 2 and 3b), where the amplitude of the HF mode increases exponentially. After leaving the region of slow motions, the point, remaining near the plane $Y=0$, rises upward and goes towards increasing X—this is the decay stage. It then crosses the plane $X=0$ and drops again towards the plane $Z=0$; this corresponds to the coalescence process. After falling into the region of slow motions, the point moves near $Z=0$ in the direction of increasing X and then, crossing the boundary of the region of slow motions, lands on the circle $X^2+X^2=\text{const}$, and so on.

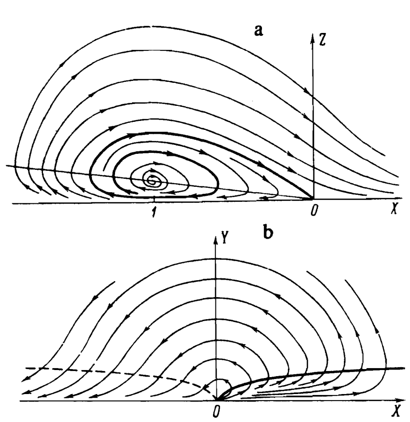

FIG. 3. Phase portraits of the integral planes: a) $Y=0$; b) $Z=0$.

To determine the properties of this motion, let us ascertain in which manner the points of the narrow vertical slit $0<Y_1<Y<Y_2\ll 1$ on the plane $X=0$ are mapped by the phase trajectories at $X>0$ into the points of the horizontal strip $0<Z_1<Z<Z_2\ll 1$ of the same plane, and then, how they are mapped for the trajectories at $X<0$ again into the points of the vertical strip. The construction of such a point mapping—of the dependence of the motion coordinates Z and Y at the end of the period on the values of Z and Y at the start of the period—makes it then possible, by using an iteration procedure, to establish such qualitative features of the process as the existence of periodic regimes and the onset of stochasticity.

The mapping of the vertical plane into the horizontal one, and again into the vertical one, is shown qualitatively in Fig. 4. This figure shows the successive transformations of a certain line in the vertical strip for two different cases, depending on the proximity of this vertical strip to the Z axis.

Let us illustrate the mapping picture for the more complicated case shown in Fig. 4b. It follows from (3) that the larger the value of Z at which the phase trajectory crosses the vertical strip, the farther from the X axis it drops on the plane $Z=0$. The trajectory that begins on the plane $X=0$ at sufficiently large Z, after dropping to the plane $Z=0$, does not fall in the region of slow motions, but moves immediately on the circle $X^2+Y^2=R^2$. It is clear that in the vicinity of the same circle there passes also another trajectory, which, af-

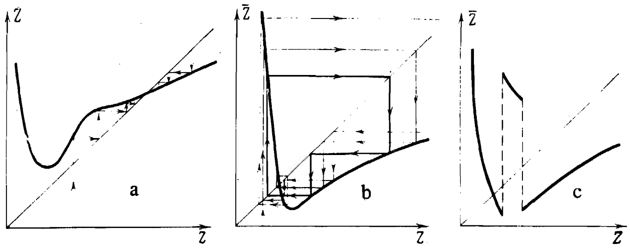

FIG. 5. Mapping of a vertical strip on the $X=0$ plane into itself by mapping of points with identical Z and close values of Y: a) mapping of a segment into itself, corresponding to a single-period regime at $\delta=0$; b) mapping in which there exists an infinite set of periodic motions at $\delta=0$; c) intermittent mapping, obtained when the mapped segment intersects the separatrix near the plane $Y=0$.

ter passing through the vertical strip at small Z, drops on the $Z=0$ plane in the region of slow motions and emerges from that region, intersecting the parabola, at $X^2 \approx R^2$. Thus, trajectories that cross the vertical strip far from each other turn out to be along-side each other after crossing the horizontal strip. It is this which explains the appearance of "horseshoes" in the mapping picture.

An analysis similar to that carried out above applies also to those $X<0$ trajectories which go off from the plane $Z=0$ and hug the plane $Y=0$, but here the role of the parabola is assumed by the straight line $Z=-\gamma X$, i.e., the character of the mapping of the horizontal strip into a vertical one is similar to that described above. Thus, a "double horseshoe" can appear in the complete mapping picture. Multiple passage of the trajectory through the plane $X=0$, corresponding to multiple application of the mapping, can make the picture of the motion very complicated and entangled.

