Calculation of the amplification of electromagnetic waves upon reflection from a rotating body

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(Submitted March 28, 1976)
Zh. Eksp. Teor. Fiz. **71**, 1003–1009 (September 1976)

The amplitudes of the waves scattered by a rotating circular cylinder are calculated by the method of perturbation theory. The scattering intensities in the partial angular harmonics and the total intensity of the scattered field are determined. It is shown that in the approximation considered, the total intensity of the scattered waves is independent of the rotation frequency of the cylinder. The change of the scattering indicatrix due to the rotation (an optical analog of the Magnus effect) is considered qualitatively.

PACS numbers: 03.50.Jj

It was noted in[1,2] that an electromagnetic wave incident on a rotating cylinder of low conductivity may be amplified upon reflection; the conditions for occurrence of such amplification were considered, and quantitative estimates were given of the intensity of the amplified waves. In the present note this question is considered in greater detail for one of the two possible polarizations of the incident field. The calculation is carried out by perturbation theory.

We consider a right circular cylinder of radius a, consisting of a medium with dielectric permittivity ε and magnetic permeability μ, and having conductivity σ (in the rest system). Let the cylinder rotate about the z axis with angular velocity Ω. Furthermore, let there be incident on the cylinder from outside an electromagnetic wave whose wave vector is perpendicular to the axis of the cylinder. For simplicity we shall suppose that the electric field of the wave is parallel to the axis of the cylinder. We shall calculate the amplitudes of the scattered waves.

We introduce a cylindrical coordinate system whose z axis coincides with the cylinder axis. In this system, the vector potential of the incident wave may be written in the form[3]

$$A_z^{(0)}=e^{i(\mathbf{kr}-\omega t)}=e^{-i\omega t}\sum_{n=-\infty}^{n=+\infty}\frac{i^n}{2}[H_n^{(1)}(kr)+H_n^{(2)}(kr)]e^{in\varphi}. \tag{1}$$

Here $\mathbf{k}$ and $\mathbf{r}$ are two-dimensional vectors in a plane perpendicular to the z axis; their coordinates, in a polar coordinate system in this plane, may be written in the form $\mathbf{k}=(k=\omega/c,\ \varphi=0)$ and $\mathbf{r}=(r,\ \varphi)$, where the angle φ is measured from the direction of the vector $\mathbf{k}$; $H_n^{(1,2)}$ is a Hankel function with argument kr.

To determine the field inside the rotating dielectric, we use the equation for the z component A_z of the vector potential $\mathbf{A}$ in the electrodynamics of moving media[4]:

$$\left\{\Delta-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\frac{\varepsilon\mu-1}{c^2}\gamma^2\left(\mathbf{u}\nabla+\frac{\partial}{\partial t}\right)^2-\frac{4\pi\sigma\mu}{c^2}\gamma^2\left(\mathbf{u}\nabla+\frac{\partial}{\partial t}\right)\right\}A_z=0, \tag{2}$$

where $\mathbf{u}$ is the vector velocity of transport of the elements of the rotating medium, ∇ is the three-dimensional gradient vector ($\Delta=\nabla^2$), and $\gamma=(1-u^2/c^2)^{-1/2}$. Equation (2) differs from the corresponding equation given in[4] in that it takes account of the conductivity of the medium.

It is obvious that in our system of coordinates, $\mathbf{u}$ has only the component $u_\varphi=r\Omega$. Then $\mathbf{u}\nabla=\Omega\partial/\partial\varphi$. Hereafter we shall consider the case of nonrelativistic rotation of the medium, $u_\varphi\ll c$, and shall set $\gamma=1$. This is correct if the condition $a\Omega/c\ll 1$ is satisfied.

