Supercritical current states in pure superconductors

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The supercritical current state in a "pure" superconducting channel is investigated. The existence of such a state is due to the difference between the responses to the nonequilibrium electric field of the superconducting condensate and of the excitation gas. The inhomogeneous longitudinal field distribution in the channel is formed by elastic scattering of the excitations. Equations are proposed which describe the resistive structure in a "pure" superconductor. The equations are used to determine the asymptotic behavior of the current-voltage characteristics.

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A current exceeding the critical value j_c flowing through a thin superconducting channel can produce in the channel a peculiar state characterized by simultaneous existence, on the one hand, of an energy gap in the excitation spectrum and a resultant superconducting current, and on the other hand a normal dissipative current producing a voltage drop in the channel. The microscopic theory of such a resistive state was developed in[1-3] for the case of dirty superconductors. The present communication is devoted to an analysis of the supercritical current state of a pure superconductor. Interest in this case is due in part to publication of reports of experiments on the measurement of the current-voltage characteristics of whiskers under supercriticality conditions (see, e.g., [4]). Principally, however, the investigation of a pure superconductor is of interest because of the more lucid manifestation (compared with the case of an alloy) of the mechanisms that cause the response of the superconductor to a nonequilibrium electric field. This lucidity is due to the possibility of representing a pure superconductor, over a sufficiently long time, in the form of a combination of two subsystems—a superconducting condensate and a gas of quasi-particle excitations. The difference in the responses of these subsystems to the longitudinal field plays the crucial role in the understanding of the nature of resistive states.[2] At the same time, the close formal analogy between the kinetic equation for the excitation in a superconductor and the kinetic equation of the normal metal makes it possible to compare in detail the character of the perturbation of their distributions by the electric field.

The process of separating the condensate and forming the spectrum of the excitations in the superconductor, described by the kinetic equation for the electron-hole density matrix $\hat{\gamma}$[5]:

$$i\dot{\hat{\gamma}}=[\hat{\varepsilon},\hat{\gamma}], \tag{1}$$

reduces to an asymptotic (over times that are large in comparison with $1/\Delta$) diagonalization of $\hat{\gamma}$ in the representation of the locally homogeneous matrix

$$\hat{\gamma}^{(0)}=\sum_\sigma \hat{E}_\sigma f_\sigma,\quad \hat{\varepsilon}=\sum_\sigma \hat{E}_\sigma \varepsilon_\sigma,\quad \sigma=\pm1,$$
$$\hat{\varepsilon}=\sigma_z(\xi+\Phi)+\mathbf{vp}_s+\sigma_x\Delta. \tag{2}$$

The approximation (2) corresponds to a transition to an "abbreviated" description of the superconductor in terms of a two-component excitation distribution function f_σ. Under the influence of the small perturbing terms, the distribution of the excitations changes slowly. The kinetic equation that describes this relaxation with allowance for the weak spatial dispersion and scattering by the impurities has been derived a number of times in preceding papers[5-8] and is of the form

$$\dot{f}_\sigma+\frac{\partial\varepsilon_\sigma}{\partial\mathbf{p}}\nabla f_\sigma-\nabla\varepsilon_\sigma\frac{\partial f_\sigma}{\partial\mathbf{p}}=I_\sigma(f), \tag{3}$$

where $I_\sigma(f)$ is the integral of the collisions with the impurities.[7] The applicability of (3) is restricted by the following conditions: the time of formation of the condensate must be short in comparison with the time of elastic relaxation: $\Delta\tau\gg1$, and the characteristic spatial inhomogeneities determined by the depth of penetration of the electric field must be large in comparison with the coherence lengths: $\delta_E\gg\xi(T)$. As shown in[2], the second condition is satisfied in the region of weak supercriticality, $j\gtrsim j_c$.

