$\alpha(q^2) \to \alpha_N(q^2)$; $K^{(1)}_{l,l+1}$ is as before given by Eq. (58), and in $K^{(0)}_{l,l+1}$ one must set $A_0(q^2) = 2q^2 - 2(N^2+1)N^{-2}m^2$.

The reggeization of the vector meson is proved in the same manner as for $SU(2)$, since Eqs. (63)–(66) remain valid with the indicated substitutions.

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Translated by M. E. Mayer

Properties of deep-lying levels in a strong electrostatic field

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(Submitted February 6, 1976)
Zh. Eksp. Teor. Fiz. 71, 856–870 (September 1976)

The properties of deep levels (real, virtual, and quasistationary) lying near the boundary of the lower continuum are investigated. The effective range expansion is generalized to the case of the Dirac equation. With its aid the motion of the levels near the boundary $\epsilon = -mc^2$ is investigated for different values of the angular momenta j and l. The formulas become simpler in the limiting cases $R \ll \hbar/mc$ and $R \gg \hbar/mc$, where R denotes the range of the forces. In particular, the case of a wide potential well $R \gg \hbar/mc$ reduces to a determination of the spectrum of the bound and quasistationary states in the Schrödinger equation with a power-law potential. The asymptotic behavior of the critical nuclear charge Z_{cr} is found in the region $R_N \gg \hbar/mc$ (R_N denotes the nuclear radius), and Z_{cr} is calculated for the muon for various distributions of electric charge inside the nucleus. Differences in the behavior of the levels near $\epsilon = -mc^2$ for scalar and spinor particles and the inapplicability of the single-particle Klein-Gordon equation for $Z \geq Z_{cr}$ are discussed.

PACS numbers: 11.10.Qr

1. INTRODUCTION

Deep levels, lying near the boundary of the lower continuum of solutions to the Dirac equation, are investigated in the present article. The appearance of levels near the boundary $\varepsilon = -mc^2$ is of interest for quantum electrodynamics (the critical nuclear charge Z_{cr} and the spontaneous production of positrons from the vacuum for $Z > Z_{cr}$ or associated with the approach of two heavy nuclei to within a distance $R < R_{cr}$ [1–8]) and for nuclear physics (the formation of a π-meson condensate and a phase transition in nuclei, the possible existence of superheavy nuclei with charge $Z \sim (137)^{3/2}$ [9,10]).

The motion of the levels near the boundary of the lower continuum has been considered by a number of authors[2,11–15] in connection with problems of quantum electrodynamics in strong external fields. In this connection the Dirac and Klein–Gordon equations were solved analytically or numerically for potentials of the following special forms: rectangular well, Coulomb potential, and Hulthén potential. We shall consider the question of the motion of the levels and the analytic properties of the S-matrix for an arbitrary potential, using a generalization of the effective range method to the relativistic case. In this connection one is able to express the energy levels and the expansion parameters of the S-matrix for $\varepsilon \to -mc^2$ in terms of the wave function at the critical point $V = V_{cr}$, i.e., at the moment when the level intersects the boundary of the lower continuum.

Let us describe the contents of this article. A generalization of the effective range expansion to the case of the Dirac equation is presented in Sec. 2. The limiting cases $R \ll \lambda_c = \hbar/mc$ and $R \gg \lambda_c$ (R denotes the characteristic range of action of the potential and m is the particle's mass) are considered in Secs. 3 and 4. By using the effective potential method,[2,3] it is shown in Sec. 4 that investigation of the levels near $\varepsilon = -mc^2$ in the case $R \gg \lambda_c$ reduces to the solution of the Schrödinger equation. In Sec. 5 the asymptotic form of Z_{cr} is found in the region $R_N \gg \lambda_c$ which has not been previously investigated (R_N is the nuclear radius), and the critical charge of the nucleus for a muon is evaluated. In Sec. 6 a comparison is made of the behavior of the levels and the analytic properties of the S-matrix near $\varepsilon = -mc^2$ for scalar and spinor particles.

Below we assume $\hbar = m = c = 1$, the energy ε is measured in units of mc^2, $\lambda = (1-\varepsilon^2)^{1/2}$, and we write the interaction potential (the time component of the 4-vector $A_\mu(x)$) in the form

$$V(r) = -Vf(r). \tag{1.1}$$

We shall assume the function $f(r)$, determining the shape of the potential, to be fixed, but the depth V can vary. The Dirac equation is given by

$$\frac{dG}{dr} = -\frac{\varkappa}{r}G + [1+\varepsilon+Vf(r)]F, \quad \frac{dF}{dr} = [1-\varepsilon-Vf(r)]G + \frac{\varkappa}{r}F, \tag{1.2}$$

where $\varkappa = \mp(j+\frac{1}{2})$ for states with $j = l_1 \pm \frac{1}{2}$; l_1 and l_2 denote the orbital angular momenta corresponding to the upper (G) and lower (F) components of the Dirac bispinor. In contrast to the nonrelativistic situation, for $\varepsilon \approx -1$ the angular momentum l_2 is more important than l_1; for the sake of brevity we denote $l_2 \equiv l$. For example, we have $\varkappa = -1$, $l_1 = 0$, and $l = 1$ for the ground level $1s_{1/2}$.

2. THE EFFECTIVE-RANGE METHOD FOR THE DIRAC EQUATION

First let us assume $\varkappa \neq 1$ (the states $np_{1/2}$ are excluded by this). In this connection the angular momentum $l \geq 1$. For $V = V_c$ let such a level drop down to the boundary of the lower continuum. For values of V close to V_{cr}, it is convenient to represent the dependence of the energy level ε on V in the form of the equation

$$V_{cr} - V = \varphi(\lambda), \quad \lambda = (1-\varepsilon^2)^{1/2}. \tag{2.1}$$

Let $\varphi_\pm(\lambda)$ denote the even and odd parts of the function $\varphi(\lambda)$: $\varphi(\lambda) = \varphi_+(\lambda) + \varphi_-(\lambda)$, $\varphi_\pm(-\lambda) = \pm\varphi_\pm(\lambda)$. For small values of $V - V_{cr}$, only the first terms of the expansion of $\varphi_\pm(\lambda)$ are essential for $\lambda \to 0$:

$$\varphi_+(\lambda) = c_2\lambda^2 + \ldots, \quad \varphi_-(\lambda) = c_{2l+1}\lambda^{2l+1} + \ldots. \tag{2.2}$$