Assuming that the considered motions are stable, let us investigate their structure in greater detail. To this end we use a cruder model, identifying the points having equal Z in the vertical strip with points having equal Y in the horizontal strip. Instead of mapping a strip into a strip we then map the line Z into itself, i.e., the dependence $\bar{Z}(Z)$ of the succeeding points $\bar{Z}$ on the initial points Z (see Fig. 5). At different positions of $\bar{Z}(Z)$ relative to the bisector $\bar{Z}=Z$ there exists either a single stable single-period regime (see Fig. 5a), or a set of regimes with modulation—multiperiod regimes, to which closed cycles correspond on the $(\bar{Z}, Z)$ plane (see Fig. 5b).[4)]

We now use the known result of the formal theory of point mapping,[9] namely, if in a system described by the mapping $\bar{Z}(Z)$ there exists periodic motion with an odd number of periods, then regardless of its stability there exists in the system one more denumerable set of unstable multiperiodic motions and a finite or denumerable number of stable motions. As seen from Fig. 5b, such odd (say three-period) motions exists in our crude model described by the mapping $\bar{Z}(Z)$. Consequently, a denumerable set of unstable periodic motions exists in this system.

One can expect at a sufficiently small γ this set to exist in a bounded region of phase space also in the initial system. If there were likewise no stable motions in such a bounded phase-space region, to which the neighboring trajectories tend, then this would apparently be sufficient for the onset of stochasticity.[5)] However, in the absence of detuning, judging from the crude mapping $\bar{Z}(Z)$, such stable motions do exist and correspond to cycles supported on one of the vertices at the "bottom" of the function $\bar{Z}(Z)$, where $|d\bar{Z}/dZ| \approx 0$.

B. Influence of the detuning on the dynamics of a triplet. Introduction of detunings destroys the integral surface $Y=0$ and transforms the equilibrium state at zero from a saddle–node into a saddle–focus (see Fig. 6). The stable separatrix of this equilibrium state, which became twisted around the point $X=-1$, $Z=\gamma$ as $t\to\infty$ in the case $\delta=0$, already goes off to the half-space $Y<\delta/2$ at $\delta>0$. The behavior of the trajectories becomes much more complicated, and a qualitative analysis without additional assumptions becomes impossible. We confine ourselves therefore to the case of small δ, when we can still approximately disregard the destruction of the integral surface $Y=0$. The role of the detuning reduces in this case, just as in a conservative system, only to a hindering of the exchange of energy between the modes, i.e., in our case, to a decrease of the maximum attainable values of Z at $X=-1$. The latter circumstance means that situations are possible in which the line carried by the trajectories at $X<0$ in the course of the mapping from the plane $X=0$ into the vicinity of the plane $Y=0$ lies on this plane near the equilibrium state $X=-1$, $Z=\gamma$. This means that the considered mapping is divided into two classes of motion by a separatrix that enters in the zero state of equilibrium. The trajectories that enter in the vicinity of the plane $Y=0$, outside the unwinding separatrix (see Fig. 2) cross the plane $X=0$ and move farther in a direction of increasing X. On the other hand the trajectories that fall inside the separatrix turn towards decreased values of X before they cross the plane $X=0$. They cross this plane, but only after one or several revolutions in the region of $X<0$, and furthermore at larger Z. The picture of the "cruder" mapping of a line into a line, corresponding to the described situation, is shown in Fig. 5c. The discontinuities on the $\bar{Z}(Z)$ curve near small Z correspond precisely to the subdivision of the motions by the separatrix. For dynamic systems described by a similar piecewise-smooth transformation under certain limitations,[12] one can

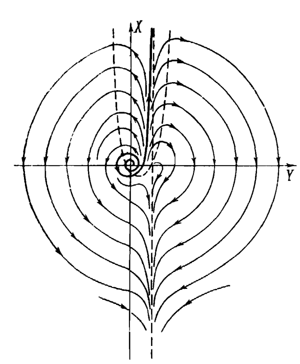

FIG. 6. Phase portrait of integral plane $Z=0$ at $\delta>0$.

