

Solitons in one-dimensional molecular chains

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A theory is developed for collective excitations, corresponding to intramolecular excitations, in one-dimensional chains of weakly bound molecules. It is shown that the excitations are described by solitary waves (solitons), propagating (without spreading) together with the local deformation along the chain. The theory is developed without using the concept of inertialess motion of molecules. It is shown that the limiting velocity of the solitons is the velocity of sound waves corresponding to longitudinal vibrations of the molecules. The stability of solitons is discussed.

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1. INTRODUCTION

A special type of collectively excited states which has been treated by the authors^[1] and called particle-like excitations or solitary excitons is possible in one-dimensional molecular chains consisting of periodically repeated molecules (or groups of strongly bound atoms) interacting through relatively weak forces. In what follows these excitations will be simply referred to as solitons. Special cases of such systems are α -helical protein molecules in which the peptide groups, consisting of four atoms (HNCO) bound by chemical forces, are arranged along the molecule in the form of three chains. The neighboring peptide groups in each chain are linked by hydrogen bonds that provide a rigid structure for the α -helix.

The collective excitations of a one-dimensional molecular chain, which correspond to intramolecular electronic or vibrational excitations with an electric moment directed along the chain, are of a very specific nature and differ from ordinary excitons in three-dimensional crystals. In protein molecules these include excitations based on valence vibrations of amide-1 peptide groups, with an energy 0.21 eV and an electric dipole moment 0.30 Debye directed along the molecular axis.

The solitons (solitary excitons) considered earlier^[1] are excitations localized in some region of the chain. They are due to changes in the distances between the molecules in this region and propagate with a certain velocity along the chain.^[1] In theories of electronic excitations^[1] and vibrational excitations of amide-1 α -helical protein molecules^[4] it is assumed that the chain deformation follows the motion of excitation without inertia. This assumption imposes a restriction on the excitation velocity. In the present paper we develop a theory that avoids the concept of inertialess molecular motion. It is shown that the limiting velocity of a soliton in a molecular chain is the velocity of sound waves corresponding to molecular vibrations along the chain. The problem of soliton stability is also discussed.

2. THE BASIC EQUATIONS AND THEIR SOLUTION IN THE FORM OF SOLITONS

Consider an infinite chain of weakly bound molecules (or groups of atoms) with a mass M and a distance R from each other. We assume that the internal excita-

tions of molecules (electronic or vibrational) in the chain are characterized by an energy and an electric dipole moment d directed along the chain. When a molecule is excited its interactions with the neighboring molecules change, resulting in a change in the equilibrium distances between molecules. Besides, since the molecules are identical, there arise additional resonance interactions due to excitation transfer from one molecule to another. Thus, the internal excitations of molecules and their motion about equilibrium positions are inseparably linked.

In a one-dimensional chain it is sufficient to take into account the interactions between neighboring molecules only. In this approximation the collective excitations of the chain, which correspond to an intramolecular excitation with an energy ϵ , can be described by the Hamiltonian

$$H = T + U + \sum_n [(e - D_n)B_n^+ B_n - J(B_{n+1}^+ B_n + B_{n+1} B_n^+)], \quad (1)$$

where the index n labels the molecules occupying the positions r_n in the chain, while B_n^+ and B_n are intramolecular-excitation creation and annihilation operators. They satisfy the commutation relations

$$[B_n, B_m^+] = \delta_{nm}, \quad [B_n, B_m] = 0.$$

The function $-D_n$ characterizes the change in the static interaction of the n -th molecule with neighboring molecules when it is excited. If ρ_n is defined by the expression

$$r_n - r_{n-1} = R - \rho_n, \quad (2)$$

we have, allowing for interactions with nearest neighbors,

$$D_n = \mathcal{D}_n(|r_{n+1} - r_n|) + \mathcal{D}_n(|r_{n-1} - r_n|) \approx \left(1 + \frac{\beta}{R} \rho_n\right) D, \quad (3)$$

where $D \equiv 2\mathcal{D}_n(R)$ and β are positive parameters of the theory. The resonance interaction determining the transition of intramolecular excitation from one molecule to another is characterized by the quantity $-J$, defined in terms of the electric dipole moment by means of the equality

$$J=2d^2R^{-3}.$$

The Hamiltonian (1) is written down in an approximation in which the inequality

$$\left| \frac{\partial D_n}{\partial \rho} \right|_{\rho=0} \approx \frac{\beta D}{R} > \left| \frac{\partial J_n}{\partial \rho} \right|_{\rho=0} \approx \frac{3J}{R} \quad (4)$$

is satisfied. Owing to the short range of the interaction D_n we have $\beta > 3$ in inequality (4).