Near the critical temperature T_c, the only region where it is meaningful to study the supercritical regime, the current flowing through the channel is weak, and consequently the electric fields produced in it are weak. This raises the question of identifying the stage, (1) or (3), at which it is necessary to take into account the perturbation of the system by the nonequilibrium longitudinal field. An analysis of the process (1) and (2) shows that a longitudinal field should be taken into account already during the stage of condensate formation. In the opposite case, the off-diagonal (in the $\hat{\varepsilon}$ representation) corrections to the matrix $\hat{\gamma}^{(0)}$ (2) introduced into the macroscopic quantities (e.g., in the particle-number density) a contribution that is comparable with the contribution of $\hat{\gamma}^{(0)}$ itself. Physically this means that the condensate adjusts itself rapidly to the longitudinal field, and a resultant appearance of a potential Φ in the spectrum of the excitations ε_σ:

$$\varepsilon_\sigma=\sigma\tilde{\varepsilon}+\mathbf{vp}_s,\quad \tilde{\varepsilon}=\sqrt{(\xi+\Phi)^2+\Delta^2}. \tag{4}$$

In formula (4), Φ and $\mathbf{p}_s$ denote the gauge-invariant scalar potential $\Phi=e\varphi+\dot{\chi}/2$ and the superfluid momentum $\mathbf{p}_s = -e\mathbf{A} + \nabla\chi/2$, where curl $\mathbf{A} = 0$ and χ is the phase of the superconducting order parameter.

Thus, the response of the condensate to a longitudinal electric field of arbitrarily small amplitude is generally speaking nonlinear (in fact Φ is of the order of Δ).

In contrast to the condensate, the relaxation of the excitations is much slower and is described by Eq. (3). To analyze this equation it is convenient to change over (with quasi-classical accuracy $T_c \ll \varepsilon_F$) to new variables[7]: $\mathbf{p}$, $\sigma \to \varepsilon \equiv \varepsilon_\sigma$, $\mathbf{n} = \mathbf{p}/p$, $\alpha \equiv \mathrm{sign}(\xi + \Phi)$. In terms of these variables, taking into account the one-dimensional character of the problem, Eq. (3) becomes

$$\dot{f}_\alpha + \frac{\partial f_\alpha}{\partial \varepsilon}(\alpha v\dot{\Phi} + u\dot{\Delta} + \eta v_F \dot{p}_s) + \alpha v \eta v_F f_\alpha' = \frac{1}{\tau}\{\varphi_0 - u\varphi_1 + \alpha v\psi - f_\alpha(\langle 1\rangle - u\langle u\rangle)\}.$$

$$\eta = (\mathbf{n}, \mathbf{p}_s)/p_s, \quad u = \frac{\Delta}{\varepsilon - \eta v_F p_s}, \quad u^2 + v^2 = 1. \tag{5}$$

In the right-hand side of (5), the angle brackets denote averaging over the angles:

$$\langle \ldots \rangle = \int \frac{do}{4\pi}\theta((\varepsilon - \eta v_F p_s)^2 - \Delta^2)|v|^{-1} \ldots . \tag{6}$$

The functions φ_0, φ_1, and ψ, which enter in (5) are isotropic and are expressed in terms of the even part f_+

$$\varphi_0 = \langle f_+\rangle, \quad \varphi_1 = \langle u f_+\rangle, \quad f_+ = \frac{1}{2}\Sigma f_\alpha,$$

and the odd part f_-

$$\psi = \langle v f_-\rangle, \quad f_- = \frac{1}{2}\Sigma \alpha f_\alpha,$$

of the distribution function.