One can express the coefficients c_2 and c_{2l+1} in terms of the wave function of the level at the critical point

$$c_2 = \left[2\int_0^\infty f(r)\chi_l^2(r)\,dr\right]^{-1}, \quad c_{2l+1} = (-1)^l[A_l/(2l-1)!!]^2 c_2, \tag{2.3}$$

$$\chi_l^2(r) = [G^2(r) + F^2(r)]_{\varepsilon=-1,\,V=V_{cr}}, \quad \int_0^\infty \chi_l^2(r)\,dr = 1, \tag{2.4}$$

where A_l is the coefficient in the asymptotic form:

$$\chi_l(r) \approx A_l r^{-l}, \quad r \to \infty. \tag{2.5}$$

The derivation of formulas (2.3)–(2.8), (2.13) and (2.14), and also of the formulas given in Sec. 6 for scalar particles is not discussed here, but has been published separately.[16,17] It is assumed that the interaction potential is regular at the origin and has a short range of interaction: $V(r) \to 0$ faster than any power r^{-n} as $r \to \infty$. If $V(r)$ behaves like r^{-n} at infinity, the effective range expansion has the form (2.6) only for $n > 2l + 5$.[18,19] We note that weaker restrictions on the rate of decrease of the potential[1)] are required for the validity of the asymptotic expressions (2.5) and (2.12) and also for the formulas which are derived below for the coefficients in the series (2.2).

Since $\varkappa \neq 1$, it follows that $l \geq 1$, and owing to the centrifugal barrier the bound state is not delocalized as $\varepsilon \to -1$. Let us emphasize that the Dirac equation has a solution with energy $\varepsilon = -1$, possessing the finite norm (2.4), only for discrete values of the well depth $V = V_{cr}(n\varkappa)$ at which one of the levels of the discrete spectrum drops down to the boundary of the lower continuum. For $k \to 0$ and values of V close to V_{cr}, the S-matrix has the form

$$S_l(k) = e^{2i\delta_l} = (\operatorname{ctg}\delta_l + i)/(\operatorname{ctg}\delta_l - i), \quad k^{2l+1}\operatorname{ctg}\delta_l(k) = -\frac{1}{a_l} + \frac{1}{2}r_l k^2 + \ldots \tag{2.6}$$

(a_l is the scattering length, r_l is the effective range, and $k = (\varepsilon^2-1)^{1/2}$). The scattering amplitude is given by

$$f_l(k) = \frac{S_l(k)-1}{2ik} = k^{2l}\left(-\frac{1}{a_l} + \frac{1}{2}r_l k^2 + \ldots - ik^{2l+1}\right)^{-1},$$

where

$$\frac{1}{a_l} = \alpha_1(V-V_{cr}) + \alpha_2(V-V_{cr})^2 + \ldots, \quad \alpha_1 = -\frac{2}{A_l^2}\int_0^\infty f(r)\chi_l^2(r)\,dr, \quad r_l = -2[(2l-1)!!/A_l]^2. \tag{2.7}$$

A connection exists between the parameters c_2, c_{2l+1} and a_l, r_l

$$c_2 = r_l/2\alpha_1, \quad c_{2l+1} = (-1)^{l+1}/\alpha_1. \tag{2.8}$$

The existence of such a connection is not surprising since a bound state corresponds to a pole of the S-matrix and is determined by the equation $\cot\delta_l(k) = i$ for $k = i\lambda$. As is evident from Eq. (2.3), $c_2 > 0$ for any attractive potential. Therefore, the discrete level enters the lower continuum with a finite slope

$$\varepsilon = -1 + \frac{1}{2c_2}(V_{cr} - V) + \ldots, \quad V \to V_{cr} - 0. \tag{2.9}$$

A collision of real and virtual levels takes place at $V = V_{cr}$, and then these levels emerge into the complex plane, being converted into a pair of Breit–Wigner poles: $k = i\lambda = k' + ik''$,

$$k' = \pm[(V-V_{cr})/c_2]^{1/2}, \quad k'' = (k')^{2l}/r_l. \tag{2.10}$$

These poles are always located on the second sheet (Fig. 1) since $k'' < 0$ for $r_l < 0$. The other sign for the effective range would lead to a contradiction with unitarity. The essentialness of the condition $r_l < 0$, which is always

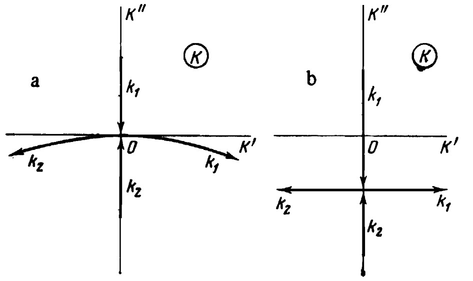

FIG. 1. Distribution of the poles of the S-matrix in the k-plane for the Dirac equation with $\varepsilon \to -1$ and $k = (\varepsilon^2 - 1)^{1/2} \to 0$; a) for $\varkappa \neq 1$, b) $\varkappa = 1$.

satisfied[2)] in virtue of (2.7), is evident from here.

The energy of the quasistationary level, which is submerged into the lower continuum for $V > V_{cr}$, is given by

$$\varepsilon = -(1+k^2)^{1/2} = \varepsilon_0 + i\gamma/2.$$

At threshold $|k''| \ll k'$, and for ε_0 we have the same formula (2.9) as in the subcritical region. The threshold behavior of the width γ has the form $(kR \ll 1)$

$$\gamma = \gamma_l (V - V_{cr})^{l+1/2},$$
$$\gamma_l = -2(r_l c_2^{l+1/2})^{-1} = (-1)^l c_{2l+1} c_2^{-(l+3/2)}. \quad (2.11)$$

For $V > V_{cr}$ the probability for the spontaneous production of positrons is equal to γ. It decreases with increasing angular momentum l, which is explained by the presence of the centrifugal barrier whose penetration factor depends on the positron's momentum $\sim k^{2l+1}$ as $k \to 0$.

Let us turn our attention to the unusual sign of the imaginary part: $\operatorname{Im} \varepsilon = \gamma/2 > 0$. Such a sign is also obtained upon solving the equation for ε in a Coulomb potential[2] and is explained by the fact that we are working in a theory which is not second-quantized, examining the unoccupied states (holes) in the lower continuum. Upon a transition to quasistationary positron states, the energy ε changes sign, and $\operatorname{Im} \varepsilon_p = -\gamma/2 < 0$.