The first two terms in (1) correspond to the potential and kinetic energies of longitudinal displacements of the molecules. The potential energy of displaced molecules in the non-excited chain is chosen in the harmonic approximation

$$U = \frac{1}{2} w \sum_n \rho_n^2. \quad (5)$$

Taking into account the equality

$$\dot{\rho}_n = - \sum_{-\infty < l < n} \dot{\rho}_l, \quad (5a)$$

the kinetic energy can be written as

$$T = \frac{1}{2} M \sum_n \dot{\rho}_n^2 = \frac{1}{2} M \sum_n \left(\sum_{-\infty < l < n} \dot{\rho}_l \right)^2. \quad (6)$$

The dots in (6) and in subsequent expressions denote time derivatives.

The collective excitations of interest can be described by the wave function

$$|\psi\rangle = \sum_n a_n(t) B_n^+ |0\rangle. \quad (7)$$

The coefficients $a_n(t)$ satisfy the condition

$$\sum_n |a_n(t)|^2 = 1 \quad (7a)$$

and characterize the excitation distribution along the molecular chain. They satisfy a system of equations which can be obtained from the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} |\psi\rangle = H |\psi\rangle. \quad (8)$$

Using the explicit form of the operator H and (7), we obtain the equation

$$\sum_n \left\{ i\hbar \frac{\partial a_n}{\partial t} - \left[\varepsilon + T + U - \left(1 + \frac{\beta}{R} \rho_n \right) D \right] a_n + J(a_{n+1} + a_{n-1}) \right\} B_n^+ |0\rangle = 0.$$

Since the functions $B_n^+ |0\rangle$ corresponding to different values of n are orthogonal, the following system of equations is obtained from this equation

$$i\hbar \frac{\partial a_n}{\partial t} = \left[\varepsilon + T + U - \left(1 + \frac{\beta}{R} \rho_n \right) D \right] a_n - J(a_{n+1} + a_{n-1}). \quad (9)$$

In what follows it is convenient to introduce the functional of the variables ρ_n , $\dot{\rho}_n$, and $a_n(t)$

$$F = \langle \psi | H | \psi \rangle = \sum_n \left\{ \left[\varepsilon + T + U - \left(1 + \frac{\beta}{R} \rho_n \right) D \right] a_n^* a_n - J a_n^* (a_{n+1} + a_{n-1}) \right\}. \quad (10)$$

Following Toda,^[5] we associate the displacements ρ_n with the canonically conjugate variables

$$s_n = \frac{\partial T}{\partial \rho_n} = -M \sum_{l>n} \dot{\rho}_l.$$

This expression yields the equality

$$\dot{\rho}_n = M^{-1} (s_n - s_{n-1}),$$

which enables us to express the kinetic energy of the molecular motion in terms of the variables s_n :

$$T = \frac{1}{2M} \sum_n (s_n - s_{n-1})^2. \quad (11)$$

Thus, for fixed values of the coefficients $a_n(t)$ the function F is the Hamilton function with respect to the molecular displacements ρ_n and their canonically conjugate quantities s_n . The Hamilton equations can, therefore, be written as

$$\begin{aligned} \dot{\rho}_n &= \frac{\partial F}{\partial s_n} = \frac{1}{M} (2s_n - s_{n+1} - s_{n-1}), \\ \dot{s}_n &= -\frac{\partial F}{\partial \rho_n} = -w\rho_n + \frac{\beta D}{R} |a_n(t)|^2. \end{aligned}$$

Eliminating s_n from these equations, we find

$$\ddot{\rho}_n = -\frac{w}{M} (2\rho_n - \rho_{n+1} - \rho_{n-1}) + \frac{\beta D}{R} (2|a_n|^2 - |a_{n+1}|^2 - |a_{n-1}|^2). \quad (12)$$