We shall suppose that the time dependence of the vector potential A_z is the same as for the incident wave $A_z^{(0)}$ in (1). Then in Eq. (2) $\partial/\partial t=(-i\omega)$. As a result, Eq. (2) takes the form

$$\left(\Delta+\frac{\omega^2}{c^2}\right)A_z(r,\varphi)=\left\{\frac{\varkappa}{c^2}\left(\Omega\frac{\partial}{\partial\varphi}-i\omega\right)^2+\frac{4\pi\sigma\mu}{c^2}\left(\Omega\frac{\partial}{\partial\varphi}-i\omega\right)\right\}A_z(r,\varphi), \tag{3}$$

where $\varkappa=\varepsilon\mu-1$. We shall treat the right member of Eq. (3) as a perturbation and shall accordingly seek a solution in the form

$$A_z=A_z^{(0)}+A_z^{(1)}+A_z^{(2)}+\ldots+A_z^{(j)}+\ldots, \tag{4}$$

where $A_z^{(0)}$ is defined by formula (1), and where the subsequence terms of the expansion are determined by the system of equations

$$\left(\Delta+\frac{\omega^2}{c^2}\right)A_z^{(j)}=\left\{\frac{\varkappa}{c^2}\left(\Omega\frac{\partial}{\partial\varphi}-i\omega\right)^2+\frac{4\pi\sigma\mu}{c^2}\left(\Omega\frac{\partial}{\partial\varphi}-i\omega\right)\right\}A_z^{(j-1)}, \tag{5}$$

where $j=1,\ 2,\ \ldots$. The conditions for validity of such an expansion will be determined below.

By means of (5) and (1), we obtain for the first approximation $A_z^{(1)}$ the equation

$$\left\{\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}\right)+\frac{\omega^2}{c^2}\right\}A_z^{(1)}(r,\varphi)=f_0(r,\varphi)$$
$$=\frac{1}{2}\sum_{n=-\infty}^{n=+\infty}i^n\Delta_n[H_n^{(1)}(kr)+H_r^{(2)}(kr)]e^{in\varphi}, \tag{6}$$

where

$$\Delta_n=-\frac{\varkappa}{c^2}(n\Omega-\omega)^2+\frac{4\pi\sigma\mu}{c^2}i(n\Omega-\omega). \tag{7}$$

As is well known (see, for example, [5]), the solution of Eq. (6) can be expressed in terms of the right member $f_0(r,\varphi)$ as

$$A_z^{(1)}(r,\varphi)=-\frac{i}{4}\int_0^a r'\,dr'\int_0^{2\pi}d\varphi'\,H_0^{(1)}\left(\frac{\omega}{c}|\mathbf{r}-\mathbf{r}'|\right)f_0(\mathbf{r}'). \tag{8}$$

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where the two-dimensional vectors $\mathbf{r}$ and $\mathbf{r}'$ are defined by the values of the coordinates (r, φ) and (r', φ') respectively. The integral extends over the region occupied by the rotating body. It is easily seen that $f_0(r, \varphi)$ vanishes outside this region.

If we now substitute in (8) the expression for $f_0(r, \varphi)$ from (6) and use the addition theorem for Hankel functions, [3]

$$H_0^{(1)}\left(\frac{\omega}{c}|\mathbf{r}-\mathbf{r}'|\right)=\sum_{m=-\infty}^{m=+\infty}H_m^{(1)}\left(\frac{\omega}{c}r\right)J_m\left(\frac{\omega}{c}r'\right)e^{im(\varphi-\varphi')} \tag{9}$$

the integration in (8) gives

$$A_z^{(1)}(r,\varphi)=-\frac{\pi i}{2}\sum_{n=-\infty}^{n=+\infty}i^n\Delta_nB_nH_n^{(1)}\left(\frac{\omega}{c}r\right)e^{in\varphi},\qquad B_n=\int_0^a J_n^2\left(\frac{\omega}{c}r'\right)r'\,dr'. \tag{10}$$

By use of this solution for $A_z^{(1)}$, it is not difficult, by means of (5), to obtain in similar fashion an expression for the next approximation:

$$A_z^{(2)}(r,\varphi)=-\frac{\pi^2}{4}\sum_{n=-\infty}^{n=+\infty}i^n\Delta_n^2B_n(B_n+iD_n)H_n^{(1)}\left(\frac{\omega}{c}r\right)e^{in\varphi},\qquad D_n=\int_0^a J_n\left(\frac{\omega}{c}r'\right)N_n\left(\frac{\omega}{c}r'\right)r'\,dr'. \tag{11}$$