The case $\varkappa = 1$ is a special case since $l = 0$ and the bound state is delocalized for $\varepsilon \to -1$. At the critical point $V = V_{cr}$ the wave function has the asymptotic form

$$G(r) = -1/2r, \quad F(r) = 1, \quad r \to \infty. \quad (2.12)$$

The power series for $\varphi(\lambda)$ starts with a term $\sim \lambda$:

$$c_1 = -1/\alpha_1, \quad c_2 = -\frac{1}{2} r_0 c_1 + \alpha_2 c_1^3, \ldots$$
$$\alpha_1 = -2\int_0^\infty f\chi_0^2\, dr, \quad r_0 = 2\int_0^\infty (1-\chi_0^2)\, dr, \quad (2.13)$$

$$\alpha_2 = -2\int_0^\infty f(r)\,(G_0 G_1 + F_0 F_1)\, dr. \quad (2.14)$$

The functions G_0 and F_0 pertain to $V = V_{cr}$ and energy $\varepsilon = -1$; G_1 and F_1 are corrections to them which are linear in $V - V_{cr}$. The equations for G_1 and F_1 are obtained[17] if the system (1.2) is expanded in powers of $V - V_{cr}$ and the terms $\sim (V - V_{cr})^2$ are discarded.

Comparison with Eqs. (2.3) shows that the case $\varkappa = 1$ is more complicated for calculations: In order to determine the first two terms of the expansion $\varphi(\lambda)$ it is not only necessary to find the wave function of the level at the edge of the lower continuum, but also the correction to it of order $V - V_{cr}$. The analytic properties of the poles of the S-matrix are also more complicated. The curve of the $p_{1/2}$-level is in contact with the boundary of the continuum

$$\varepsilon = -1 + \frac{1}{2c_1^2}(V - V_{cr})^2 + \ldots, \quad V \to V_{cr} - 0. \quad (2.15)$$

Solving Eq. (2.1) with respect to λ, we obtain the two roots

$$\lambda_{1,2} = -\mu \pm (\mu^2 - k_0^2)^{1/2}, \quad (2.16)$$

where $\mu = c_1/2c_2$ and $k_0^2 = (V - V_{cr})/c_2$. The location of the poles $k_1 = i\lambda_1$ and $k_2 = i\lambda_2$ in the k-plane depends on the signs of the coefficients c_1 and c_2. It follows from Eqs. (2.13) and (2.14) that $c_1 > 0$ for any arbitrary attractive potential; as to c_2, it is not possible to establish its sign in the general case. However, $c_2 > 0$ in all of the examples we have considered, and also in the limiting cases $R \ll 1$ and $R \gg 1$ (see below). Therefore, we shall assume that c_1 and c_2 are positive. Then, for $V < V_{cr}$ the first level is real and the second is virtual: $\lambda_1 > 0$ and $\lambda_2 < 0$. For $V > V_{cr}$ the real level turns into a virtual level, moving away onto the second sheet

$$k = -\frac{i}{c_1}(V - V_{cr}) + \ldots .$$

A collision of the two virtual levels takes place for $k_0^2 = \mu^2$ (or $V - V_{cr} = c_1^2/4c_2$), after which they move away along the imaginary axis of the k-plane (Fig. 1b). The picture of the motion of the poles of the S-matrix for $\varkappa = 1$ is analogous to that obtained in the article by Migdal *et al.*[20] for the S-levels in the nonrelativistic Schrödinger equation with a wide barrier.

Further simplification of the formulas is possible in the limiting cases of narrow and wide (in comparison with $\hbar/mc$) attractive potentials. For this purpose we introduce the range of action R of the forces, assuming $f = f(r/R)$ in Eq. (1.1); R is measured in units of $\lambda\!\!\!^{-}_c = \hbar/mc$.

3. THE CASE $R \ll 1$

Changing to the variable $x = r/R$, assuming $V = g^2/R$, and neglecting terms of order R in Eq. (1.2), we obtain for G and F a system of equations which do not depend on the energy ε. For $\varkappa \neq 1$ the function $\chi_l(x) \equiv (G^2 + F^2)^{1/2}$ is normalized by the condition

$$\int_0^\infty \chi_l^2(x)\, dx = 1,$$

where $\chi_l(x) = B_l x^{-l}$ as $x \to \infty$. The dependence of the quantities of interest to us on the radius R may be introduced in explicit form

$$c_2 = \left\{2\int_0^\infty f(x)\chi_l^2(x)\, dx\right\}^{-1}, \quad c_{2l+1} = (-1)^l [B_l/(2l-1)!!]^2 c_2 R^{2l-1}, \quad (3.1)$$

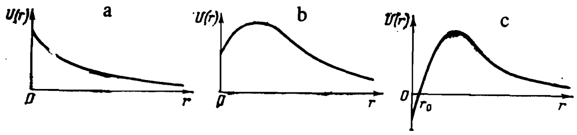

FIG. 2. The effective potential for the Dirac equation: a) $\varepsilon > 0$; b) $0 > \varepsilon > -1$; c) $\varepsilon > -1$.

$$\gamma = \gamma_l R^{2l-1}(V - V_{cr})^{l+1/2} \quad \text{as} \quad V \to V_{cr}. \tag{3.2}$$

The quantities c_2, γ, and B_l are constants of the order of unity, which depend on the shape of the potential, but do not depend on R.

As is clear from Eq. (3.2), for states with large angular momenta l the probability for spontaneous production of a positron in the threshold region $V - V_{cr} \ll V_{cr}$ falls off rapidly with increasing l. We note that the case of the ground state ($\varkappa = -1$, $l_1 = 0$, $l = 1$) in a narrow well was investigated earlier.[2]

For levels having the quantum number $\varkappa = 1$, $\chi_0(r)$ is not a function of only the single variable $x = r/R$. From Eqs. (2.13) and (2.14) we have $c_1 = 0(R)$ and $c_2 = 0(1)$. Therefore, in the case $R \ll 1$ a collision of the virtual levels takes place near the boundary of the lower continuum

$$V - V_{cr} = c_1^2/4c_2 = O(R^2).$$

4. THE CASE $R \gg 1$

The case of a wide potential well is more difficult in connection with both the analytic solution and the numerical solution of the problem.[3)] First, let us assume monotonicity of the potential: $f'(r) < 0$ for $0 < r < \infty$ (below we shall show how this restriction can be avoided).