The system of equations (9) and (12) defines the collective excitations of the chain. If we are interested only in excitations distributed over some region of the chain, we can transform to the continuum model. We introduce the dimensionless variable $\xi = r/R$ and the continuous functions $\rho(\xi, t)$ and $a(\xi, t)$ such that

$$\rho(n, t) = \rho_n(t), \quad a(n, t) = a_n(t).$$

Expanding, then, $\rho(\xi \pm 1)$ and $a(\xi \pm 1)$ in series, for example,

$$\rho(\xi \pm 1, t) \approx \rho(\xi, t) \pm \frac{\partial \rho(\xi, t)}{\partial \xi} + \frac{1}{2} \frac{\partial^2 \rho(\xi, t)}{\partial \xi^2}$$

we can transform (9) and (12) to the form

$$i\hbar \frac{\partial a(\xi, t)}{\partial t} - \left[\lambda - \frac{\beta D}{R} \rho(\xi, t) \right] a(\xi, t) + J \frac{\partial^2 a(\xi, t)}{\partial \xi^2} = 0, \quad (13)$$

$$\frac{\partial^2 \rho(\xi, t)}{\partial t^2} - v_a^2 \frac{\partial^2 \rho}{\partial \xi^2} + \frac{\beta D}{MR} \frac{\partial^2}{\partial \xi^2} |a(\xi, t)|^2 = 0, \quad (14)$$

$$\lambda = \varepsilon + T + U - D - 2J, \quad (15)$$

$$T + U = \frac{M}{2} \left\{ \int_{-\infty}^{\infty} d\xi \left(\int_{-\infty}^{\infty} d\eta \frac{\partial}{\partial t} |a(\eta, t)|^2 \right)^2 + v_a^2 \int_{-\infty}^{\infty} d\xi |a(\xi, t)|^4 \right\}, \quad (16)$$

$v_a = (w/M)^{1/2}$, and $v_a R = V_a$ is the longitudinal sound velocity in the chain.

We look for solutions of the system (13), (14) in the form of excitations propagating along the chain with ve-

locity $V = \nu R$, i.e., we set

$$\rho(\xi, t) = \rho(\xi - \nu t), \quad a(\xi, t) = \Phi(\xi - \nu t) \exp\{i\theta(\xi, t)\}. \quad (17)$$

We then obtain from (14)

$$\rho(\xi, t) = \frac{\beta D |a(\xi, t)|^2}{MR(\nu_a^2 - \nu^2)}. \quad (18)$$

Substituting this value into (13) and (16), we transform them to the form

$$i\hbar \frac{\partial a(\xi, t)}{\partial t} - \lambda a(\xi, t) + J \frac{\partial^2 a(\xi, t)}{\partial \xi^2} + G_0 |a(\xi, t)|^2 a(\xi, t) = 0, \quad (19)$$

$$T + U = \frac{G_0}{2(\nu_a^2 - \nu^2)} \left[\int_{-\infty}^{\infty} d\xi \left(\int_{-\infty}^{\xi} d\eta \frac{\partial}{\partial t} |a(\eta, t)|^2 \right)^2 + \nu_a \int_{-\infty}^{\infty} d\xi |a(\xi, t)|^2 \right], \quad (20)$$

$$G_0 = \frac{\beta^2 D^2}{MR^2(\nu_a^2 - \nu^2)} = G_0 \left[1 - \frac{V^2}{V_a^2} \right]^{-1}, \quad (21)$$

$$G_0 = \beta^2 D^2 (MV_a^2)^{-1} = \beta^2 D^2 / wR^2. \quad (22)$$

The exact solution of the nonlinear equation (19), normalized by

$$\int_{-\infty}^{\infty} |a(\xi, t)|^2 d\xi = 1,$$

is

$$a(\xi, t) = \frac{(\mu_w/2)^{1/2} \exp\{i[(\hbar\nu/2J)(\xi - \xi_0) - \Omega t]\}}{\text{ch}[\mu_w(\xi - \xi_0 - \nu t)]}, \quad (23)$$

where ξ_0 is an arbitrary constant resulting from the translational invariance of Eqs. (13) and (14). The other quantities in (23) are defined by

$$\mu_w = G_0/4J, \quad (24)$$

$$\hbar\Omega_w = \lambda + \frac{\hbar^2\nu^2 - G_0^2/4}{4J}. \quad (25)$$

When (23) is taken into account Eq. (20) acquires the form

$$T + U = G_0^2(\nu^2 + \nu_a^2)/24J(\nu_a^2 - \nu^2).$$

Substituting this value into (15), we obtain the final expression for the energy of the soliton described by the function (23) when its velocity along the chain ($V = \nu R$) is given:

$$\hbar\Omega_w = \hbar\Omega_0 + \frac{\hbar^2 V^2}{4JR^2} + \frac{G_0^2(x^2 + 1/2x^4 - 1/2x^6)}{24J(1-x^2)^3}, \quad (26)$$

$$x = V/V_a, \quad \hbar\Omega_0 = \epsilon - D - 2J - G_0^2/48J, \quad (27)$$

where $\hbar\Omega_0$ is the internal energy of the soliton.