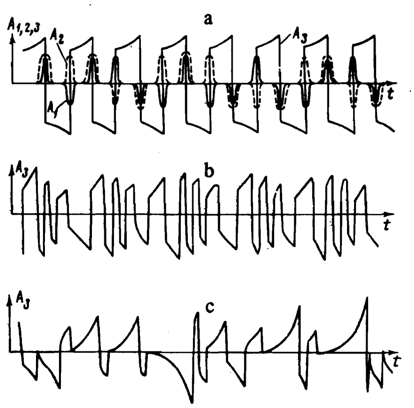

FIG. 7. The steady-state motion observed in the computer experiment: a) single-period regime at $\delta=0$, $\Delta\omega=0$, $\nu\gamma>1$; b) multiperiodic regime at $\delta=0$, $\Delta\omega=0$, $\nu\gamma\le 1$; c) stochastic regime at $\delta=0.001$ and $\Delta\omega=0.001$.

prove a very strong statement, namely that mixing exists in the system.

Thus, our analysis shows that if the regime of stabilization of the unstable mode as a result of decay into damped modes is possible, then in the presence of detuning it can be stochastic—in a purely dynamic system!

2. Decay triplet. Computer experiments

A detailed numerical experiment[6] performed with the system (1) has confirmed all the principal results of the qualitative analysis. We emphasize that the dynamic processes are observed only in the absence of detuning, $\delta=0$, while the stochastic processes, conversely, only at $\delta\ne 0$. We obtained approximately 100 realizations of which $\frac{2}{3}$ were at $\delta\ne 0$. We chose the following values of the parameters: $\gamma=0.15$, 0.1, and 0.01; $\delta=0$, 0.001, 0.01, and 0.1; the initial conditions were chosen to be $A_{1,2,3}=5$–10 and $\Phi(0)=0$–2.

A. Resonant case, $\delta=0$. In the resonant case, depending on the initial energies, we observed two types of motion—single-period (Fig. 7a), for which the maximum value of A_3 were ~ 13–18, and multiperiodic (see Fig. 7b). Only three- and four-periodic motions were

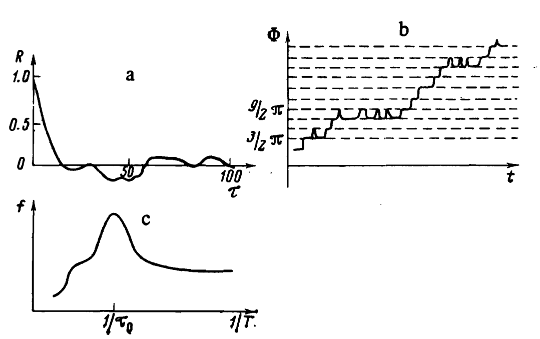

FIG. 8. Characteristics of the regime of total stochasticity: a) form of the autocorrelation function $R(\tau)$; b) character of the variation of the phase difference Φ with time, with accumulation of Φ observed; c) distribution function in the reciprocal duration between spikes $f(1/T)$.

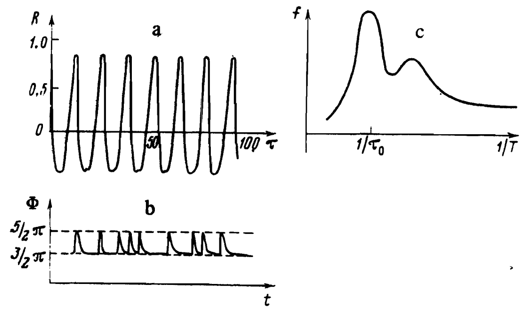

FIG. 9. Characteristics of partial stochasticity regime: a) autocorrelation function $R(\tau)$; b) plot of $\Phi(t)$, it is seen that there is no phase accumulation; c) distribution function $f(1/T)$.

stably observed, while others, for example five- and six-period motions, had a small stability region and from time to time were transformed into one another. The amplitude of the multiperiodic motions was noticeably smaller, $A_{3\,max}\sim 5$–8, and the period was larger by a factor 1.5–3 than for the single-period motions. The instability of the mode could be eliminated only at a sufficiently large relative damping and a small growth rate, namely, at $\gamma<\frac{1}{6}$. At $\gamma\sim\frac{1}{6}$ the form of the oscillations was described approximately by elliptic functions, and with decreasing γ it became more and more relaxational—the motions took the form of discontinuous exponentials for A_3 and of narrow solitons for A_1 and A_2 already at $\gamma=0.1$ (see Fig. 7a), i.e., sharply pronounced rapid and slow motions were observed. The period of the oscillations was ~ 1–1.5, and the time of the fast motions was <0.1. With the exception of these time intervals, the phase Φ remained approximately equal to $\pm\pi/2$ regardless of the initial values of $\Phi(0)$.