The effective potential[2,3] is given by

$$U(r) = U_0 + U_s,$$

where U_0 is the potential for a scalar particle and U_s is the spin correction:

$$U_0(r) = \varepsilon V(r) - \frac{1}{2}V^2(r) + \frac{\varkappa(\varkappa+1)}{2r^2},$$
$$U_s(r) = \frac{1}{4}\left[\frac{V''}{W} + \frac{3}{2}\left(\frac{V'}{W}\right)^2 - \frac{2\varkappa V'}{rW}\right] \tag{4.1}$$
$$W(r) = 1 + \varepsilon - V(r).$$

With increasing depth of the well V, the effective potential varies as shown in Fig. 2. When V becomes greater than $2mc^2$, a region of effective attraction appears near $r = 0$.

For $x = r/R \to 0$, let

$$f(x) = 1 - \frac{1}{2}ax^\alpha + \ldots, \quad a, \alpha > 0. \tag{4.2}$$

Let us set $V = 2 + vR^{-\nu} + \ldots$, where $v = 0(1)$, and let us assume that the spin term U_s can be neglected in comparison with U_0 (this is justified by the result). Then for $\varepsilon = -1$, $\varkappa = -1$ we have

$$U_0(r) = -V - {}^1/_2 V^2 = -vR^{-\nu} + ax^\alpha.$$

The turning point is

$$x_0 = r_0/R = (v/aR^\nu)^{1/\alpha} \ll 1.$$

The existence of a bound state at the edge of the lower continuum is equivalent to the appearance of a level with zero binding energy in the effective potential. The condition for the appearance of a level[21,22]

$$\xi \geq 1, \quad \xi = \frac{2m}{\hbar^2}\int_0^{r_0} |U(r)|r\,dr,$$

determines the exponent ν:

$$\xi \sim |U(0)|r_0^2 \sim R^{2(1-\nu/\alpha)-\nu}.$$

In order for ξ to remain bounded ($\xi = 0(1)$ as $R \to \infty$), it is necessary that

$$\nu = 2\alpha/(\alpha+2). \tag{4.3}$$

Knowing ν, let us analyze the problem more rigorously and determine the behavior of the level near $\varepsilon = -1$. Assuming

$$r = \rho R^{\nu/2}, \quad V = 2 + vR^{-\nu}, \quad \varepsilon = -1 + \bar{\varepsilon}R^{-\nu}, \tag{4.4}$$

we have

$$U_0 = -\frac{2\bar{\varepsilon}+v}{R^\nu} + \frac{\varkappa(\varkappa+1)}{2r^2} + a\left(\frac{r}{R}\right)^\alpha + \ldots$$
$$U_s = \frac{a\alpha}{8R^2}(\alpha + 2\varkappa - 1)x^{\alpha-1}[1 + O(x^\alpha)].$$

Hence

$$U_s(r)/U_0(r) \sim R^{-2(\alpha+1)/(\alpha+2)} \ll 1$$

for $r \ll R$; in this region one can neglect the spin term U_s in comparison with U_0. The squared Dirac equation assumes the form

$$\frac{d^2\chi}{d\rho^2} + 2\left[\mu - a\rho^\alpha - \frac{\varkappa(\varkappa+1)}{2\rho^2}\right]\chi = 0, \quad 0 < \rho < \infty, \tag{4.5}$$

where $\mu = \bar{\varepsilon} + v$. This equation has a discrete spectrum

$$\mu = \mu_{n\varkappa}, \quad n = 1, 2, 3 \ldots.$$

Finally we obtain

$$V_{cr} = 2 + \mu_{n\varkappa}R^{-\nu}, \quad \varepsilon = -1 + (\mu_{n\varkappa} - v)R^{-\nu} = V_{cr} - 1 - V, \tag{4.6}$$

i.e., a linear dependence of the energy level ε on the depth V of the well.

An imaginary part appears in ε for $V > V_{cr}$; the threshold behavior of this part is determined by formula (2.11). The constant γ_l is related to the barrier penetration and is exponentially small for $R \gg 1$

$$\gamma_l \propto \exp\left\{-2\int [2U_0(r)]^{1/2}\,dr\right\} = e^{-\beta R},$$
$$\beta = 4\int_0^\infty dx[f(x)(1-f(x))]^{1/2}. \tag{4.7}$$

(We recall that $f(x) \leq 1$, see Eq. (4.2).) The influence of a finite penetration barrier on $\varepsilon_0 = \mathrm{Re}\,\varepsilon$ can be neglected since $\exp(-\beta R) \ll R^{-\nu}$ for $R \gg 1$.

The wave function is mainly concentrated in the classically allowed region $r \lesssim r_0 \sim R^{\nu/2}$, which is small in comparison with the range R of the forces (since $\nu/2 < 1$ for $\alpha > 0$). Because of this, in order to determine the asymptotic ($R \to \infty$) spectrum of the levels it was found to be sufficient to give the first terms of the expansion of the interaction potential $V(r)$ for small r.

If $\alpha = 2$, Eq. (4.5) coincides with the Schrödinger equation for a three-dimensional harmonic oscillator, from which

$$\mu_{n\varkappa} = [2n + j + {}^1/_2(\operatorname{sign}\varkappa - 1)](2a)^{1/2}, \quad n = 1, 2, 3, \ldots . \qquad (4.8)$$

The other example is a rectangular well, $V(r) = -V\theta(R - r)$. Expressions for V_{cr} for arbitrary values of n, $\varkappa$, and R are cited in[14]; for $R \gg 1$ one finds

$$V_{cr}(n, \varkappa) = 2 + {}^1/_2 \xi_{n\varkappa}^2 R^{-2} + O(R^{-4}), \qquad (4.9)$$

where $\xi_{n\varkappa}$ denotes the n-th positive zero of the Bessel function $J_p(\xi)$,

$$p = j + {}^1/_2(\operatorname{sign}\varkappa - 1) = |\varkappa| - 1 + {}^1/_2 \operatorname{sign}\varkappa.$$

This result agrees with (4.6) since a rectangular well corresponds to $\alpha \to \infty$ and $\nu = 2$.

Let us proceed to the coefficients c_2 and c_{2l+1}. Since

$$\left[\frac{d\varepsilon}{dV}\right]_{\varepsilon=-1} = -(2c_2)^{-1},$$

it follows from (4.6) that $c_2 = \frac{1}{2}$ in the case $R \gg 1$, independently of the shape of the potential. From a comparison of Eqs. (2.11) and (4.7) we obtain $c_{2l+1} \propto \exp(-\beta R)$. Thus, in a wide well the probability of the spontaneous production of positrons for $V > V_{cr}$ contains the exponentially small factor $\exp(-\beta R)$.