From the point of view of effects connected with the even part of the distribution function f_+, the behaviors of the excitations in a superconductor and in a normal metal are closest to each other. An electric field causes the distribution of the excitations to deviate from equilibrium, and this deviation can be calculated by perturbation theory. The principal term in the corresponding expansion of f_+ is the homogeneous isotropic function $\varphi_0^{(0)}$, which is determined by the inelastic-scattering processes unaccounted for in (3):

$$\varphi_0^{(0)} = (e^{-\varepsilon/T} + 1)^{-1}. \tag{7}$$

This function is not at equilibrium, since the field Φ enters in ε. Accurate to terms quadratic in Φ, the equilibrium distribution function φ_p is connected with $\varphi_0^{(0)}$ by the relation

$$\varphi_p = \varphi_0^{(0)} - \frac{\Phi^2}{2}\langle v^2\rangle \frac{\partial^2 \varphi_0^{(0)}}{\partial \varepsilon^2}. \tag{8}$$

The noted singularity ($\varphi_p \neq \varphi_0^{(0)}$) is not peculiar to superconductors. It is connected with the change of the systematics of the electron-hole states (shift by Φ) when the field is turned on, a change that can be realized as well also in a normal metal. However, whereas in the normal metal this change of the systematics is purely formal in character, meaning in fact only a redefinition of the chemical potential, in a superconductor it reflects a real physical process, the restructuring of the condensate, and manifests itself in a change of the value of the energy gap Δ. The self-consistency equation for Δ in the quasi-particle approximation (2) is[5]

$$1 = \frac{|g|}{2}\int \frac{d^3\mathbf{p}}{(2\pi)^3}\sum_\sigma \frac{\sigma}{\tilde{\varepsilon}} f_\sigma.$$

Substituting in this equation the equilibrium distribution function (8) (ideal heat removal) and expanding near T_c in terms of the small Δ, we obtain

$$\frac{T_c - T}{T_c} = \frac{7\zeta(3)}{8\pi^2 T_c^2}\left(\Delta^2 + 2\Phi^2 + \frac{2}{3}(p_s v_F)^2\right).$$

The difference between the function (8) and the exact solution of (3) for the isotropic part of f_+ is connected, just as in a normal metal, with the release of Joule heat. This effect is proportional to $E^2 \sim \Phi^2\delta_E^{-2}$ and near T_c, owing to a large depth of penetration δ_E of the electric field into the superconductor, makes a contribution of higher order than Φ^2 to Eq. (9). These considerations enable us to separate the problem of calculating the resistive structure from questions connected with allowance for thermal effects.

The behavior of the odd part of the distribution function f_- in a superconductor is more distinctive. The point is that Eq. (3), while close to the kinetic equation of a normal metal, differs from the latter in one essential aspect: the integral of the elastic collisions (5) is made to vanish in the general case not by an arbitrary isotropic function, but also by a function that is even in α. Its application to an odd (asymmetrical in ξ) isotropic distribution function in the form $f_\alpha = \alpha\psi$ yields

$$\langle v I_-(\alpha\psi)\rangle = -\frac{\lambda^2}{\tau}\psi, \quad \lambda^2 = \langle u^2\rangle\langle v\rangle - \langle u\rangle\langle uv\rangle. \tag{10}$$

The function λ differs from zero at $p_s \neq 0$ and takes near T_c the form

$$\lambda = \sqrt{\frac{1}{3}}\frac{\Delta p_s v_F}{\varepsilon^2}. \tag{11}$$

Relation (10) shows that when the condensate moves the process of elastic scattering in the superconductor, besides making the distribution isotropic, equalizes also the asymmetry between its electronic and hole branches. This "branch mixing" (in accordance with the terminology of Tinkham and Clarke[9] is characterized by a time

$$\tau_n = \frac{\tau}{\lambda^2} \sim \tau\left(\frac{T_c^2}{\Delta p_s v_F}\right)^2. \tag{12}$$