By this method it is possible to calculate an approximation of arbitrary order; then the series (4) is easily summed:

$$A_z(r,\varphi)=A_z^{(0)}(r,\varphi)+\left(-\frac{\pi i}{2}\right)\sum_{n=-\infty}^{n=+\infty}\frac{2i^n\Delta_nB_n}{2+\pi i\Delta_n(B_n+iD_n)}H_n^{(1)}\left(\frac{\omega}{c}r\right)e^{in\varphi}. \tag{12}$$

The condition for convergence of the perturbation-theory series may now be written in the form of an inequality:

$$\xi={}^1/_2\pi|\Delta_n(B_n+iD_n)|\ll 1. \tag{13}$$

As is seen from the definition of Δ_n (see (7)), the inequality (13) can be satisfied if the value of $\varepsilon\mu$ is sufficiently close to unity and the conductivity σ of the medium close to zero. In this case we shall restrict ourselves to the first three terms of the series (4):

$$A_z(r,\varphi)\approx A_z^{(0)}+A_z^{(1)}+A_z^{(2)}$$
$$=\frac{1}{2}\sum_{n=-\infty}^{n=+\infty}i^ne^{in\varphi}\{[1-\pi i\Delta_nB_n-{}^1/_2\pi^2\Delta_n^2B_n(B_n+iD_n)]H_n^{(1)}(kr)+H_n^{(2)}(kr)\}, \tag{14}$$

where $k=\omega/c$. In the sum over n, the coefficient before the function $H_n^{(1)}(kr)$ gives the amplitude of the diverging wave, the coefficient before $H_n^{(2)}(kr)$ the amplitude of the converging wave. The energy currents in the converging and diverging waves are proportional to the squares of the moduli of the corresponding coefficients. If the rotating body is absent, then $\Delta_n=0$, and the amplitudes before $H_n^{(1)}$ and $H_n^{(2)}$ are equal, while the expression (14) reduces to the expression (1) for the incident wave.

We surround the rotating cylinder with a coaxial cylindrical surface of radius $r>a$. The flow of energy of the electromagnetic field across an element of surface at frequency ω may be written in the form

$$dS_\omega=c\,\mathrm{Re}(E_{z,\omega}H_{\varphi,\omega}^*)r\,d\varphi\,dz. \tag{14'}$$

From formula (14) for A_z we have

$$E_{z,\omega}=\sum_{n=-\infty}^{n=+\infty}i\frac{\omega}{c}\left[a_nH_n^{(1)}(kr)+b_nH_n^{(2)}(kr)\right]e^{in\varphi},$$
$$H_{\varphi,\omega}=\sum_{m=-\infty}^{m=+\infty}\frac{\omega}{c}[a_mH_m'^{(1)}(kr)+b_mH_m'^{(2)}(kr)]e^{im\varphi}, \tag{15}$$

where

$$a_n={}^1/_2i^n[1-\pi i\Delta_nB_n-{}^1/_2\pi^2\Delta_n^2B_n(B_n+iD_n)],\quad b_n={}^1/_2i^n,$$
$$H_n'^{(1,2)}(x)=dH_n^{(1,2)}/dx. \tag{16}$$

If we substitute the expressions (15) and (16) in formula (14′) for dS_ω and carry out the integration over $d\varphi$, we get the flow of energy W_ω at frequency ω across a cylindrical surface of unit height:

$$W_\omega\,dz=\left(\sum_{n=-\infty}^{n=+\infty}W_{n,\omega}\right)dz, \tag{17}$$

where

$$W_{n,\omega}=4\omega(|a_n|^2-|b_n|^2). \tag{18}$$

By use of the expressions (16) for the coefficients a_n and b_n, we find, restricting ourselves to terms of not over the second order in the expansion parameter ξ (see formula (13)),

$$W_{n,\omega}=2\pi\omega\beta_nB_n(1+\pi\alpha_nD_n+\pi\beta_nB_n), \tag{19}$$

where

$$\alpha_n=-\frac{\varepsilon\mu-1}{c^2}(n\Omega-\omega)^2,\quad \beta_n=\frac{4\pi\sigma\mu}{c^2}(n\Omega-\omega), \tag{20}$$

and the quantities B_n and D_n are determined by formulas (10) and (11).