B. Nonresonant case. In the nonresonant case, when the synchronism conditions between the modes were not exactly satisfied ($\delta\ne 0$), all the observed motions were stochastic. Their properties, in contrast to the dynamic processes, depended essentially on the initial conditions with respect to Φ. At $\delta>0$, $A_{1,2,3}\sim 5$–10, and $\Phi=0$–2π we observed three qualitatively different groups of stochastic regimes, the autocorrelation functions of which are shown[7] in Figs. 8, 9a, and 10a. The same figures show the distribution functions, corresponding to these regimes, of the intervals between the jumps in terms of their reciprocal duration $f(1/T)$, and oscillograms of the difference between the phases $\Phi=\Phi(t)$. The amplitude realizations corresponding to the different stochastic regimes are qualitatively similar, and only the maximum values of the amplitudes change. One of the typical realizations is the depen-

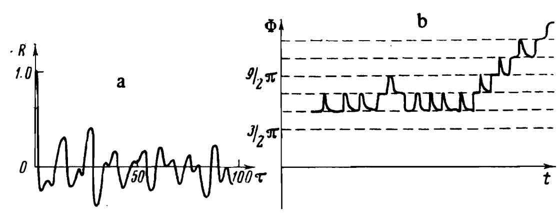

FIG. 10. Characteristics of transient stochasticity: a) $R(\tau)$, b) $\Phi(t)$.

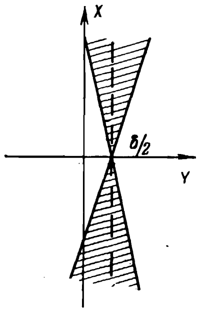

FIG. 11. Limits of the existence of different types of stochastic regimes on the (X, Y) plane.

dence of the amplitude of the unstable mode on the time and is shown in Fig. 7c. The regions of the initial values of $\Phi(0)$, at which these different regimes were observed (we shall call them arbitrarily "complete stochasticity"—Fig. 8, "partial stochasticity"—Fig. 9, and "transient stochasticity"—Fig. 10) are plotted in the (X, Y) plane, see Fig. 11. We add that the average period—the time of the slow motions of all the stochastic processes—was larger by one order of magnitude than the time of the slow processes in the dynamic case.

3. Comparison with theory

We recall that the stochastic regimes were observed only after introduction of the detuning δ, and even at small δ the intensity of these motions Z_{max} or $(X^2 + Y^2)_{max}$ was smaller by approximately one order of magnitude than at $\delta = 0$. We have thus confirmed the statement used in the qualitative analysis that the detuning lowers the level of the maximum attainable Z, on the basis of which the intermittent mapping $\bar{Z}(Z)$ was obtained.

As follows from the qualitative analysis, disordered motions of two types are possible in the system (3) at $\delta > 0$. One of them corresponds to trajectories in the right half-space, for which Y is always larger than $\delta/2$ and the accumulation of the phase difference Φ in the periodic or quasi-periodic motion is impossible. The other types of motion correspond to trajectories in the left half-space—$Y < \delta/2$, and for these motions accumulation of a phase difference is already possible, because the trajectories can go from the region $Y < 0$ into the layer $0 < Y < \delta/2$, and consequently encircle the Z axis.

It is easy to verify that the experimentally observed partial-stochasticity regimes correspond precisely to "right-hand" disordered motions and to those "left-hand" motions whose trajectories do not enter the layer $0 < Y < \delta/2$. Indeed, during the slow stage of the right-hand motion the phase difference Φ is close to $\pm\pi/2$, after which it rapidly changes to π—motion near the plane $Y = \delta/2 + 0$, after which it resumes its previous value—motion near the plane $Z = 0$ outside the region of the slow motions. The situation is similar for the left-hand motions lying in the half-space $Y < 0$. This is precisely the time-variation of motion observed experimentally in the partial-stochasticity regime, see Fig. 9b.