The generalization of these results to the case of nonmonotonic potentials is simple. Let $V(r)$ have a minimum at the point $r = r_0$

$$f(r) = 1 - \frac{a}{2}\left(\frac{r - r_0}{R}\right)^2 + \ldots, \quad r_0 > 0 \qquad (4.10)$$

(we assume r_0 and R to be quantities of the same order). In order to eliminate R from the Dirac equation, we change to the variable

$$\rho = (r - r_0) R^{-1/2}, \quad -\infty < \rho < \infty.$$

We define the quantities v and $\tilde{\varepsilon}$ according to Eqs. (4.4) for $\nu = 1$. Finally we arrive at the equation for a one-dimensional oscillator,[4)] from which

$$\varepsilon = V_{cr} - 1 - V = -1 + \frac{v_n - v}{R} + \ldots, \quad v_n = (2a)^{1/2}(n - {}^1/_2). \qquad (4.11)$$

The wave function of the n-th level is mainly concentrated in the region

$$|\rho| \lesssim (n/\omega)^{1/2}, \quad \omega = (2a)^{1/2}, \text{ i.e.,}$$
$$|r - r_0| \lesssim (nR)^{1/2} a^{-1/4} \ll R.$$

This justifies neglect of the following terms of the expansion in Eq. (4.10).

5. THE CRITICAL CHARGE OF THE NUCLEUS FOR A MUON

That value of Z for which the discrete level with quantum numbers n, j, $\varkappa$ drops down to the boundary of the lower continuum[1-4] is called the critical charge $Z_{cr} = Z_{cr}(j, j, \varkappa)$. The spontaneous production of positrons by a Coulomb field becomes possible for $Z > Z_{cr}$. In recent years many articles have been devoted to these questions (see[5-8] and the additional references cited there).

The value of Z_{cr} depends on the nuclear radius R_N. The calculations carried out up to the present time pertain to the region $R_N \ll 1$. This is a good approximation for the electron ($\hbar/m_e c = 386$ F), but it is not appropriate for the muon ($\hbar/m_\mu c = 1.87$ F). Let us find the asymptotic value of Z_{cr} in the region $R_N \gg 1$.

For $Z > 137$ it is necessary to solve the Dirac equation in the presence of a Coulomb potential which is cutoff at small distances

$$V(r) = -\zeta \tilde{f}(x)/R_N, \quad x = r/R_N, \qquad (5.1)$$

where $\zeta = Ze^2 = Z/137$ and R_N is the nuclear radius in units of $\hbar/mc$. The cutoff function $\tilde{f}(x)$ depends on the electric charge distribution over the volume of the nucleus.

According to the results of Sec. 4, the asymptotic form of Z_{cr} for $R_N \gg 1$ is determined by the behavior of the electrostatic potential φ near the center of the nucleus

$$\varphi(r) = \varphi(0) - \frac{2}{3}\pi\rho(0) r^2 + O(r^3),$$

$$\tilde{f}(x) = \begin{cases} a_0 - \dfrac{1}{2} a_2 x^2 + \ldots, & x \to 0 \\ x^{-1}, & x \to \infty \end{cases}, \qquad (5.2)$$

$$a_0 = R_N I_1/I_2, \quad a_2 = \rho(0)/\rho_0, \qquad (5.3)$$

here $\rho_0 = 3Ze/4\pi R_N^3$, $\rho(r)$ denotes the charge density in a spherical nucleus, and

$$I_n = \int_0^\infty \rho(r) r^n \, dr.$$

For example, for a small sphere which is uniformly charged throughout its volume (cutoff model II, see[2]) we have

$$\rho(r) = \rho_0 \theta(R_N - r), \quad a_0 = {}^3/_2, \quad a_2 = 1.$$

Assuming

$$|V_{cr}(0)| = 2 + v_{n\varkappa} R_N^{-1}, \quad \rho = r R_N^{-1/2},$$

for $r \ll R_N$ we arrive at Eq. (4.5) in which $\mu = v_{n\varkappa}$, $a = a_2/a_0$, and $\alpha = 2$. For $R_N \gg 1$ the asymptotic behavior of $\zeta_{cr} = Z_{cr}/137$ has the form

$$\zeta_{cr}=\beta_1 R_N+\beta_2(n+\gamma)+O(R_N^{-1/3}),$$
$$\beta_1=\frac{2}{a_0},\quad \beta_2=\left(\frac{2}{a_0}\right)^{2/3} a_2^{1/3},\quad 2\gamma=j+\frac{1}{2}(\operatorname{sign}\varkappa-1). \tag{5.4}$$

Let us compare expression (5.4) with the numerical solution of the problem. Outside of the nucleus the Dirac equation with energy $\varepsilon = -1$ has an exact solution, expressible in terms of Bessel functions of imaginary order. Denoting the logarithmic derivative of the interior ($r < R_N$) wave function at the edge of the nucleus by ξ, we have the following equation for the determination of Z_{cr}[2]:

$$zK_{i\nu}'(z)/K_{i\nu}(z)=2\xi,$$
$$z=(8\zeta_{cr}R_N)^{1/2},\quad \nu=2(\zeta_{cr}^2-\varkappa^2)^{1/2}, \tag{5.5}$$

where $K_{i\nu}$ denotes the Macdonald function.

Let us consider two models of cutoff:

$$\text{I}\ f(x)=1\quad \text{for}\ 0<x<1,\quad f(x)=x^{-1}\quad \text{for}\ x>1,$$
$$\text{II}\ f(x)=(3-x^2)/2\quad \text{for}\ 0<x<1,\quad f(x)=x^{-1}\quad \text{for}\ x>1.$$

Model I corresponds to a surface charge distribution; model II corresponds to a uniform volume density of charge inside a nucleus having a sharp boundary.

The results of the calculation are shown in Fig. 3, where the difference $\zeta_{cr} - \tilde{\zeta}_{cr}$ is given for the level $1s_{1/2}$. Here $\tilde{\zeta}_{cr}$ denotes the value of the critical charge according to the asymptotic formula (5.4). For the ground state: $\varkappa = -1$, $n = 1$, $\gamma = -\frac{1}{4}$,

$$\tilde{\zeta}_{cr}=\beta_1 R_N+{}^3/_4\beta_2, \tag{5.6}$$

where $\beta_1 = 2$, $\beta_2 = 0$ for model I and $\beta_1 = \frac{4}{3}$, $\beta_2 = (\frac{4}{3})^{3/2} = 1.5396$ for model II. It is interesting to note that, for model II the approach of ζ_{cr} to the asymptotic value (5.6) takes place much more rapidly than for model I. The accuracy of the asymptotic form amounts to $\approx 1\%$ for $R_N = 5$ and $\approx 0.3\%$ for $R_N = 10$ (for model II).