The quantity $W_{n,\omega}$ has the following physical meaning. We consider that part of the total field for which the dependence on the azimuthal angle φ is determined by the factor $e^{in\varphi}$ (that is, the azimuthal harmonic of number n). For this component of the total field, the difference between the flow of energy away from the body and the flow of energy on to the body is exactly determined by the value of $W_{n,\omega}$ (19). Obviously if $W_{n,\omega}>0$, then one may say that upon reflection from the rotating body there occurs an amplification of the harmonic of number n; but if $W_{n,\omega}<0$, then what occurs is absorption of this harmonic of the field by the rotating body.

In the expression (19) for $W_{n,\omega}$, the terms in parentheses $\pi\alpha_nD_n$ and $\pi\beta_nB_n$ come from the second approximation (11). By virtue of the condition (13), we shall hereafter disregard these terms. Then we obtain for $W_{n,\omega}$ the expression

$$W_{n,\omega}=\frac{8\pi^2\omega\sigma\mu}{c^2}(n\Omega-\omega)\int_0^a J_n^2\left(\frac{\omega}{c}r\right)r\,dr. \tag{21}$$

Formula (21) corresponds to the case that was considered in[1,2]. It is evident that the sign of $W_{n,\omega}$ is determined by the sign of the difference $(n\Omega-\omega)$. When $n\Omega<\omega$, the rotating body absorbs energy from the nth harmonic, whereas when $n\Omega>\omega$ it amplifies this harmonic. The expression (21), in contrast to the results of references,[1,2] where the proportionality between $W_{n,\omega}$ and $(n\Omega-\omega)$ was pointed out, gives also the value of the coefficient of proportionality and thereby enables us to estimate not only the sign but also the absolute value of the amplification. We note that the quantity $W_{n,\omega}$ has the dimensions of energy per unit length of the cylinder and per unit frequency interval. We add furthermore that we have supposed the amplitude of the incident wave to be $A_0=1$. Otherwise the value of $W_{n,\omega}$ is proportional to $|A_0|^2$.

After substituting the expression (21) in (17), one can carry out the summation over all the harmonics of the radiation. By use of the formulas[3]

$$\sum_{n=-\infty}^{n=+\infty} J_n^2(x)=1,\quad \sum_{n=-\infty}^{n=+\infty} nJ_n^2(x)=0, \tag{22}$$

we get

$$W_\omega=\sum_{n=-\infty}^{n=+\infty} W_{n,\omega}=-\frac{4\pi^2a^2\omega^2\sigma\mu}{c^2}. \tag{23}$$

In the derivation of formula (23) we have assumed that the value of the conductivity σ is independent of the frequency ω (no dispersion).

The expression (23) shows that although in certain harmonics $(n\Omega-\omega>0)$ amplification occurs upon reflection from the rotating body, nevertheless, on the whole, the energy of the incident plane wave is absorbed by this body (in the absence of dispersion).

This result can be understood by consideration of formula (21). As is evident from this formula, $W_{0,\omega}<0$. Furthermore, $W_{n,\omega}+W_{-n,\omega}<0$; thus the addition of the two harmonics with numbers n and $-n$ leads to cancellation of the term that is proportional to the velocity of rotation Ω and that can lead to amplification.