It is easily seen that the function ψ is responsible for the formation of the particle-number density fluctuation in the system:

$$\delta N = \delta\int \frac{d^3\mathbf{p}}{(2\pi)^3}\left(1 - \sum_\sigma \frac{\xi + \Phi}{\sigma\tilde{\varepsilon}} f_\sigma\right) = -\delta\int \frac{d^3\mathbf{p}}{(2\pi)^3}\sum_\sigma \frac{\xi + \Phi}{\sigma\tilde{\varepsilon}}\varphi_0^{(0)} - \frac{m p_F}{\pi^2}\int d\varepsilon\, \psi. \tag{13}$$

It follows therefore that relation (10) should introduce a contribution, which is peculiar to superconductors, to the detailed-balance equation of the charge; this contribution characterizes the accumulation of a self-consistent charge density in the channel. As a result, the distribution of the electric field becomes inhomogeneous. As shown by further calculations, the characteristic length of this inhomogeneity, which has the meaning of the depth of penetration of the electric field into the superconductor, is given by the estimate

$$\delta_E \sim l/\lambda \gg l. \tag{14}$$

At $l \sim 10^{-3}$ cm and $T \sim 0.9T_c$, the length δ_E is of the order of $\delta_E \sim 10^{-2}$ cm.

The connection between the function ψ and the density fluctuations δN (13) makes it possible to examine the nature of the onset of the charge distribution in the superconductor during the branch mixing from a different point of view—as the manifestation of the fact that the elastic-collision integral has the property of not conserving the total number of particles. We recall that the kinetic equation (3) does not include the continuity equation[5–7]:

$$e\dot{N} + \operatorname{div} \mathbf{j} = 0. \tag{15}$$

The particle source that appears in this approximation in the right-hand side of (15) is made up, besides elastic collisions, also of convective terms and inelastic-scattering processes, i.e., it consists of those parts of the kinetic equation which violate the symmetry between the electron and hole states (in the case of elastic scattering, this violation occurs precisely at $p_s \neq 0$). Conservation of the particle number (15) is ensured by the self-consistency equation for the phase χ, which in the approximation (2) degenerates into a trivial identity and becomes meaningful only when account is taken of the corrections to the matrix $\hat{\gamma}^{(0)}$ (2) which are not diagonal in the $\hat{\varepsilon}$ representation.

Thus, in the principal approximation (2), the conservation law (15) is an independent equation, which jointly with the Poisson equation $\nabla E = 4\pi e \delta N$ determines the distribution of the fields Φ and p_s. In a metal, owing to the high density of the electronic states, these equations can be replaced by electroneutrality equation $\delta N = 0$ and by the condition that total current be constant:

$$j = eNv_s + ev_F \int \frac{d^3\mathbf{p}}{(2\pi)^3} \eta \left(1 - \sum f_0\right) = eNv_s - \frac{ep_F^2}{\pi^2} \int d\varepsilon \langle \eta f_+ \rangle = \text{const}. \tag{16}$$

Using formula (13) in the linear approximation in Φ, the electroneutrality condition takes the form

$$\Phi + \int d\varepsilon\, \psi = 0. \tag{17}$$

Before we proceed to find the functions ψ and f_+ from (5), we note that the inelastic scattering processes, as is clear from the foregoing, also give rise to a non-uniform distribution of the field.[9] However, owing to the low frequencies of the energy relaxation, these processes are not very effective in comparison with the elastic processes, if the latter exist. In the problem of supercritical current states, where p_s is of necessity different from zero, the elastic mechanisms assume the leading role. An example of an independent manifestation of "inelastic mixing" may be the singularity observed in[6] of the second-viscosity coefficient of a superfluid Fermi liquid near T_c.

The kinetic equation (5) was formulated in a patently gauge-invariant form with the aid of the functions Φ and p_s. The appearance in (5) of these gauge-invariant functions, which is connected with the procedure of expansion in the gradient in the course of its derivation, introduces singularities into the equation, inasmuch as Φ and p_s are singular. Indeed, in accordance with the main idea of[1,2], the phase of the parameter of the superconducting ordering in the resistive state contains discontinuities. In an infinite channel it is given by

$$\chi(x,t) = \chi(x) + 2\Phi_0 t \left[\frac{x}{d} + \frac{1}{2}\right], \tag{18}$$

where $\chi(x) = \chi(x+d)$ is a continuous periodic function, while the square brackets denote the integer part.