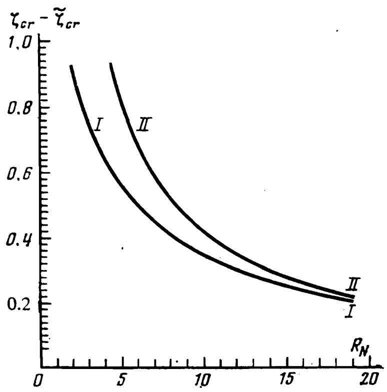

FIG. 3. The difference between the exact value ζ_{cr} and the asymptotic value (5.6) for the ground state $1s_{1/2}$. The nuclear radius R_N is measured in units of $\hbar/mc$. The numerals I and II on the curves refer to cutoff models I and II. The values of $\zeta_{cr} - \tilde{\zeta}_{cr}$ for model II are increased by a factor of ten in the figure.

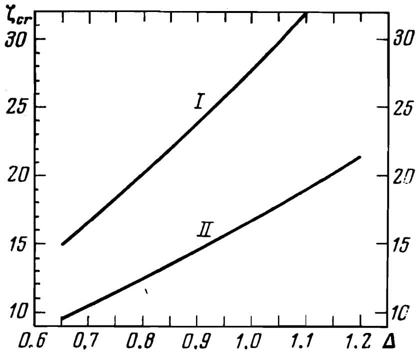

FIG. 4. The critical charge of a nucleus for a muon in cutoff models I and II.

In order to determine the critical charge for muons, it is necessary to know the dependence of the radius R_N on Z. For ordinary heavy nuclei $R_N = r_0 A^{1/3}$ and $A = 2.6\ Z$. The possibility of the existence of nuclei with $Z \sim 137^{3/2}$ was predicted by Migdal[9]; however, their density and the relationship between A and Z remain quite indeterminate. Let us use $R_N = r_0 A^{1/3}$ and let us set $r_0 = 1.2\delta$ F, $A/Z = 2.6\eta$; the nuclear density $n = 3A/4\pi R_N^3 = n_0\delta^{-3}$, where n_0 is the density of ordinary nuclei.

The value of ζ_{cr} depends on the parameter $\Delta = \delta\eta^{1/3}$ (Fig. 4). If $\Delta = 1$ and model II is assumed, $\zeta_{cr} = 16.7$ and $Z_{cr} = 2280$. The value of Z_{cr} strongly depends on the model of the superheavy nucleus, i.e., on the nuclear density n, the relationship between A and Z, and on the form of the cutoff function in Eq. (5.1). It is not difficult to find the critical charge for the subsequent levels of the muon spectrum with the aid of the asymptotic form (5.4), which gives

$$\Delta\zeta_{cr}\simeq\beta_2\left[N-1-\frac{1}{2}\left(j+\frac{1}{2}\operatorname{sign}\varkappa\right)\right]\left[1+\frac{1}{3}\beta_1 r_0\left(\frac{A}{Z\alpha}\right)^{1/3}\zeta_{cr}^{-2/3}\right],$$

where $\Delta\zeta_{cr} = \zeta_{cr}(N, \varkappa) - \zeta_{cr}(1, -1)$, N is the principal quantum number, and $r_0 = 0.64$ is the parameter $r_0 = 1.2$ F in units of $\hbar/m_\mu c$. For example, the difference between the values of Z_{cr} for the levels $1s_{1/2}$ and $2p_{1/2}$ amounts to ≈ 270.

For comparison we recall that, for an electron $\zeta_{cr} = 1.25$ and $Z_{cr} = 170$ (for the ground level[2,4]). It is obvious from here how Z_{cr} depends on the mass of a light particle moving in the Coulomb field of a nucleus.

The asymptotic behavior (5.4) enables one to rapidly determine the change in Z_{cr} associated with a variation of different parameters. Let us estimate, for example, the influence due to a smearing of the edge of the nucleus. Selecting the charge distribution in the form

$$\rho(r)=\text{const}\,[1+\exp((r-R_N)/d)]^{-1}, \tag{5.7}$$

where $R_N \gg d$, we find

$$\beta_1=\frac{4}{3}\left(1+\frac{2}{3}t^2+\ldots\right),\quad \beta_2=\left(\frac{4}{3}\right)^{3/2}\left(1+\frac{1}{2}t^2+\ldots\right),$$

where $t = \pi d/R_N \ll 1$. Hence

$$\frac{\Delta Z_{cr}}{Z_{cr}}=\frac{2\pi^2}{3}\left(\frac{d}{R_N}\right)^2. \qquad (5.8)$$

For $R_N = 11.6\ \hbar/m_\mu c = 21.7$ F, which corresponds to $\delta = \eta = 1$ and $d = 0.5$ F, taking the diffuse nature of the nuclear boundary into consideration increases the value of Z_{cr} by 0.35%.

The value of Z_{cr} obtained above for a muon should be regarded as a first approximation, since we did not take into consideration the screening of the bare nucleus' potential by the electron cloud, which nucleus draws to itself from the vacuum as a result of the spontaneous production of e^+e^- pairs.[3] The number of electrons in the vacuum shell of a supercritical atom is given by

$$N_e \approx \frac{4}{3\pi}\zeta^3 \ln\frac{\zeta}{R_N}, \quad \zeta \gg 1. \qquad (5.9)$$

Hence it is clear that, for $Ze^3 \gtrsim 1$ or $\zeta \gtrsim 10$ the number of electrons N_e becomes comparable with the nuclear charge Z; therefore, screening significantly modifies the bare potential and increases the value of Z_{cr} for a muon. A calculation of Z_{cr} with the corrections due to screening taken into consideration is being carried out at the present time.

6. CONCLUDING REMARKS

1. The Dirac equation admits an exact solution in very few cases: rectangular well and Coulomb field. The formulas of Sec. 2 provide a description of the motion of the levels near $\varepsilon = -mc^2$ for a potential of arbitrary form in terms of the two parameters c_2 and c_{2l+1}. For their evaluation it is sufficient to find the wave function at the critical point. By this means the problem is substantially simplified.

2. Analogous results are obtained for levels close to the boundary of the upper continuum. In this case it is convenient to change the sign in Eq. (2.1)

$$V-V_0=c_2\lambda^2+\ldots+c_{2l+1}\lambda^{2l+1}+\ldots. \qquad (6.1)$$

Here V_0 denotes the well depth at which a bound state appears; the orbital angular momentum l_1 plays the role of l. Expressions (2.3) and (2.13) for the coefficients c_2 and c_{2l+1} retain their form. If $l_1 \geqslant 1$ the emerging level deepens linearly with respect to $V - V_0$. Now $\varkappa = -1$ ($l_1 = 0$, the levels $ns_{1/2}$) is a special case, when the level deepens $\sim (V - V_0)^2$.