On the other hand, in the left-hand motion, in which the representative point goes from the half-space $Y < 0$ into the layer $0 < Y < \delta/2$, the value of Φ increases, in this case Φ changes during the fast motion by π, after which the trajectory again goes off to the left half-plane and moves near the circle, while Φ increases by another π. This explains the accumulation of Φ observed in the regime of complete stochasticity—see Fig. 8b.

We emphasize that the properties of the regimes of complete and partial stochasticity, for example the form of the autocorrelation function, did not depend on the calculation accuracy in a wide range. This proves the dynamic origin of such a stochasticity in the computer experiment. All that depended on the calculation accuracy, i.e., in fact on the fluctuation level, in narrow limits, was the thickness of the boundary separating the regions of the existence of the two regimes. At the boundary itself there was observed the stochasticity regime represented in Fig. 10. Its properties depended on the calculation accuracy and it obviously does not characterize the stochasticity of the dynamic system itself.

Besides experiments with the initial system (1), experiments were performed also on the degenerate interaction of a stable mode and its unstable second harmonic. In the presence of detuning, the observed results agree fully with those described, and by the same token confirm the validity of the transition from the system (1) to the system (2) in the qualitative analysis.

4. Structure of turbulence

We examine this problem, using as an example a parametrically excited one-dimensional acoustic turbulence in a dissipative medium with dispersion. This can be, in particular, acoustic waves in a solid, ion sound in a plasma, and other types of low frequency waves with analogous dispersion law $\omega(k)$—helicons, internal waves, etc. We assume that the frequency of the unstable mode is close to the limit of the strong dispersion, so that no energy is transferred upward in the spectrum. Then, bearing in mind the linearity of $\omega(k)$ at low frequencies, we can confine ourselves to an analysis of only decay processes. In the general case, for a continuous spectrum of the decay pairs, the initial system is represented in the form

$$\frac{\partial b}{\partial t} + v_b \frac{\partial b}{\partial x} = -\beta\omega_0 \int\limits_{\Delta}^{k_0/2} \int\limits_{-\delta k}^{\delta k} a_k a_{k_0-k+q} e^{i[\delta\omega(k,q)t-qx]}\, dk\, dq + \tilde{\gamma} b,$$

$$\frac{\partial a_k}{\partial t} + v_k \frac{\partial a_k}{\partial x} = \beta_k b \int\limits_{-dk}^{\delta k} \dot{a}_{k_0-k+q} e^{i[\delta\omega(k,q)t-qx]}\, dq - \nu_k a_k,$$

where b is the complex amplitude of the wave of frequency ω_0, excited with a growth rate $\tilde{\gamma}$, a_k is the complex amplitude of the k-th component of the spectrum, and ν_k is its decrement ($k_0 = k(\omega_0)$, $0 < |q| < \delta k \ll k_0$).

According to (4), in the linear approximation there will be excited predominantly pairs with $k \sim k_0/2$, which are symmetrical with respect to $\omega_0/2$ and $k_0/2$—for these the decay increment

$$\gamma_p = \beta[\omega(k)\omega(k_0 - k)|b|^2]^{1/2}$$

is maximal, while pairs with $|k_0 - k| \sim k_0/2$ do not grow at all, a fact taken into account into (4), where $\Delta/k_0 \ll 1$.

The problem of the nonlinear evolution of the spectrum of the initial fluctuations was solved for a particular but discrete pair spectrum. As shown by a computer experiment in which the interaction of different decay pairs was investigated, close pairs become mutually synchronized in phase, forming bound states—domains in k-space.[8] In the dynamic case, when the dispersion is neglected, $\Delta\omega = 0$, the width $\Omega = V_0 \delta k$ of these domains decreases slowly with increasing pair damping—at $\gamma/\tilde{\gamma}$ we have $\Omega = 0.1$ and at $\gamma/\tilde{\gamma} = 30$ we have $\Omega = 0.06$. When the dispersion is taken into account (stochastic triplets), the width of the domains, in a wide range, no longer depends on ν/γ and increases linearly with increasing detuning $\Delta\omega$ inside the triplet. For example, at $10 < \nu/\tilde{\gamma} < 100$ we have $\Omega \approx 30\ \Delta\omega$.