It is interesting to estimate the dependence of the value of $W_{n,\omega}$ (21) on the harmonic number n. On calculating the integral that occurs in (21), we get

$$W_{n,\omega}=\frac{4\pi^2a^2\omega\sigma\mu}{c^2}(n\Omega-\omega)\left[J_n^2\left(\frac{\omega}{c}a\right)-J_{n-1}\left(\frac{\omega}{c}a\right)J_{n+1}\left(\frac{\omega}{c}a\right)\right]. \tag{24}$$

The expression (24) can be simplified for two limiting cases. In the first case we suppose that the harmonic number n is sufficiently large $(n\to\infty)$ and that the argument of the Bessel function satisfies the inequality $\omega a/c<n$. In the second case, that the radius of the rotating cylinder is less than the wavelength of the radiation in free space $(\omega a/c\ll1)$. In both cases the asymptotic expression for $W_{n,\omega}$ has the same form and can be written as follows:

$$W_{n,\omega}=\frac{4\pi^2a^2\omega\sigma\mu}{c^2}(n\Omega-\omega)\left(\frac{\omega a}{2c}\right)^{2n}\left(\frac{1}{n!}\right)^2\frac{1}{n+1}. \tag{25}$$

From the expression (25) it follows that the series (23) converges very rapidly, and that its calculation requires only a small number of terms with numbers $n=0,\pm1,\pm2,\ldots$.

In conclusion, we shall discuss the role of dispersion in the scattering of electromagnetic waves by a rotating body. We have so far assumed that the permittivity ε and magnetic permeability μ of the body are independent of frequency. We shall now consider what effect allowance for dispersion has on the results obtained. We shall consider the simplest form of dispersion,

$$\varepsilon=1+\frac{\omega_0^2}{(\omega_s^2-\omega^2)-i\Gamma\omega},\quad \mu=1, \tag{26}$$

where ω_s is the natural frequency of the atom in the classical oscillator model, Γ is the damping of this oscillator, and ω_0 is the plasma frequency of the medium. The expressions (26) are valid in the rest system of the rotating cylinder. From formulas (26) it follows that

$$\begin{gathered}\varkappa=\operatorname{Re}\varepsilon-1=\frac{\omega_0^2(\omega_s^2-\omega^2)}{(\omega_s^2-\omega^2)^2+\omega^2\Gamma^2},\\ \sigma=\frac{\omega}{4\pi}\operatorname{Im}\varepsilon=\frac{\Gamma\omega_0^2\omega^2}{4\pi[(\omega_s^2-\omega^2)^2+\Gamma^2\omega^2]}.\end{gathered} \tag{27}$$

In substituting these values in the expressions for the nth harmonic of the scattered field, which is proportional to the factor $\exp\{i(n\varphi-\omega t)\}$, one must replace the frequency ω in formulas (27) by the Doppler shifted frequency $\omega'=(\omega=n\Omega)$. Then the values of α_n and β_n in (20) take the form

$$\begin{gathered}\alpha_n=-\frac{(n\Omega-\omega)^2}{c^2}\frac{\omega_0^2[\omega_s^2-(\omega-n\Omega)^2]}{[\omega_s^2-(\omega-n\Omega)^2]^2+\Gamma^2(\omega-n\Omega)^2},\\ \beta_n=\frac{1}{c^2}\frac{\Gamma\omega_0^2(n\Omega-\omega)^3}{[\omega_s^2-(\omega-n\Omega)^2]^2+\Gamma^2(\omega-n\Omega)^2}.\end{gathered} \tag{28}$$

It is now also necessary to use these expressions in calculating the intensity of the scattered radiation. As is seen from (28), at high harmonics $(n\to\infty)$ the value of β_n decreases in inverse proportion to the harmonic number n, while the value of α_n approaches a constant limit independent of the number n. For comparison we note that in the absence of dispersion, as is seen from (20), both these quantities increase with increase of the harmonic number n: β_n linearly, α_n quadratically. Thus allowance for dispersion decreases the intensity of the scattered radiation at high harmonics (see, for example, formula (19)). We note further that allowance for dispersion decreases the intensity near the amplification threshold $(\omega=n\Omega)$.