It follows from (26) that when the soliton velocity approaches the longitudinal sound velocity V_a the soliton energy tends to infinity. Consequently, at all finite values of the soliton energy its velocity is always smaller than V_a . The soliton energy, thus, cannot be transformed into an energy of longitudinal sound waves.

At small soliton velocities, when the inequality $V \ll V_a$ is satisfied, the soliton energy (26) transforms into

$$\hbar\Omega_w = \hbar\Omega_0 + \frac{\hbar^2 V^2}{2R^2 J} + \frac{G_0^2 V^2}{24V_a^2 J} \left[1 - \frac{9}{2} \left(\frac{V}{V_a} \right)^2 + \dots \right].$$

In the same approximation the parameter

$$\mu_w \approx (G_0/4J) [1 + (V/V_a)^2] \approx G_0/4J = \mu_0 \quad (28)$$

depends very weakly on the soliton velocity.

Substituting (23) into (18), we find the distribution of the changes in the relative distance between molecules

$$\rho(\xi, t) = \frac{(G_0^3/w)^{1/2}}{8J(1-V^2/V_a^2)^2} \text{sch}^2[\mu_w(\xi - \xi_0 - \nu t)]. \quad (29)$$

According to (29) the maximum deviation

$$\rho_{\max} = \frac{(G_0^3/w)^{1/2}}{8J(1-V^2/V_a^2)^2} \quad (30)$$

moves at $\xi = \xi_0 + \nu t$ along the chain with velocity $V = \nu R$. The internal excitation moves together with the deformation along the chain. The region of the chain involved in the internal excitation and deformation is defined by the quantity

$$R\Delta\xi \approx \pi R/\mu_w = 4\pi J R/G_0. \quad (31)$$

The applicability condition for the continuum approximation is associated with the validity of the inequality $\Delta\xi = 4\pi J/G_0 > 1$, from which follows the inequality

$$4\pi J > \beta D(\beta D/wR^2).$$

The validity of this inequality, even under condition (4), is favored by a very small value of the factor $\beta D/wR^2$ that characterizes the ratio of the change of the molecular interaction with its neighbors upon excitation to the lattice deformation energy for a molecular displacement equal to the lattice constant.

In a very rigid lattice the parameter w is very large. According to expressions (28) and (22) the inequality $\mu \ll 1$ is satisfied in this case. Therefore the region involved in the excitation is significantly broadened, and in the limiting case ($w \rightarrow \infty$) expression (23) transforms to the form

$$a(\xi, t) \sim \exp\{i[(\hbar\nu/2M)(\xi - \xi_0) - \Omega t]\}.$$

This excitation is not localized, and its energy $\hbar\Omega_w = \epsilon - 2D - 2J + \hbar V^2/2JR^2$ coincides with the exciton energy in a rigid lattice.

3. THE STABILITY OF SOLITONS

A remarkable property of solitons is that at fixed velocity the excitation and deformation distributions described by Eqs. (23) and (29) remain unchanged during the soliton motion. For $V \ll V_a$ we have $G_0 \approx G_0$, so that the size and shape of the soliton are independent of its velocity, i.e., the soliton slows down without changing its shape.

To study the stability of solitons, we consider expression (10) at vanishing velocity ($\nu = 0$). In this case

$a(\xi, t)$ represents a state with a simple time dependence, i. e.,

$$a(\xi, t) = \Phi(\xi) \exp(-i\Omega_0 t). \quad (32)$$

Passing to the continuum approximation in (10) and taking (32) into account, we obtain

$$F_0\{\Phi\} = \varepsilon - D - 2J - \int_{-\infty}^{\infty} \Phi(\xi) \left[J \frac{\partial^2 \Phi(\xi)}{\partial \xi^2} + \frac{1}{2} G_0 \Phi^3(\xi) \right] d\xi. \quad (33)$$