The singularities connected with these discontinuities, although playing a role in the intermediate stages of the calculations, do not manifest themselves in the formulas for the observed quantities. In the normal terms (which do not vanish in the limit $\Delta = 0$), the corresponding terms combine into an electric field intensity:

$$e\mathbf{E} = \dot{\mathbf{p}}_s - \nabla\Phi. \tag{19}$$

In the superconducting terms, the singularities are suppressed by the factor Δ, which vanishes at the point where the potential is discontinuous. This behavior of Δ follows from an analysis of the resistive structure in small [$\sim \xi(T)$] vicinities of the singular points, which is presented in[3] (where the non-stationary Josephson effects connected with the time dependence of the phase (18) is also discussed). The condition $\Delta = 0$ can be regarded as an effective boundary condition for the large-scale equation (3). The magnitude of the jump of the potential Φ_0 is given by the total current j, and the period of the structure d can be determined from the additional condition that the entropy production be minimal.

To solve Eq. (5) we use the smallness of the ratio l/δ_E (14). Confining ourselves to the case of a static structure, we omit from (5) all the derivatives with respect to time, with the exception of the term proportional to $\dot{p}_s$, which is not equal to zero by virtue of (18). We then solve the equation for f_α, by inverting the operator

$$\frac{d}{dx} + (\langle 1 \rangle - u\langle u \rangle)/\alpha\eta v l.$$

In the obtained integral equation, we expand the slowly-varying integrands up to terms $\sim (l/\delta_E)^2$. The subsequent averaging of (8) leads to a system of differential equations for the determination of the functions φ_0, φ_1, and ψ (the diffusion approximation). In the approximation that is linear in Φ and is of lowest order in Δ, this system reduces to a single equation

$$\psi'' - \frac{\lambda^2}{\langle\eta^2\rangle l^2}\psi = \frac{\partial\varphi_0}{\partial\varepsilon}\dot{p}_s'. \tag{20}$$

In the same approximation, the connection between the anisotropic even part of the distribution f_+, which causes the appearance of the dissipative current, and the function ψ is given by

$$f_+ = \eta l\left(\psi' - \frac{\partial\varphi_0^{(0)}}{\partial\varepsilon}\dot{p}_s\right). \tag{21}$$

Substituting (17), (19), and (21) in formula (16) for the current and taking into account the smallness of Δ, we obtain

$$j = \frac{7\zeta(3)}{4\pi^2}\frac{eN}{m}\frac{\Delta^2 p_s}{T_c^2} + \sigma E = \text{const}, \qquad \sigma = \frac{ep_F^2 l}{3\pi^2}. \tag{22}$$

From (22), with allowance for the order of the gradients determined by Eq. (20), we obtain for the potential the estimate $\Phi \sim \Delta$.

The derivation of Eqs. (20)–(22) is essentially based on an expansion in terms of Δ, the correctness of which in the nonequilibrium terms of these equations calls for explanation (in fact, the parameter is Δ/ε). The possibility of this expansion is ensured by the behavior of the functions ψ at low energies, namely $\psi(\varepsilon) \sim \exp(-1/\varepsilon^2)$, which cuts off all the power-law divergences connected with the singularity in the density of states. As a result, the characteristic energies assume an "equilibrium" order $\varepsilon \sim T_c$.

Equations (9), (17), (20), and (22) constitute a complete system, that describes the resistive structure in a pure superconductor. These equations are analogous to the corresponding equations for an alloy[2] and differ from the latter essentially only in the character of the energy dependence of the coefficient in the equation (20) for ψ.