3. Let us compare the behavior of the levels near $\varepsilon = -mc^2$ for scalar and spinor particles (in both cases the interaction with the external field is introduced by the principle of minimal electromagnetic interaction, $p_\mu \to p_\mu - eA_\mu$, and the electrodynamics is renormalizable). Let a bound state with energy $\varepsilon = -1$ appear in the Klein–Gordon equation for $V = V_0$. For values of V close to V_0, the energy level ε is determined from the equation

$$V_0-V=\varphi(\lambda)=\sum_{n=n_0}^{\infty}c_n\lambda^n, \quad n_0=\begin{cases}1, & l=0\\ 2, & l\geqslant 1\end{cases}, \qquad (6.2)$$

For the sake of simplicity, we shall confine our attention to the case[5] $R \ll 1$. Assuming $V_0 = \zeta/R$ and discarding terms $\sim R$ and $\sim R^2$ in the Klein–Gordon equation, we obtain

$$\chi_l''+[\zeta^2f^2(x)-l(l+1)x^{-2}]\chi_l=0, \quad x=r/R. \qquad (6.3)$$

From here the spectrum $\zeta = \zeta_{nl}$ is determined. For $\varepsilon = -1$ the normalization conditions have the form

$$\chi_0(\infty)=1, \quad \text{if} \quad l=0,$$
$$\int_0^\infty \chi_l^2(x)\,dx=1, \quad \text{if} \quad l\geqslant 1. \qquad (6.4)$$

Let us introduce the notation

$$\overline{f^n}=\int_0^\infty [f(x)]^n\chi_l^2(x)\,dx.$$

Then for $l = 0$

$$c_1=-1/2\zeta_{n0}\overline{f^2}, \quad c_2=\overline{f}/2\overline{f^2}, \qquad (6.5)$$

and for $l \geqslant 1$

$$c_2=\overline{f}/2\overline{f^2}, \quad c_{2l+1}=(-1)^{l+1}[B_l/(2l-1)!!]^2R^{2l}/(2\zeta_{nl}\overline{f^2}),$$
$$B_l=\lim x^l\chi_l(x) \quad \text{as} \quad x\to\infty. \qquad (6.6)$$

Since the radius R does not explicitly appear in Eq. (6.3) or in the boundary conditions, ζ_{nl}, c_1, and c_2 are quantities of the order of unity. In contrast to the Dirac equation, $c_1 < 0$. Therefore, a bound level is not present for $V < V_0$; this level emerges from the lower continuum at $V = V_0$, and its energy

$$\varepsilon=-1+\frac{1}{2}\lambda^2+\ldots$$

increases with increasing depth V. According to Migdal,[9] such levels should be interpreted as bound states of antiparticles. Neglecting terms $\sim\lambda^3$ in Eq. (6.2) and changing from V, ε to dimensionless variables v, w

$$v=(V-V_0)/(V_{cr}-V_0), \quad \lambda=-c_1w^{1/2}/2c_2,$$
$$\varepsilon=-(1-\lambda^2)^{1/2}=-1+\frac{c_1^2}{8c_2^2}w+\ldots \qquad (6.7)$$

$(V_{cr} - V_0 = c_1^2/4c_2)$, we reduce Eq. (6.2) to the form

$$v=2w^{1/2}-w.$$

There are two branches of the single-particle spectrum, corresponding to particles (w_+) and antiparticles (w_-)

$$w_\pm=[1\pm(1-v)^{1/2}]^2, \quad 0<v<1. \qquad (6.8)$$

The two branches merge together at the critical point $v = w = 1$. The curve of the s-level has a bend (see Fig. 5a) for $c_1 < 0$, $c_2 > 0$ (these conditions are satisfied for any attractive potential, when $f(x) \geqslant 0$). Analytic continuation of the one-particle solutions into the region $V > V_{cr}$ (i.e., $v > 1$) leads to states with complex energies on the physical sheet. This indicates that the Hamiltonian of the Klein–Gordon equation ceases to be self-

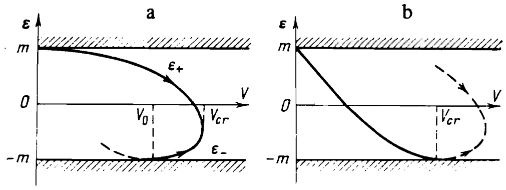

FIG. 5. Dependence of the energy level on the depth of the potential: a) s-level for scalar particles; b) $p_{1/2}$ level for the Dirac equation. The continuous curves correspond to real levels, and the dashed lines correspond to virtual levels. The potential is assumed to be short-range.

adjoint for $V > V_{cr}$. Taking vacuum polarization into account separates the levels $\varepsilon_\pm$, and their collision at the point $V = V_{cr}$ does not occur.[9]

For states with $l \geq 1$, it follows from Eq. (6.6) that $c_2 > 0$. Therefore, boson levels with angular momenta $l \geq 1$ enter the lower continuum with a finite slope.[6)] However, a comparison of formulas (3.1) and (6.6) shows that c_{2l+1} and r_l have different signs for bosons and fermions. As a consequence of this, the picture of the motion of boson poles with increasing V differs from Fig. 1a by the fact that the colliding poles emerge on the first sheet: $\mathrm{Im}\, k_{1,2} > 0$.