Expression (33) can be regarded as a functional of $\Phi(\xi)$; its absolute minimum defines the soliton energy at vanishing velocity, i. e.,

$$\hbar\Omega_{\min} = \min F_0\{\Phi\}. \quad (33a)$$

If $\Phi(\xi)$ is taken to be a normalized function that depends on a single parameter μ

$$\Phi(\xi) = (\mu/2)^{1/2} \operatorname{sch} [\mu(\xi - \xi_0)], \quad (34)$$

we obtain

$$F_0(\mu) = \varepsilon - D - 2J + 1/2 [J\mu^2 - 1/2 G_0 \mu]. \quad (35)$$

The minimum of (35) corresponds to $\mu_0 = G/4J$, and the energy

$$(\hbar\Omega)_{\min} = \varepsilon - D - 2J - G_0^2/48J \quad (36)$$

coincides with the value of (27). Functions differing from (34) or values $\mu \neq \mu_0$ correspond to higher energies. In particular, the second value of the functional (35) at $\mu = 0$ corresponds to nonlocalized excitations, i. e., excitons. Their energy (at zero velocity) equals $\varepsilon - D - 2J$ and, consequently, exceeds the soliton energy by $G_0^2/48J$. These excitations are unstable in one-dimensional molecular systems.

If the electric dipole moment d of the intramolecular excitation is perpendicular to the chain, then the parameter J characterizing the resonance interaction is negative ($\sim -d^2/R^3$). In this case the functional (35) has no minimum, the continuum approximation is not valid, and there are no stable excitations such as the soliton. The excitation tends to be completely localized on a single molecule.

In conclusion we investigate the stability of solitons with respect to an external perturbation. As the perturbation we consider a low-frequency sound wave

$$A \cos \omega(t - \xi/v_s) \quad (37)$$

at which the wavelength of sound propagations with velocity $v_s R$ exceeds significantly the size of the soliton, i. e.,

$$2\pi v_s R / \omega \gg R \Delta \xi = 4\pi J R / G_0. \quad (38)$$

As mentioned above, at finite values of the soliton energy its velocity is significantly lower than that of longitudinal sound waves. Therefore, in investigating the in-

teractions of longitudinal sound waves with solitons the latter can be regarded as immobile.

To take into account the role of the interaction between the sound wave and the exciton we investigate the solutions of Eqs. (13) and (14) at

$$\begin{aligned} \rho(\xi, t) &= \rho_0(\xi) + A \cos \omega(t - \xi/v_s), \\ \rho_0(\xi) &= \frac{G_0}{8J} \left(\frac{G_0}{w} \right)^{1/2} \operatorname{sch}^2 \mu_0 \xi, \quad \mu_0 = \frac{G_0}{4J}, \end{aligned} \quad (39)$$

where ρ_0 is the solution for a free soliton at rest. In this case Eq. (14) is exactly satisfied by the function

$$a(\xi, t) = \Phi(\xi) \exp[-i\theta(t)] \quad (40)$$

if

$$\Phi(\xi) = (\mu_0/2)^{1/2} \operatorname{sch} \mu_0 \xi. \quad (40a)$$

At $|\xi| > \Delta \xi$, Eq. (13) is approximately satisfied because the function (40) and its derivatives tend rapidly to zero. In the region $|\xi| \leq \Delta \xi$ the sound wave has the same phase under condition (38). Therefore Eq. (13) reduces to

$$\frac{\partial \theta(t)}{\partial t} + \frac{\beta D A}{\hbar R} \cos \omega t - \Omega_0 = 0. \quad (41)$$

The soliton energy is defined by the integral

$$E = i\hbar \int a^*(\xi, t) \frac{\partial a(\xi, t)}{\partial t} d\xi.$$

Therefore, taking into account (40) and (41), we have

$$E = \hbar \frac{\partial \theta(t)}{\partial t} = \hbar \Omega_0 - \frac{\beta D A}{R} \cos \omega t. \quad (42)$$

The soliton energy averaged over the period of the sound wave is thus equal to the energy $\hbar \Omega_0$ of a free soliton. According to (40) the sound wave does not alter the soliton shape, defined by the function

$$\Phi^2(\xi) = 1/2 \mu_0 \operatorname{sch}^2 \mu_0 \xi.$$

The stability of solitons results from a competition between two opposite trends: broadening of the excitation region due to the resonance interaction (J), and excitation localization characterized by the nonlinear term in Eq. (19) proportional to the parameter G . According to (22) this parameter is proportional to $(\beta D)^2$, i. e., to the square of the interaction energy of intramolecular excitations with the molecular displacements (phonons). For $\mu = \beta^2 D^2 / 4wJR^2 \ll 1$ the elementary excitations of the chain are excitons. The interaction of the excitons with the phonons results in their scattering. At strong interactions of the intramolecular excitations with the phonons the elementary excitations of the chain are solitons.