If the growth rate of the unstable mode is spectrally narrow, there can exist at each instant of time, as shown by experiment, only one pair of domains. The obvious reason is that the different pairs are fed from a single spectrally narrow source and suppress each other when they interact.[9] The presence of fluctuations causes the lifetime of the arbitrary pair of domains to be finite, and the system goes over randomly from a state with one excited pair into a state with another excited pair. Physically this result seems quite obvious and is explained by the fact that the decay pairs really exist only during a fraction of the period τ_0 of the triplet—their intensities take the form of sequences of narrow spikes, see Fig. 7. The random bursts of intensity of other spectral components in the interval between the spikes induce decay of the unstable mode into a new pair, etc. In the computer experiment, the lifetime of the pair turns out to be $\tau_{\text{pair}} \sim 10\text{–}50\tau_0$. The distribution function of the pair in frequency had a maximum near $\omega_0/2$.

2. KINETIC EQUATION FOR TRIPLETS

If the instability region of the decaying modes in k-space, while remaining small in comparison with k_0, is broad in comparison with the width of the domain, then different domain pairs can already coexist.[10] The turbulence in this case is an ensemble of different decay triplets, and its spectrum depends on the character of the interaction between them. If there is no dispersion in the entire considered k-interval, domain pairs from different triplets will be resonantly coupled via a third unstable mode, different triplets apparently synchronize each other, and the turbulent spectra becomes much more complicated.

We consider here a simpler case, when the dispersion is appreciable within the instability interval and the spectrum of the unstable mode is not equidistant. The damped pairs from different triplets are then not at resonance with the foreign unstable modes and can interact only with one another. But since the short-duration bursts of the damped modes from the different triplets are not correlated, their interaction can be neglected in comparison with the interaction between the long-lived unstable high-frequency modes. This interaction will be due to the low-frequency acoustic perturbations, the frequencies and wave numbers of which are smaller than or of the order of the spectral width of the instability region.

Since in our case the width of the instability region is assumed to be much smaller than the frequency of the HF modes, the sought equations are obtained within the framework of the adiabatic approximation.[15] In this approximation, the high-frequency modes with random phases propagate in a medium with parameters that vary slowly as a result of the low-frequency sound. Consequently, for parametrically excited one-dimensional sound waves, we have a kinetic equation for the high-frequency mode and a hydrodynamic equation for the low-frequency sound:

$$\begin{gathered} \frac{\partial N_k}{\partial t} + v_k \frac{\partial N_k}{\partial x} - S \frac{\partial^2 \xi}{\partial x^2} \frac{\partial N_k}{\partial k} = \gamma_k N_k - \rho_k N_k^2, \\ \frac{\partial^2 \xi}{\partial t^2} - v_S^2 \frac{\partial^2 \xi}{\partial x^2} = -S \frac{\partial}{\partial x} \sum_k N_k + \nu_S \frac{\partial^3 \xi}{\partial t\, \partial x^2}. \end{gathered} \tag{5}$$

Here N_k is the intensity of the k-th HF mode, ξ is a variable characterizing the low-frequency acoustic perturbation, S is a constant that depends on the normalization, ν_s is the viscosity, γ_k is the linear increment of the HF modes, and $\rho_k N_k^2$ is a model term describing the stabilization of the HF mode on account of the decay into modes that are damped within the triplet, $\gamma_k/\rho_k = N_k^{(0)}$ being the average level of the high-frequency mode in the regime of stationary stabilization in the autonomous triplet. Replacement of the real collision integral by such model terms is possible if the time of establishment of the quasistationary state within the triplet is much shorter than the time of interaction between the triplets.

An "ideal" gas of non-interacting triplets corresponds to a state with $\xi = 0$, which exists only in the case when the energy of the high-frequency mode has a uniform distribution in space. Such a state, however, may turn out to be unstable to long-wave perturbations.