In the dimensionless variables

$$\Delta \to \Delta_0\Delta, \quad \Phi \to (\Delta_0/\sqrt{2})\Phi, \quad p_s v_F \to \Delta_0\sqrt{3/2}\,p_s, \quad \varepsilon \to 2T_c\varepsilon,$$

$$j \to \frac{7\sqrt{6}\,\zeta(3)}{8\pi^2}\frac{\Delta_0^3 eN}{T_c^2 p_F}j, \quad x \to \frac{4\pi^2}{7\sqrt{3}\,\zeta(3)}\frac{T_c^2 l}{\Delta_0^2}x,$$

$$\psi \to \frac{\Delta_0}{\sqrt{2}}\frac{\partial\varphi_0^{(0)}}{\partial\varepsilon}\psi, \quad \Delta_0^2 = 8\pi^2 T_c(T_c - T)/7\zeta(3)$$

Equations (9) and (22) take the form

$$\Delta^2 + p_s^2 + \Phi^2 = 1, \quad j = \Delta^2 p_s - d\Phi/dx.$$

In the equation for ψ, it is convenient to transfer the singular inhomogeneous term to the boundary condition:

$$\frac{\partial^2\psi}{\partial x^2} - \left(\frac{1}{7\sqrt{2}\,\zeta(3)}\frac{\Delta p_s}{\varepsilon^2}\right)^2\psi = 0, \quad \psi\left(\pm\frac{d}{2}\right) = \mp\frac{\Phi_0}{2}. \tag{23}$$

Equation (7), which makes the system closed, takes in terms of the dimensionless variables the form

$$\Phi = -\int d\varepsilon\,\psi\,\mathrm{ch}^{-2}\,\varepsilon. \tag{24}$$

The solution of the obtained system of equations is a very difficult task, principally because of the non-local connection between ψ and Φ (or, equivalently, between Φ and E). The dependence of the coefficient of Eq. (23) leads to a "floating" length scale in this equation, which hinders an effective construction of asymptotic solutions. Nonetheless, qualitative information on the behavior of the current-voltage characteristics in the limiting cases of small ($j - j_c \ll j_c$) and large ($j \gg j_c$) currents (but still far from the current j_{c_2}[1,2]) can be obtained from these equations.

At large currents $j \gg j_c$, as shown in[2], the period of the resistive structure decreases. A simple analysis shows that the characteristic energies that determine the deviation of the potential from the linear spatial distribution $\Phi = -\Phi_0 x/d + \varphi(x)$ also decrease in this case. This makes it possible to omit from the integral with respect to the energy, which determines φ, the factor $\cosh^{-2}\varepsilon$. An investigation of the group properties that the solutions acquire in this case establishes the following asymptotic relations: $d \sim j^{-2}$, $\Phi_0 \sim j^{-1}$; Δ, $p_s \sim 1$.

From the obtained formulas follows a connection between the current and the average field intensity $\bar{E} = \Phi_0/d$ in the form

$$j = \sigma\bar{E}\left(1 + \text{const}\cdot\frac{j_c}{\sigma\bar{E}}\right).$$

In the immediate vicinity of j_c, when $j - j_c \ll j_c$, the period of the structure increases[2] and becomes much less than the depth of penetration of the electric field. The functions Δ and p_s are significantly altered near the singular centers (over distances on the order of δ_E) and assume constant values in the interior of the period. The distribution of the potential in this internal region of the period can be obtained in elementary fashion and it is easy to calculate from this distribution the (minimum) value of the electric field intensity at the midpoint of the period, $E_m(d)$. The use of the minimum-entropy production principle reduces to a physically obvious conclusion: at the point where the field intensity is minimal, the value of the superconducting current should be maximal and equal to j_c. The foregoing considerations determine the current-voltage characteristic in the form

$$\frac{j - j_c}{j_c} \sim \exp\left[-\text{const}\cdot\left(\frac{j_c}{\sigma\bar{E}}\right)^{1/2}\right].$$

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