In some sense this situation is reminiscent of the role of electron-phonon interactions in metals. In the normal state of a metal this interaction leads to electron scattering and is the cause for resistivity. In the superconducting state of the metal, however, it is precisely

this electron-phonon interaction which gives rise to the effect of superconductivity. The superconducting state occurs only in metals for which the electron-phonon interaction is sufficiently strong. A strong electron-phonon interaction, giving rise to a large resistivity in the normal state, contributes to the formation of the non-resistive superconducting state. This analogy is verified by the formal relation between the nonlinear Schrödinger equation (19) and the Ginzburg-Landau equation^[7] for a one-dimensional model with no magnetic fields or currents, well-known in the theory of superconductivity^[8], Chap. VI.

Thus, excitations of a molecular chain in the form of solitons are very stable with respect to retaining the shape, magnitude, and size of the region involved in the excitation. This fact has been used by one of us^[4] to explain the high efficiency of the transfer of energy released in the hydrolysis of ATP molecules in living organisms along the α -helical protein molecule and to ex-

plain the contraction mechanism of transversely striated animal muscles at the molecular level.^[8]

¹The deformation of a molecular chain during intramolecular excitations (without taking their motion into account) was first treated by Rashba^[2] (see also^[3]).

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Spin susceptibility of the pseudoisotropic (*B*) phase of superfluid ³He in the acoustic limit (spin waves)

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From a theory of the Larkin-Migdal type the spin-susceptibility tensor at zero temperature is calculated in the acoustic limit and in the collisionless regime. No restrictions are imposed on the effective interaction of the quasi-particles in the particle-hole channel. With regard to the effective interaction in the particle-particle channel it is assumed that only the $l = 0$ and $l = 1$ amplitudes can be close to each other. It is shown that the spin susceptibility always has two poles, corresponding to the frequencies of spin waves of different polarization. It is shown that the spin-wave frequencies depend on the Landau exchange amplitudes for $0 \leq l \leq 3$; the form of this dependence is found. If in the effective interaction in the particle-particle channel the $l = 1$ amplitude is greater than the $l = 0$ amplitude and these amplitudes are close to each other, the spin susceptibility also has a pole corresponding to excitations with a small gap (i.e., much smaller than the gap in the two-particle spectrum).

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The purpose of the present article is to consider the spin waves for systems with Balian-Werthamer^[1] (BW)²⁾ pairing without any restrictions imposed on the effective interaction of the quasi-particles in the particle-particle channel, but in the collisionless regime and at zero temperature. Interest in this topic has arisen again following the identification of the *B* phase of superfluid ³He^[3,4] with the BW state, as originally proposed by Anderson and Brinkman.^[5] This identification gave rise to a number of objections, although the proposal of spin-singlet *D*-pairing^[6] is also in disagreement with recent measurements^[7,8] of the spin susceptibility, and serious objections against spin-triplet *F*-pairing^[9] have been put forward in a paper by Mermin.^[10] Therefore, the BW state is, as yet, the best state for the description of the *B* phase of superfluid ³He.

Collective excitations of the BW state were first considered by Vdovin^[2] for weak coupling. We did not impose this restriction in our previous papers,^[11,12] in which, for BW pairing, we developed theories analogous to those of Larkin and Migdal^[13] and Larkin,^[14] which had been proposed earlier for systems with isotropic *S*-pairing. We^[11] solved the vertex-function equations describing the scalar and vector vertices in the acoustic limit ($|\omega|, kv \ll \Delta$, where ω and \mathbf{k} are the frequency and wave vector, v is the speed of the quasi-particles, and Δ is the energy gap). In addition, we showed^[11] that all these vertex functions have a single pole at the zero-sound frequency, which coincides with the first-sound frequency obtained using the thermodynamic formulas for a Fermi liquid.^[15] Our results have recently been generalized by Maki^[16] to nonzero temperatures. On