Consider, for example, the stationary regime $N_k(x, t) = N_k^{(0)}$ with a dependence $N_k^{(0)}(k)$, for which $N_k^{(0)}(k_2) - N_k^{(0)}(k_1) = \alpha$. Here $k_{1,2}$ are the limits of the instability region of the high-frequency modes in k-space. It is easily seen that at $v_k \sim v_S$ such a stationary spatially-homogeneous regime is unstable with respect to breakup into clusters with characteristic dimensions

$$\Lambda > 2\pi \left[\frac{v_k \nu (v_k v_S^2 - \alpha S^2)}{\gamma v_S^2 (v_S^2 + \nu\gamma)} \right]^{1/2}, \tag{6}$$

if $\alpha S^2 < v_k v_S^2$. If $\alpha S^2 > v_k v_S^2$, then the short-wave perturbations are also unstable.

CONCLUSION

We call attention once more to the fact that the onset of the stochastic behavior in a nonlinear dissipative medium is not necessarily connected with a large number of interactions. Disordered motions, which require statistical methods for their descriptions, can arise even in a system containing three or even two coupled nonconservative modes. The turbulence—the disordered motion of the medium with a large number of excited degrees of freedom—is in this case the result of an interaction of ready-made stochastic formations, for example triplets.

The problem considered above can simultaneously serve also as a finite-dimensional model of turbulence in a viscous liquid.[11)] The existence of stochasticity in this example confirms in essence the recently advanced ideas concerning the onset of turbulence,[10,11] according to which onset of a disordered behavior is possible as a result of loss of stability of only doubly-periodic motion.

We emphasize also that owing to the randomization of the phases of the modes inside the triplet, when describing an ensemble of triplets with the aid of the kinetic equation for such complicated quasiparticles, the description can be obtained without resorting to additional hypotheses such as the random-phase approximation.

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[1)] For waves with random phase, stabilization by decay is possible if $\nu_1+\nu_2>\tilde{\gamma}$ (we assume that $\sigma=1$). Stationary values of the mode intensities are then established, i.e., the energy flux over the spectrum is constant.

[2)] We note that at $\gamma=1$ Eq. (3) satisfies the Liouville theorem on the conservation of the phase volume $\operatorname{div}\dot{u}=0$ ($u=(X, Y, Z)$), i.e., in this case our nonequilibrium system is conservative.

[3)] At high initial energies, processes are also possible in which the point goes around the region of the slow motions and, disengaging from the circle, falls immediately at $X<-1$ into the region $Z>\gamma\,|X|$ with $X<0$ and small Y. At small γ, however, these motions should be unstable—the loss of the energy of the HF mode during the coalescence and decay is not offset by the enhancement of the HF mode.

[4)] The $\bar{Z}(Z)$ curve becomes deformed, drops or rises, depending on the proximity of the vertical strip that hugs the line to the plane $Y=0$, i.e., the form of this cruder mapping depends on the initial conditions on the plane $X=0$. This is the price we have to pay for making the problem cruder—a decrease in the dimensionality of the mapping.

[5)] Phase-space regions to which all neighboring trajectories are attracted but in which there are no stable motions—cycles or equilibrium state—are called "strange attractors".[10,11]

[6)] Preliminary results of this experiment are reported in[13].

[7)] The autocorrelation function

$$R_{A_3}(m\tau)=\frac{D^{-1}}{n-m}\sum_{i=1}^{n-m}\bar{A}_3(t_i)\bar{A}_3(t_{i+m})$$

was calculated for each realization at $n=2000$ (here $\tau=t_{i+1}-t_i$, $\bar{A}_3=A_3-M_{A3}$, M_{A_3} is the mathematical expectation value and D is the variance of the quantity A_3).

[8)] The process of concentration of the energy of the decaying modes in domains recalls the appearance of jets in k-space, for example, when waves are scattered by particles.[14] The difference between these processes lies particularly in the fact that the phases of the different modes in a jet are random, while in a domain the phase of the spectral components is constant and is independent of k.

[9)] This process is analogous to mode competition in a laser with homogeneously broadened line of the active medium.

[10)] Continuing the analogy with lasers, it can be noted that this case is similar to simultaneous generation of many modes in a medium with an inhomogeneously broadened active-medium line—the different modes emit different active frequencies.

[11)] The onset of disordered motions in finite-dimensional nonconservative systems was deduced numerically also in[16,17].

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