Independently of whether bending of the boson level exists near the boundary $\varepsilon = -1$ or not, continuation of the Klein–Gordon equation into the region $V > V_{cr}$ leads to poles on the physical sheet, i.e., runs into a contradiction with unitarity. At the same time the poles $k_{1,2}$ in the Dirac equation move away onto a nonphysical sheet for $V > V_{cr}$. Thus, the statistics of the particles is essentially manifested in the analytic properties of the S-matrix near the boundary of the lower continuum. The one-particle Klein–Gordon equation cuts itself off at the critical point, but the Dirac equation does not contradict unitarity for $V > V_{cr}$. Therefore, one can anticipate that, as a zero-order approximation, it retains meaning in the trans-critical region, but taking account of vacuum polarization is essential in a narrow region near $\varepsilon = -1$ (since $e^2 \ll Ze^2$). This conclusion is important for calculations of the probability for the spontaneous creation of positrons during a collision of nuclei, the energy spectrum of e^+, and other similar quantities.[2,7,8]

4. Up until now we have assumed that $|V(r)| < r^{-n}$ for $r \to \infty$ and $n > 2$. Let us show how the limiting transition to an unscreened Coulomb field takes place. Cutting off the potential $V(r) = -\zeta/r$ at large distances $r \sim R \gg 1$, we have ($s = 0, \frac{1}{2}$)

$$c_1 \sim (-1)^{2s+1} \exp[-(8\zeta R)^{1/2}], \quad c_2 \sim 1. \tag{6.9}$$

$c_1 < 0$ for scalar particles, which leads to a bend in the s-level curve. Since $\zeta_{cr} - \zeta_0 \sim c_1^2$, the bend is located exponentially close to the boundary of the lower continuum and vanishes in the limit $R \to \infty$. In analogous fashion the tangency of the $p_{1/2}$-level curve with the boundary $\varepsilon = -1$ disappears as $R \to \infty$ for fermions (Fig. 5b). In an unscreened Coulomb potential, all levels of the discrete spectrum enter the lower continuum with finite slope.[2] However, continuation of a discrete level into the region $Z > Z_{cr}$ corresponds, in the boson case, to an S-matrix pole on the physical sheet, but in the fermion case—it corresponds to a pole on a nonphysical sheet. Thus, the principle difference between the Klein–Gordon and Dirac equations is maintained in supercritical fields and for an unscreened Coulomb interaction.

The question of level bending near $\varepsilon = -1$, arising in connection with a discussion of π-meson condensate in nuclei, appeared recently as a topic of discussion.[23,24] In this connection various methods (the virial theorem,[24] numerical calculations[23]) indicated that level bending is characteristic for a short-range potential, and vanishes upon transition to a Coulomb field.[7)] However, the analytic properties of the poles k_1, k_2 in the trans-critical region were not investigated in[23,24].

[1)] For example, expressions (2.3) are valid for all values $l \geq 1$ provided that $\lim V(r) r^{2+\delta} = 0$ for $r \to \infty$ and some $\delta > 0$. This is explained by the fact that we are considering the special case $V = V_{cr}$, when a bound state with zero effective energy $E = (\varepsilon^2 - 1)/2 = 0$ appears in the potential. Among the potentials which do not satisfy the enumerated conditions, potentials having a Coulomb tail at infinity[2] are of special interest. They may be treated by the present method with the aid of a cutoff for $r > R$ and the limiting transition $R \to \infty$ (see Sec. 6 for further detials).

[2)] Complex poles of the S-matrix on the physical sheet arise for the Klein-Gordon equation when $V > V_{cr}$. The fact that such a difficulty does not arise for fermions is a remarkable property of the Dirac equation.

[3)] See, for example, article[15] in which the Klein-Gordon equation is solved in the presence of an exponential potential.

[4)] In this equation one can discard the centrifugal and spin terms since in the important region $r \sim r_0$

$$U_s(r)/U_0(r) \sim R^{-1}, \quad \varkappa(\varkappa+1)/r_2 U_0(r) \sim R^{-1},$$
$$U_0(r) = -\varepsilon V f(r) - {}^1/_2 V^2 f^2(r).$$

Because of this, the asymptotic spectrum (4.11) is degenerate with respect to the angular momentum j. In the preceding case ($r_0 = 0$) the centrifugal energy is essential as $r \to 0$, and the dependence of the level spectrum on j is retained.

[5)] The general case is investigated in[16], where an effective range expansion is obtained for the Klein-Gordon equation in the presence of an arbitrary potential.

[6)] This conclusion is undoubtedly valid in the case $R \ll 1$. For potentials having a sharp edge (of the rectangular well type) not only a bending of the s-level, but also of the p-level[16] is possible for sufficiently large values of R. In the absence of level bending, $V_{cr} = V_0$.

[7)] This fact was established earlier[2] for a narrow ($R \ll 1$) potential of arbitrary form $f(x)$.

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Translated by H. H. Nickle

Radiative effects in the field of an electromagnetic wave with allowance for the action of a stationary magnetic field

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(Submitted March 2, 1976)
Zh. Eksp. Teor. Fiz. 71, 871–883 (September 1976)

Radiative effects due to the combined action of stationary and alternating electromagnetic fields are considered by the analytic continuation technique in the first order with respect to the fine structure constant $\alpha = e^2/\hbar c$. The real parts of the corrections to the electron and photon masses are determined for the case of low wave intensities ($\xi = eE_0/cm\omega < 1$) by means of the expressions for the probabilities of the multiphoton electron scattering and electron–positron pair production by photons in the wave field or magnetic field. In the overlap region the derived formulas are in agreement with results obtained in an investigation of radiative processes in stationary crossed fields and also with the results of an analysis of the mass and polarization operators in the field of an electromagnetic wave. Polarization effects are studied. It is shown that in this case the anomalous magnetic moment of the electron is a function of all the parameters that characterize the total field (ω, E_0, and H). Corrections proportional to the wave intensity are obtained for the Schwinger value of the anomalous electron moment.

PACS numbers: 12.20.Ds, 13.10.+q, 14.60.Cd

The interaction of electrons and phonons with an electromagnetic vacuum in a constant electromagnetic field has been studied in sufficient detail (see, e.g. [1–5]). Interest has arisen recently in the study of similar processes in the field of an intense electromagnetic wave, in view of the added possibilities afforded by the use of laser techniques. It turns out that the presence of a sufficiently intense electromagnetic wave in a vacuum is capable of leading to effects analogous to those that take place in constant fields. The emission of a photon by an electron and the production of electron-positron pairs by a photon in the field of a wave[6] are processes that give rise to radiative effects leading to changes in the masses of the electron and the photon. The question of the change in the photon mass in the field of a wave was considered in[7] by using the electron Green's function obtained by Schwinger.[8] An analogous method was used in[9] to calculate the polarization operator. Quite recently[10,11] the method of operator diagram technique has yielded the corrections to the mass and polarization operators in the field of a wave of rather general form.

In this paper we use the dispersion-relation method to consider radiative processes in the field of an electromagnetic wave, with account taken of the action of a constant magnetic field. The development of a method of analytic continuation as applied to the case of an investigation of radiative effects in constant crossed fields was described by Ritus.[4,12] The main deviation from[4,7,9–12] is that the external field was chosen by us to be a superposition of fields, namely a constant magnetic field of intensity H and the field of a plane electromagnetic wave propagating along a magnetic field,

$$\mathbf{H} = (0, 0, H), \tag{1}$$