Scattering of light on a rotating body leads to still another interesting effect—a change of the scattering indicatrix.[1)] Specifically, the maximum of the intensity of the scattered radiation deviates from the direction of the incident wave in the direction of rotation of the body (that is, the direction in which the scattered energy is greatest makes a positive angle with the direction of the incident light if the body is rotating in the positive direction, and a negative angle if the direction of rotation is negative). Simple calculations show that the angle of deflection is proportional to the cylinder radius a and to the angular velocity of rotation Ω. The reason for this deflection is the effect of entrainment of light by a moving medium: if the direction of propagation of the wave coincides with the direction of motion of the medium, then the phase velocity of the wave is

$$c/\sqrt{\varepsilon\mu}+u(1-1/\varepsilon\mu),$$

but if the velocities of the wave and of the medium are opposite, then the phase velocity is

$$c/\sqrt{\varepsilon\mu}-u(1-1/\varepsilon\mu).$$

In our case, u is in order of magnitude equal to $a\Omega$. An estimate with allowance for these relations leads to the following expression for the angle θ between the maximum of the scattering indicatrix and the initial direction of propagation of the wave:

$$\theta\approx\frac{u}{c}\sqrt{\varepsilon\mu}\left(1-\frac{1}{\varepsilon\mu}\right)=\frac{a\Omega}{c}\sqrt{\varepsilon\mu}\left(1-\frac{1}{\varepsilon\mu}\right). \tag{29}$$

From this result it follows that the field exerts on the rotating cylinder a force numerically equal to the change of the quantity of motion of the light per unit time in the scattering process. The direction of this force is opposite to the vector change of momentum of the wave during the scattering. This phenomenon may be regarded as an analog of the well-known Magnus effect in the mechanics of continuous media.

The effect considered occurs in the absence of absorption (conductivity). Allowance for conductivity presumably leads to a weakening of this effect.

The authors are grateful to V. L. Ginzburg, Ya. B. Zel'dovich, D. A. Kirzhnits, and S. I. Syrovatskiĭ for a number of valuable comments made during discussion.

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Translated by W. F. Brown, Jr.

The ponderomotive force of a high-frequency electromagnetic field in a dispersive medium

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(Submitted March 31, 1976)
Zh. Eksp. Teor. Fiz. **71**, 1010–1016 (September 1976)

A general expression is obtained for the ponderomotive force of a high-frequency field with slowly varying amplitude in a transparent dispersive fluid and, in particular, in a plasma. For electromagnetic waves in an isotropic plasma, and also for irrotational oscillations, this expression coincides with that obtained earlier. In the general case, however, our expression contains time derivatives of the field amplitude; these may play a significant role, for example, in a magnetically active plasma.

PACS numbers: 52.40.Db

1. Since the formulation of the general problem of finding the time-average stress tensor and ponderomotive force acting on a dispersive medium in a high-frequency field ([1], §61), this problem has been the subject of a large number of papers. We cite first of all studies[2-5] in which this force was obtained for an isotropic collisionless plasma:

$$\mathbf{f}=-\frac{\omega_p^2}{16\pi\omega^2}\nabla|\mathbf{E}|^2. \tag{1}$$

Here $\mathbf{f}$ is the ponderomotive force acting on unit volume, $\omega_p=(4\pi ne^2/m)^{1/2}$ is the plasma frequency, and $\mathbf{E}$ is the amplitude of the high-frequency (hf) field:

$$\tilde{\mathbf{E}}={}^1/_2(\mathbf{E}e^{-i\omega t}+\text{c.c.}),\quad \tilde{\mathbf{H}}={}^1/_2(\mathbf{H}e^{-i\omega t}+\text{c.c.}). \tag{2}$$

A general phenomenological approach to the determination of the ponderomotive force of a hf field in a dispersive transparent medium was proposed by Pitaevskiĭ.[6] The expression obtained in[6], however, was limited by several conditions that will be discussed below. The one most important for us here is the assumption that the hf field is stationary (amplitude constant in time). In[7,8], on the basis of quasimicroscopic considerations, an expression was obtained for the ponderomotive force of a hf field in a magnetically active plasma. Here again the effects of nonstationarity of the hf field amplitudes were disregarded. (For a more detailed discus-

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